Discussiones Mathematicae
Graph Theory 29 (2009) 499–510

THE LIST LINEAR ARBORICITY OF PLANAR GRAPHS *

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Abstract

The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests which partition the edges of G. An and Wu introduce the notion of list linear arboricity $lla(G)$ of a graph G and conjecture that $lla(G) = la(G)$ for any graph G. We confirm that this conjecture is true for any planar graph having $\Delta \geqslant 13$, or for any planar graph with $\Delta \geqslant 7$ and without i-cycles for some $i \in \{3, 4, 5\}$. We also prove that $\lceil \frac{\Delta(G)}{2} \rceil \leqslant lla(G) \leqslant \lceil \frac{\Delta(G)+1}{2} \rceil$ for any planar graph having $\Delta \geqslant 9$.

Keywords: list coloring, linear arboricity, list linear arboricity, planar graph.

2000 Mathematics Subject Classification: 05C10, 05C70.

1. Introduction

All graphs considered here are finite, undirected and simple. We refer to [4] for unexplained terminology and notations. For a real number x, $\lceil x \rceil$ is the least integer not less than x. Let $G = (V(G), E(G))$ be a graph. $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of G, respectively. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and the minimum degree of G, respectively. Let v be a vertex of G. The neighborhood of v, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The degree of v, denoted

*The work is supported by NSFC (No.10601044), XJEDU2006S05 and Scientific Research Foundation for Young Scholar of Xinjiang University.

by $d_G(v)$, is the number of edges incident with v in G. Since G is simple, $d_G(v) = |N_G(v)|$. If there is no confusion, we use $N(v)$ and $d(v)$ for the neighborhood and degree of v instead of $N_G(v)$ and $d_G(v)$, respectively. Let $N_k(v) = \{u|u \in N(v) \text{ and } d(u) = k\}$. The girth of G is the minimum length of cycles in G. A k- or k^+-vertex is a vertex of degree k, or at least k.

A *linear forest* is a graph in which each component is a path. A map φ from $E(G)$ to $\{1, 2, \ldots, k\}$ is called a *k-linear coloring* if $(V(G), \varphi^{-1}(i))$ is a linear forest for $1 \leqslant i \leqslant k$. The *linear arboricity* $la(G)$ of a graph G, introduced by Harary [8], is the minimum number k for which G has a k-linear coloring. Akiyama, Exoo and Harary [1] conjectured that $la(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$ for any regular graph G. It is obvious that for a graph G, $la(G) \geqslant \lceil \frac{\Delta(G)}{2} \rceil$ and $la(G) \geqslant \lceil \frac{\Delta(G)+1}{2} \rceil$ when G is regular. So it is equivalent to the following conjecture, known as the linear arboricity conjecture.

Linear Arboricity Conjecture. For any graph G,

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leqslant la(G) \leqslant \left\lceil \frac{\Delta(G)+1}{2} \right\rceil.$$

The linear arboricity has been determined for complete bipartite graphs [1], series-parallel graphs [10], and regular graphs with $\Delta = 3$ [1], 4 [2], 5, 6, 8 [6], 10 [7]. The LAC also has already been proved to be true for any planar graphs in [9] and [12]. In particular, the author proved that if G is a planar graph with $\Delta \geqslant 13$, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. In [9] and [11], the authors showed that the same also holds for a planar graph with $\Delta \geqslant 7$ and without i-cycles for some $i \in \{3, 4, 5\}$.

A list assignment L to the edges of G is the assignment of a set $L(e) \subseteq N$ of colors to every edge e of G, where N is the set of natural numbers. If G has a coloring φ such that $\varphi(e) \in L(e)$ for every edge e and $(V(G), \varphi^{-1}(i))$ is a linear forest for any $i \in C_\varphi$, where $C_\varphi = \{\varphi(e)|e \in E(G)\}$, then we say that G is *linear L-colorable* and φ is a *linear L-coloring* of G. We say that G is *linear k-list colorable* if it is linear L-colorable for every list assignment L satisfying $|L(e)| = k$ for all edges e. The *list linear arboricity* $lla(G)$ of a graph G is the minimum number k for which G is linear k-list colorable. It is obvious that $la(G) \leqslant lla(G)$. In [3], the authors raised the following conjecture, and confirmed that it is true for any series-parallel graph.

List Linear Arboricity Conjecture. For any graph G,

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leqslant la(G) = lla(G) \leqslant \left\lceil \frac{\Delta(G)+1}{2} \right\rceil.$$

Little was known for this conjecture. In this paper, we will prove that it is true for any planar graph having $\Delta \geqslant 13$, or for any planar graph with $\Delta \geqslant 7$ and without i-cycles for some $i \in \{3, 4, 5\}$. We also prove that $\lceil \frac{\Delta(G)}{2} \rceil \leqslant lla(G) \leqslant \lceil \frac{\Delta(G)+1}{2} \rceil$ for any planar graph having $\Delta \geqslant 9$.

2. Planar Graphs with $la(G) = lla(G)$

For convenience, we introduce two definitions. The *weight* $w(e)$ of an edge $e = uv$ is $d(u) + d(v)$. An even cycle $v_1 v_2 \cdots v_{2t} v_1$ is called *k-alternating* if $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = k$.

Let L be a list assignment of G, and φ be a coloring of G such that $\varphi(e) \in L(e)$ for any edge e of G. For a vertex $v \in V(G)$, we denote by $C_\varphi(v)$ the set of colors that appear on the edges incident with v in G.

$$C^i_\varphi(v) = \{j \mid \text{the color } j \text{ appears } i \text{ times at edges incident with } v\},$$

for any positive integer i. Observe that φ is a linear L-coloring of G if and only if G does not contain a monochromatic cycle under coloring φ and $|C^i_\varphi(v)| = 0$ for every vertex v of G and any $i \geqslant 3$. Thus, if φ is a linear L-coloring of G then $C_\varphi(v) = C^1_\varphi(v) \cup C^2_\varphi(v)$.

The following two lemmas can be found in [9].

Lemma 2.1. *Let G be a planar graph with $\delta(G) \geqslant 2$. Then either there is an edge e with $w(e) \leqslant 15$ or there is a 2-alternating cycle $v_0 v_1 \cdots v_{2n-1} v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max_{0 \leqslant i < n} |N_2(v_{2i})| \geqslant 3$.*

Lemma 2.2. *Let G be a planar graph with girth at least g and maximum degree Δ, and assume that $\delta(G) \geqslant 2$. If $g = 4, 5$ or 6, then either there is an edge e with $w(e) \leqslant 17 - 2g$ or there is a 2-alternating cycle $v_0 v_1 \cdots v_{2n-1} v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max_{0 \leqslant i < n} |N_2(v_{2i})| \geqslant 3$.*

Under the same conditions as given in the next theorem, Wu [9] proved that $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

Theorem 2.3. *Let G be a planar graph having girth at least g and maximum degree Δ. Then $la(G) = lla(G) = \lceil \frac{\Delta(G)}{2} \rceil$, provided that one of the following holds:*

(1) $\Delta \geqslant 13$, (2) $\Delta \geqslant 7$ *and* $g \geqslant 4$,

(3) $\Delta \geqslant 5$ *and* $g \geqslant 5$, (4) $\Delta \geqslant 3$ *and* $g \geqslant 6$.

Proof. Since $\lceil \frac{\Delta(G)}{2} \rceil \leqslant la(G) \leqslant lla(G)$, we show (1) by proving somewhat a stronger statement: any planar graph G is linear k-list colorable for $k = \max\{7, \lceil \frac{\Delta(G)}{2} \rceil\}$.

We shall prove it by induction on $|E(G)|$. The result holds trivially if $|E(G)| \leqslant 7$. Next we assume G be a graph with $|E(G)| \geqslant 8$, and let L be a list assignment of G with $|L(e)| = k$ for any $e \in E(G)$.

Suppose that G has an edge xy such that $w(xy) \leqslant 2k + 1$. Then by induction hypothesis, $G^* = G - xy$ has a linear L-coloring φ. Let $C_\varphi = C^2_\varphi(x) \cup C^2_\varphi(y) \cup (C^1_\varphi(x) \cap C^1_\varphi(y))$. Since $2|C_\varphi| \leqslant d_{G^*}(x) + d_{G^*}(y) = w(xy) - 2 \leqslant 2k - 1$, $|C_\varphi| < k$. We can extend φ to a linear L-coloring of G by taking $\varphi(xy) \in L(xy) \backslash C_\varphi$.

Hence, we assume that $w(xy) > 2k + 1$ for any edge $xy \in E(G)$. Since $k = \max\{7, \lceil \frac{\Delta(G)}{2} \rceil\}$, we have $\delta(G) \geqslant 2$ and $2k + 1 \geqslant 15$. Therefore, for any edge $xy \in E(G)$, $w(xy) > 15$. By Lemma 2.1, G contains a 2-alternating cycle $C = v_0 v_1 \cdots v_{2n-1} v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max\limits_{0 \leqslant i < n} |N_2(v_{2i})| \geqslant 3$.

Without loss of generality, let $|N_2(v_0)| \geqslant 3$. Let $u \in N_2(v_0) \backslash \{v_{2n-1}, v_1\}$ and $v \in N(u) \backslash \{v_0\}$. By induction hypothesis, $G^* = G - \{v_1, v_3, \ldots, v_{2n-1}\} - v_0 u$ has a linear L-coloring σ. Next, we shall extend σ to a linear L-coloring φ of G by setting $\varphi(e) = \sigma(e)$ for each $e \in E(G^*)$, and assigning some appropriate colors for the remaining edges as follows. We consider two cases.

Case 1. $|C_\sigma(v_0)| < k$.
Since $2|C^2_\sigma(v_0)| \leqslant d_{G^*}(v_0) = d(v_0) - 3 \leqslant \Delta(G) - 3 \leqslant 2k - 3$, we have $|C^2_\sigma(v_0)| \leqslant k - 2$.

Subcase 1.1. $|C_\sigma(v_{2j})| < k$ for each $2j$ with $j \in \{1, 2, \ldots, n-1\}$.
We take

$\varphi(v_0 u) \in L(v_0 u) \backslash C_\sigma(v_0)$,
$\varphi(v_0 v_1) \in L(v_0 v_1) \backslash C_\sigma(v_0)$,
$\varphi(v_0 v_{2n-1}) \in L(v_0 v_{2n-1}) \backslash (C^2_\sigma(v_0) \cup \{\varphi(v_0 v_1)\})$, and furthermore

$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash\, C_\sigma(v_{2j})$ for any $j \in \{1, 2, \ldots, n-1\}$.

To check that φ is a linear L-coloring of G, we need to show that there exists no monochromatic cycle containing at least one edge of $E(C) \cup \{v_0u\}$ in G and $|C^i_\varphi(x)| = 0$ for any vertex $x \in V(C) \cup \{u\}$ and any $i \geqslant 3$.

First note that if there is a monochromatic cycle C' in G, then C' does not contain any edges of C since $\varphi(v_0v_{2n-1}) \neq \varphi(v_0v_1)$, $\varphi(v_{2j-1}v_{2j}) \notin C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \notin C_\sigma(v_{2j})$ for each $j \in \{1, 2, \ldots, n-1\}$. Thus C' must contain the edges v_0u and uv. However, since $\varphi(v_0u) \notin C_\sigma(v_0)$, C' cannot be monochromatic.

Now let $x \in V(C) \cup \{u\}$ and i be an integer at least 3. We show that $|C^i_\varphi(x)| = 0$. Since $d(u) = 2$ and $d(v_{2j-1}) = 2$ for each $j \in \{1, 2, \ldots, n-1\}$, the result is trivially true when $x \in \{u, v_1, v_3, \cdots v_{2n-1}\}$. Since $\varphi(v_{2j-1}v_{2j}) \notin C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \notin C_\sigma(v_{2j})$, we have $|C^i_\varphi(v_{2j})| = 0$ for any $j \in \{1, 2, \ldots, n-1\}$. The selection of colors for v_0u, v_0v_1 and v_0v_{2n-1} ensure that $|C^i_\varphi(v_0)| = 0$.

Subcase 1.2. $|C_\sigma(v_{2j})| \geqslant k$ for some $2j$ with $j \in \{1, 2, \ldots, n-1\}$.
We take

$\varphi(v_0u) \in L(v_0u)\backslash(C^2_\sigma(v_0) \cup \{\sigma(uv)\})$,
$\varphi(v_0v_1) \in L(v_0v_1)\backslash(C^2_\sigma(v_0) \cup \{\varphi(v_0u)\})$,
$\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1})\backslash\, C_\sigma(v_0)$.

For $j \in \{1, 2, \cdots, n-1\}$, if $|C_\sigma(v_{2j})| < k$, we take
$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash\, C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash\, C_\sigma(v_{2j})$; otherwise,

$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash(C^2_\sigma(v_{2j}) \cup \{\varphi(v_{2j-2}v_{2j-1})\})$ and
$\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash(C^2_\sigma(v_{2j}) \cup \{\varphi(v_{2j-1}v_{2j})\})$.

Note that $|C^2_\sigma(v_{2j})| \leqslant k-2$ since $k+|C^2_\sigma(v_{2j})| \leqslant |C^1_\sigma(v_{2j})|+2|C^2_\sigma(v_{2j})| = d(v_{2j}) - 2 \leqslant 2k-2$.

We can check that $|C^i_\varphi(x)| = 0$ for any vertex $x \in V(C) \cup \{u\}$ and any $i \geqslant 3$ by a similar argument as in Subcase 1.1. Now, suppose that there is a monochromatic cycle C' in G. Clearly, C' cannot contain the edge v_0u since $\varphi(v_0u) \neq \sigma(uv)$. Thus C' must contain the edges of C. Since there exist some $2j$ such that $\varphi(v_{2j-1}v_{2j}) \neq \varphi(v_{2j-2}v_{2j-1})$, $C' \neq C$. Then C' must contain the path $v_{2l}v_{2l+1}v_{2l+2}\cdots v_{2r-1}v_{2r}$ of C since $\varphi(v_{2l-1}v_{2l}) \neq \varphi(v_{2l-2}v_{2l-1})$ and $\varphi(v_0v_{2n-1}) \notin C_\sigma(v_0)$, where $2 \leqslant 2l < 2r \leqslant 2n-2$ and $min\{|C_\sigma(v_{2l})|, |C_\sigma(v_{2r})|\} \geqslant k$. But $\varphi(v_{2r}v_{2r-1}) \neq \varphi(v_{2r-1}v_{2r-2})$ leads to the contradiction that C' is monochromatic. Thus φ is a linear L-coloring of G.

Case 2. $|C_\sigma(v_0)| \geqslant k$.
Since $k + |C_\sigma^2(v_0)| \leqslant |C_\sigma^1(v_0)| + 2|C_\sigma^2(v_0)| = d(v_0) - 3 \leqslant 2k - 3$, we have $|C_\sigma^2(v_0)| \leqslant k - 3$.

Subcase 2.1. $L(v_0v_1)\backslash C_\sigma^2(v_0) \nsubseteq L(v_0u)\backslash C_\sigma^2(v_0)$.
We take $\varphi(v_0v_1) \in L(v_0v_1)\backslash(C_\sigma^2(v_0) \cup L(v_0u))$. Furthermore, for any $j = \{1, 2, \dots, n-1\}$, we take
$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash C_\sigma(v_{2j})$
if $|C_\sigma(v_{2j})| < k$; otherwise,
$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash(C_\sigma^2(v_{2j}) \cup \{\varphi(v_{2j-2}v_{2j-1})\})$ and
$\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash(C_\sigma^2(v_{2j}) \cup \{\varphi(v_{2j-1}v_{2j})\})$, and finally
$\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1})\backslash(C_\sigma^2(v_0) \cup \{\varphi(v_0v_1), \varphi(v_{2n-1}v_{2n-2})\})$ and
$\varphi(v_0u) \in L(v_0u)\backslash(C_\sigma^2(v_0) \cup \{\varphi(v_0v_{2n-1}), \sigma(uv)\})$.

Subcase 2.2. $L(v_0v_1)\backslash C_\sigma^2(v_0) \subseteq L(v_0u)\backslash C_\sigma^2(v_0)$.
Since $|C_\sigma^2(v_0)| \leqslant k - 3$, we have $|L(v_0u)\backslash C_\sigma^2(v_0)| \geqslant |L(v_0v_1)\backslash C_\sigma^2(v_0)| \geqslant 3$.

We take $\varphi(v_0v_1) = \sigma(uv)$ if $\sigma(uv) \in L(v_0v_1)\backslash C_\sigma^2(v_0)$, and $\varphi(v_0v_1) \in L(v_0v_1)\backslash C_\sigma^2(v_0)$, otherwise. For $j \in \{1, 2, \dots, n-1\}$, we assign a color $v_{2j-1}v_{2j}$ and $v_{2j}v_{2j+1}$ by the way as described in Subcase 2.1.

And then $\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1})\backslash(C_\sigma^2(v_0) \cup \{\varphi(v_{2n-1}v_{2n-2}), \varphi(v_0v_1)\})$. If $\sigma(uv) \in L(v_0u)\backslash C_\sigma^2(v_0)$, but $\sigma(uv) \notin L(v_0v_1)\backslash C_\sigma^2(v_0)$, then $|L(v_0u)\backslash C_\sigma^2(v_0)| \geqslant 4$. So, we take
$\varphi(v_0u) \in L(v_0u)\backslash(C_\sigma^2(v_0) \cup \{\varphi(v_0v_{2n-1}), \varphi(v_0v_1), \sigma(uv)\})$; otherwise,
$\varphi(v_0u) \in L(v_0u)\backslash(C_\sigma^2(v_0) \cup \{\varphi(v_0v_{2n-1}), \varphi(v_0v_1)\})$.
It is easy to check that φ is a linear L-coloring of G both in Subcase 2.1 and Subcase 2.2 by a similar argument as in Subcase 1.2. So we complete the proof of (1).

By using Lemma 2.2, one can similarly prove (2), (3), and (4). ■

For a plane graph G, $F(G)$ denotes the set of faces of G. The degree of a face f, denote by $d(f)$, is the number of edges incident with it, where each cut edge is counted twice. A k-face is a face of degree k.

Theorem 2.4. *Let G be a planar graph with maximum degree $\Delta \geqslant 7$ and without i-cycle for some $i \in \{4, 5\}$. Then $la(G) = lla(G) = \lceil \frac{\Delta(G)}{2} \rceil$.*

Proof. We prove the theorem by contradiction. Let $G = (V, E)$ be a counterexample with the minimum size to the theorem, and be embedded in the plane. Set $k = \lceil \frac{\Delta(G)}{2} \rceil$. Then $k \geqslant 4$ since $\Delta \geqslant 7$. By a similar argument as in proof of Theorem 2.3, we can obtain the following claims.

Claim 1. For any edge $xy \in E(G)$, $w(xy) \geqslant 2k+2$.

Claim 2. G has no even cycle $v_0v_1\cdots v_{2n-1}v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max_{0 \leqslant i < n} |N_2(v_{2i})| \geqslant 3$.

Let G' be the subgraph induced by edges incident with 2-vertices. Since G does not contain two adjacent 2-vertices by Claim 1, G' does not contain any odd cycle. So it follows from Claim 2 that any component of G' is either an even cycle or a tree. So it is easy to find a matching M in G saturating all 2-vertices. Thus if $xy \in M$ and $d(x) = 2$, y is called a 2-master of x. Note that every 2-vertex has a 2-master.

We define a weight function ch on $V(G) \cup F(G)$ by letting $ch(v) = 2d(v) - 6$ for each $v \in V(G)$ and $ch(f) = d(f) - 6$ for each $f \in F(G)$. Applying Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have

$$\sum_{x \in V(G) \cup F(G)} ch(x) = \sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

In the following, we will reassign a new weight $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to some discharging rules. Since we discharge weight from one element to another, the total weight is kept fixed during the discharging. Thus

$$\sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12.$$

We shall show that $ch'(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$, a contradiction, completing the proof.

If G contains no 4-cycles, then we give the following discharging rules.

R1-1. Each 2-vertex receives 2 from its 2-master.

R1-2. Each 3-face f receives $\frac{3}{2}$ from each of its incident 5^+-vertex.

R1-3. Each 5-face f receives $\frac{1}{3}$ from each of its incident 5^+-vertex.

We can obtain that $ch'(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$ by using the same argument in [11]. This complete the proof of the case that G contains no 4-cycles.

Now assume that G contains no 5-cycles. The discharging rules are defined as follows.

R2-1. Each 2-vertex receives 2 from its 2-master.

R2-2. For a 3-face f and its incident vertex v, f receives $\frac{1}{2}$ from v if $d(v) = 4$, 1 if $d(v) = 5$, $\frac{5}{4}$ if $d(v) = 6$ and $\frac{3}{2}$ if $d(v) \geqslant 7$.

R2-3. For a 4-face f and its incident vertex v, f receives $\frac{1}{2}$ from v if $4 \leqslant d(v) \leqslant 6$, 1 if $d(v) \geqslant 7$.

By the same argument in [11], $ch'(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$. Hence, the proof was done for the case that G contains no 5-cycles. ∎

3. Planar Graphs with $\Delta \geqslant 9$

Lemma 3.1 ([5], Lemma 1). *Let G be a planar graph with $\delta(G) \geqslant 3$. Then there is either an edge $e \in E(G)$ with $w(e) \leqslant 11$ or a 3-alternating 4-cycle.*

Theorem 3.2. *Let G be a planar graph with $\Delta(G) \geqslant 9$. Then $\lceil \frac{\Delta(G)}{2} \rceil \leqslant la(G) \leqslant lla(G) \leqslant \lceil \frac{\Delta(G)+1}{2} \rceil$.*

Proof. We prove the theorem by proving somewhat a stronger statement that any planar graph G is linear k-list colorable for $k = \max\{5, \lceil \frac{\Delta(G)+1}{2} \rceil\}$.

We shall prove it by induction on $|E(G)|$. Let L be a list assignment of G with $|L(e)| = k$ for any $e \in E(G)$. Clearly, the result is true when $|E(G)| \leqslant 5$. Next we assume $|E(G)| \geqslant 6$.

Suppose that G has an edge xy such that $w(xy) \leqslant 2k + 1$. Then by induction hypothesis, $G - xy$ has a linear L-coloring φ. Let $C_\varphi = C^2_\varphi(x) \cup C^2_\varphi(y) \cup (C^1_\varphi(x) \cap C^1_\varphi(y))$. Since $2|C_\varphi| \leqslant d_{G-xy}(x) + d_{G-xy}(y) = w(xy) - 2 \leqslant 2k - 1$, $|C_\varphi| < k$. We can extend φ to a linear L-coloring of G by setting $\varphi(xy) \in L(xy) \backslash C_\varphi$.

Hence, we assume that $w(xy) > 2k + 1$ for any edge $xy \in E(G)$ as follows. Since $k = \max\{5, \lceil \frac{\Delta(G)+1}{2} \rceil\}$, we have $\delta(G) \geqslant 3$ and $2k + 1 \geqslant 11$. Thus for any edge $xy \in E(G)$, $w(xy) > 11$. By Lemma 3.1, there is a 4-cycle $v_1v_2v_3v_4v_1$ of G such that $d(v_1) = d(v_3) = 3$. Let $\{u\} = N(v_1) \backslash \{v_2, v_4\}$ and $\{w\} = N(v_3) \backslash \{v_2, v_4\}$. Note that u and w might be the same vertex. By induction hypothesis, $G^* = G - \{v_1, v_3\}$ has a linear L-coloring σ. Next, we shall extend σ to a linear L-coloring φ of G. To do this, set $\varphi(e) = \sigma(e)$ for each $e \in E(G^*)$, and we consider three cases.

Case 1. $\max\{|C_\sigma(v_2)|, |C_\sigma(v_4)|\} < k$.

Since $2|C^2_\sigma(v_2)| \leqslant d_{G^*}(v_2) = d(v_2) - 2 \leqslant \Delta(G) - 2 \leqslant 2k - 3$, we have $|C^2_\sigma(v_2)| \leqslant k - 2$, and similarly $|C^2_\sigma(v_4)| \leqslant k - 2$. We take

$\varphi(v_1v_2) \in L(v_1v_2)\backslash C_\sigma(v_2)$,
$\varphi(v_3v_4) \in L(v_3v_4)\backslash C_\sigma(v_4)$,
$\varphi(v_2v_3) \in L(v_2v_3)\backslash(C^2_\sigma(v_2) \cup \{\varphi(v_3v_4)\})$ and
$\varphi(v_1v_4) \in L(v_1v_4)\ \backslash(C^2_\sigma(v_4) \cup \{\varphi(v_1v_2)\})$.

Subcase 1.1. $u \neq w$.
If $|C_\sigma(w)| \geqslant k$ then $k + |C^2_\sigma(w)| \leqslant |C^1_\sigma(w)| + 2|C^2_\sigma(w)| = d(w) - 1 \leqslant 2k - 2$, and so $|C^2_\sigma(w)| \leqslant k - 2$. Then we assign v_3w a color
$\varphi(v_3w) \in L(v_3w)\backslash(C^2_\sigma(w) \cup \{\varphi(v_2v_3)\})$ if $|C_\sigma(w)| \geqslant k$, and
$\varphi(v_3w) \in L(v_3w)\backslash C_\sigma(w)$, otherwise. Finally,
$\varphi(v_1u) \in L(v_1u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_1v_4)\})$ if $|C_\sigma(u)| \geqslant k$, and
$\varphi(v_1u) \in L(v_1u)\backslash C_\sigma(u)$, otherwise.
To see that φ is a linear L-coloring of G, we shall check that $|C^i_\varphi(x)| = 0$ for any vertex $x \in \{v_1, v_2, v_3, v_4, u, w\}$ and any $i \geqslant 3$, and there exists no monochromatic cycle containing at least one edge of $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1u, v_3w\}$.

Since $d(v_1) = d(v_3) = 3$, $\varphi(v_1v_4) \neq \varphi(v_1v_2)$ and $\varphi(v_2v_3) \neq \varphi(v_3v_4)$, $|C^i_\varphi(x)| = 0$ for $x \in \{v_1, v_3\}$ and any $i \geqslant 3$. $|C^i_\varphi(v_2)| = 0$ for any $i \geqslant 3$ since $\varphi(v_1v_2) \notin C_\sigma(v_2)$ and $\varphi(v_2v_3) \notin C^2_\sigma(v_2)$. Similarly, $|C^i_\varphi(v_4)| = 0$ for any $i \geqslant 3$. Since $\varphi(v_1u) \notin C^2_\sigma(u)$ and $\varphi(v_3w) \notin C^2_\sigma(w)$, $|C^i_\varphi(u)| = |C^i_\varphi(w)| = 0$ for any $i \geqslant 3$.

By contradiction, suppose C is a monochromatic cycle in G. Since $\varphi(v_4v_1) \neq \varphi(v_1v_2)$ and $\varphi(v_4v_1) \neq \varphi(v_1u)$ or $\varphi(v_1u) \notin C_\sigma(u)$, C cannot contain the edge v_4v_1. Similarly, C cannot contain the edge v_2v_3. Thus C must contain the path uv_1v_2 or the path wv_3v_4. However, since $\varphi(v_1v_2) \notin C_\sigma(v_2)$ and $\varphi(v_3v_4) \notin C_\sigma(v_4)$, C cannot be monochromatic.

Subcase 1.2. $u = w$.
Since $2|C^2_\sigma(u)| \leqslant d(u) - 2 \leqslant 2k - 3$, we have $|C^2_\sigma(u)| \leqslant k - 2$.

Assign v_3u a color $\varphi(v_3u) \in L(v_3u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_2v_3)\})$. A choice for a color for v_1u is somewhat complicated.

If $\varphi(v_3u) = \varphi(v_3v_4) = \varphi(v_1v_4)$ then $\varphi(v_1u) \in L(v_1u)\backslash(C^2_\sigma(u)\cup\{\varphi(v_3u)\})$.

If it is not, $\varphi(v_1u) \in L(v_1u)\backslash C_\sigma(u)$ when $|C_\sigma(u)| < k$. For the case $|C_\sigma(u)| \geqslant k$, we have $k + |C^2_\sigma(u)| \leqslant |C^1_\sigma(u)| + 2|C^2_\sigma(u)| = d(u) - 2 \leqslant 2k - 3$, and thus $|C^2_\sigma(u)| \leqslant k - 3$. Then assign a color $\varphi(v_1u) \in L(v_1u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_1v_4), \varphi(v_3u)\})$ for v_1u.

To see φ is a linear L-coloring of G, we verify that $|C^i_\varphi(x)| = 0$ for any vertex $x \in \{v_1, v_2, v_3, v_4, u\}$ and any $i \geqslant 3$, and and show that there exists no monochromatic cycle containing at least one edge of $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1u, v_3u\}$. We can check that $|C^i_\varphi(x)| = 0$ for any vertex $x \in \{v_1, v_2, v_3, v_4\}$ and any $i \geqslant 3$ by a similar argument as Subcase 1.1. The selection of colors for v_1u and v_3u ensure that $|C^i_\varphi(u)| = 0$ for any $i \geqslant 3$. By contradiction, suppose G contains a monochromatic cycle C. One can see that C cannot contain the edge v_2v_3 since $\varphi(v_2v_3) \neq \varphi(v_3u)$ and $\varphi(v_2v_3) \neq \varphi(v_3v_4)$. Clearly, $C \neq v_1uv_3v_4v_1$ by the choice of the color of v_1u. Since $\varphi(v_1v_2) \notin C_\sigma(v_2)$ and $\varphi(v_3v_4) \notin C_\sigma(v_4)$, C cannot contain the edges v_1v_2 and v_3v_4. Thus C must contain the path v_4v_1u, but $\varphi(v_1u) \notin C_\sigma(u)$ or $\varphi(v_1u) \neq \varphi(v_1v_4)$, C is not monochromatic.

Case 2. $|C_\sigma(v_i)| < k$ and $|C_\sigma(v_j)| \geqslant k$ for $\{i, j\} = \{2, 4\}$.
By the symmetry of the roles of v_2 and v_4, assume $|C_\sigma(v_2)| < k$ and $|C_\sigma(v_4)| \geqslant k$. By the similar argument as in proof of Case 1, we have $|C^2_\sigma(v_2)| \leqslant k-2$ and $|C^2_\sigma(v_4)| \leqslant k-3$. We take
$\varphi(v_1v_2) \in L(v_1v_2)\backslash C_\sigma(v_2)$,
$\varphi(v_2v_3) \in L(v_2v_3)\backslash(C^2_\sigma(v_2) \cup \{\varphi(v_1v_2)\})$,
$\varphi(v_3w) \in L(v_3w)\backslash C_\sigma(w)$ if $|C_\sigma(w)| < k$, and $\varphi(v_3w) \in L(v_3w)\backslash(C^2_\sigma(w) \cup \{\varphi(v_2v_3)\})$, otherwise. Then we successively take
$\varphi(v_3v_4) \in L(v_3v_4)\backslash(C^2_\sigma(v_4) \cup \{\varphi(v_2v_3), \varphi(v_3w)\})$ and
$\varphi(v_1v_4) \in L(v_1v_4)\backslash(C^2_\sigma(v_4) \cup \{\varphi(v_3v_4), \varphi(v_1v_2)\})$.
Finally we assign a color for v_1u as follows. If $|C_\sigma(u)| < k$, $\varphi(v_1u) \in L(v_1u)\backslash C_\sigma(u)$. If $|C_\sigma(u)| \geqslant k$, $\varphi(v_1u) \in L(v_1u)\backslash\ (C^2_\sigma(u) \cup \{\varphi(v_1v_4)\})$ if $u \neq w$; $\varphi(v_1u) \in L(v_1u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_1v_4), \varphi(v_3w)\})$, otherwise.

It is easy to check that φ is a linear L-coloring of G by a similar argument as in proof of Case 1.

Case 3. $|C_\sigma(v_i)| \geqslant k$ for each $i \in \{2, 4\}$.
Then $|C^2_\sigma(v_2)| \leqslant k-3$ and $|C^2_\sigma(v_4)| \leqslant k-3$. We take $\varphi(v_1u) \in L(v_1u)\backslash C_\sigma(u)$ if $|C_\sigma(u)| < k$, and $\varphi(v_3w) \in L(v_3w)\backslash C_\sigma(w)$ if $|C_\sigma(w)| < k$. Next we suppose that $|C_\sigma(u)| \geqslant k$ and $|C_\sigma(w)| \geqslant k$.
If $L(v_1v_2)\backslash C^2_\sigma(v_2) \nsubseteq C^1_\sigma(v_2) \cap C^1_\sigma(v_4)$, we take
$\varphi(v_1v_2) \in L(v_1v_2)\backslash(C^2_\sigma(v_2) \cup (C^1_\sigma(v_2) \cap C^1_\sigma(v_4)))$, and then
$\varphi(v_1u) \in L(v_1u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_1v_2)\})$,
$\varphi(v_3w) \in L(v_3w)\backslash(C^2_\sigma(w) \cup \{\varphi(v_1u)\})$,
$\varphi(v_2v_3) \in L(v_2v_3)\backslash(C^2_\sigma(v_2) \cup \{\varphi(v_1v_2), \varphi(v_3w)\})$,
$\varphi(v_3v_4) \in L(v_3v_4)\backslash(C^2_\sigma(v_4) \cup \{\varphi(v_2v_3), \varphi(v_3w)\})$,
$\varphi(v_1v_4) \in L(v_1v_4)\backslash(C^2_\sigma(v_4) \cup \{\varphi(v_3v_4), \varphi(v_1u)\})$.

By the similar argument as in the proof of Case 1, one can show that φ is a linear L-coloring of G.

By symmetry, we consider that $L(v_1v_2)\backslash C_\sigma^2(v_2)$, $L(v_2v_3)\backslash C_\sigma^2(v_2)$, $L(v_3v_4)\backslash C_\sigma^2(v_4)$ and $L(v_4v_1)\backslash C_\sigma^2(v_4)$ are all contained in $C_\sigma^1(v_2) \cap C_\sigma^1(v_4)$.

We claim that $(L(v_1v_2)\backslash C_\sigma^2(v_2)) \cap (L(v_3v_4)\backslash C_\sigma^2(v_4)) = \emptyset$. Suppose it is false, and $|C_\sigma^2(v_2)| \geqslant |C_\sigma^2(v_4)|$, without loss of generality. Therefore,

$$\begin{aligned} 2k - 2|C_\sigma^2(v_2)| &\leqslant k - |C_\sigma^2(v_2)| + k - |C_\sigma^2(v_4)| \\ &\leqslant |L(v_1v_2)\backslash C_\sigma^2(v_2)| + |L(v_3v_4)\backslash C_\sigma^2(v_4)| \\ &\leqslant |C_\sigma^1(v_2) \cap C_\sigma^1(v_4)| \\ &\leqslant |C_\sigma^1(v_2)| \\ &\leqslant d(v_2) - 2|C_\sigma^2(v_2)| \\ &\leqslant 2k - 1 - 2|C_\sigma^2(v_2)|. \end{aligned}$$

It follows that $2k \leqslant 2k - 1$, a contradiction.
Thus we take
$\varphi(v_1v_2) = \varphi(v_3v_4) \in (L(v_1v_2)\backslash C_\sigma^2(v_2)) \cap (L(v_3v_4)\backslash C_\sigma^2(v_4))$ and
$\varphi(v_1u) \in L(v_1u)\backslash(C_\sigma^2(u) \cup \{\varphi(v_1v_2)\})$. And then
$\varphi(v_3w) \in L(v_3w)\backslash\ (C_\sigma^2(w) \cup \{\varphi(v_3v_4)\})$ if $u \neq w$;
$\varphi(v_3w) \in L(v_3w)\backslash\ (C_\sigma^2(w) \cup \{\varphi(v_3v_4), \varphi(v_1u)\})$, otherwise. Finally,
$\varphi(v_2v_3) \in L(v_2v_3)\backslash(C_\sigma^2(v_2) \cup \{\varphi(v_3v_4), \varphi(v_3w)\})$ and
$\varphi(v_1v_4) \in L(v_1v_4)\backslash(C_\sigma^2(v_4) \cup \{\varphi(v_3v_4), \varphi(v_1u)\})$.
One can verify that φ is a linear L-coloring of G.
The proof is complete. ■

Acknowledgement

The authors are grateful to the referees for their helpful comments, which has greatly improved the original presentation of this paper. The work was partially done when the second author was visiting department of computer science and software engineering, Concordia university, and is very grateful to the hospitality of Professor Vasek Chvatal. The financial support from Chinese Scholarship Council is also greatly appreciated.

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Received 12 February 2008
Revised 12 January 2009
Accepted 28 April 2009