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BOUNDS ON THE GLOBAL OFFENSIVE *k*-ALLIANCE NUMBER IN GRAPHS

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Abstract

Let $G = (V(G), E(G))$ be a graph, and let $k \geq 1$ be an integer. A set $S \subseteq V(G)$ is called a *global offensive k-alliance* if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $N(v)$ is the neighborhood of v. The global offensive k-alliance number $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k-alliance in G. We present different bounds on $\gamma_o^k(G)$ in terms of order, maximum degree, independence number, chromatic number and minimum degree.

Keywords: global offensive k-alliance number, independence number, chromatic number.

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1. Terminolgy

Let $G = (V, E) = (V(G), E(G))$ be a finite and simple graph. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $d_G(v)$, is $|N(v)|$. By $n(G) = n$, $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *order*, the *maximum degree* and the *minimum degree* of the graph G, respectively. If $A \subseteq V(G)$, then $G[A]$ is the graph induced by the vertex set A. We denote by K_n the *complete graph* of order n, and by $K_{r,s}$ the *complete bipartite graph* with partite sets X and Y such that $|X| = r$ and $|Y| = s$. A set $D \subseteq V(G)$ is a *k-dominating set* of G if every vertex of $V(G) - D$ has at least $k \geq 1$ neighbors in D. The *k-domination number* $\gamma_k(G)$ is the cardinality of a minimum k-dominating set. The case $k = 1$ leads to the classical *domination number* $\gamma(G) = \gamma_1(G)$.

In [11], Kristiansen, Hedetniemi and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive k-alliances, given by Shafique and Dutton [14, 15]. A set S of vertices of a graph G is called a *global offensive k-alliance* if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $k \geq 1$ is an integer. The *global offensive k-alliance number* $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k-alliance in G. If S is a global k-offensive alliance of G and $|S| = \gamma_o^k(G)$, then we say that S is a $\gamma_o^k(G)$-set. A global offensive 1-alliance is a global offensive alliance and a global offensive 2-alliance is a global strong offensive alliance. In [7], Fernau, Rodríguez and Sigarreta show that the problem of finding optimal global offensive k-alliances is NP-complete.

If $k \geq 1$ is an integer, then let $L_k(G) = \{x \in V(G) : d_G(x) \leq k - 1\}$. Denote by $\alpha(G)$ the *independence number,* by $\chi(G)$ the *chromatic number,* and by $\omega(G)$ the *clique number* of G, respectively. The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. Next assume that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

2. UPPER BOUNDS

We begin this section by giving an upper bound on the global offensive k -alliance number for an r-partite graph G in terms of its order and $|L_k(G)|$.

Theorem 1. *Let $k \geq 1$ be an integer. If G is an r-partite graph, then*

$$\gamma_o^k(G) \leq \frac{(r-1)n(G) + |L_k(G)|}{r}.$$

Proof. Clearly, the set $L_k(G)$ is contained in every $\gamma_o^k(G)$-set. In the case that $|L_k(G)| = |V(G)|$, we are finished. In the remaining case that $|L_k(G)| < |V(G)|$, let $V_1, V_2, \ldots, V_r$ be a partition of the r-partite graph $G - L_k(G)$ such that $|V_1| \geq |V_2| \geq \cdots \geq |V_r|$, where $V_i = \emptyset$ is possible for $i \geq 2$. Then every vertex of V_1 has degree at least k in G, and all its neighbors are in $V(G) - V_1$. Thus $V(G) - V_1$ is a global offensive k-alliance of G. Since

$$|V_1| \geq \frac{|V_1| + |V_2| + \cdots + |V_r|}{r} = \frac{n(G) - |L_k(G)|}{r},$$

we obtain

$$\gamma_o^k(G) \leq n(G) - |V_1| \leq n(G) - \frac{n(G) - |L_k(G)|}{r} = \frac{(r-1)n(G) + |L_k(G)|}{r},$$

and the proof is complete. ■

The case $k = r = 2$ in Theorem 1 leads to the next result.

Corollary 2 (Chellali [4]). *If G is a bipartite graph, then*

$$\gamma_o^2(G) \leq \frac{n(G) + |L_2(G)|}{2}.$$

Observation 3. *If $k \geq 1$ is an integer, then $\gamma_o^k(G) \geq \gamma_k(G)$ for any graph G.*

Proof. If S is any $\gamma_o^k(G)$-set, then every vertex of $V(G) - S$ has at least k neighbors in S. Thus S is a k-dominating set of G and so $\gamma_k(G) \leq |S| = \gamma_o^k(G)$. ■

Using Theorem 1 for $r = 2$ and Observation 3, we obtain the known theorem by Blidia, Chellali and Volkmann [2].

Corollary 4 (Blidia, Chellali, Volkmann [2] 2006). *Let k be a positive integer. If G is a bipartite graph, then*

$$\gamma_k(G) \leq \frac{n(G) + |L_k(G)|}{2}.$$

Since every graph G is $\chi(G)$-partite and $n(G) \leq \chi(G)\alpha(G)$, we obtain also the following corollaries from Theorem 1.

Corollary 5. *If G is a graph and k a positve integer, then*

$$\gamma_o^k(G) \leq \frac{(\chi(G) - 1)n(G) + |L_k(G)|}{\chi(G)}.$$

Corollary 6. *Let $k \geq 1$ be an integer. If G is a graph with $\delta(G) \geq k$, then*

$$\gamma_o^k(G) \leq (\chi(G) - 1)\alpha(G).$$

Theorem 7 (Brooks [3] 1941). *If G is a connected graph different from the complete graph and from a cycle of odd length, then $\chi(G) \leq \Delta(G)$.*

Combining Brooks' Theorem and Corollary 6, we can prove the following result.

Theorem 8. *Let $k \geq 1$ be an integer, and let G be a connected graph with $\delta(G) \geq k$. Then*

$$\gamma_o^k(G) \leq (\Delta(G) - 1)\alpha(G) \tag{1}$$

if and only if G is neither isomorphic to the complete graphs K_{k+1} or K_{k+2} nor to a cycle of odd length when $1 \leq k \leq 2$.

Proof. If G is the complete graph K_n, then $\Delta(G) = \delta(G) = n - 1 \geq k \geq 1$ and $\alpha(G) = 1$. Since $\gamma_o^k(K_{k+1}) = k$ and $\gamma_o^k(K_{k+2}) = k + 1$, inequality (1) is not true for these two complete graphs. However, in the remaining case that $n \geq k + 3$, we observe that $\gamma_o^k(G) \leq n - 2$, and we arrive at the desired bound

$$\gamma_o^k(G) \leq n - 2 = \Delta(G) - 1 = (\Delta(G) - 1)\alpha(G).$$

Assume next that $1 \le k \le 2$. If G is a cycle of odd length, then $\Delta(G) = 2$, $\gamma_o^1(G) = \gamma_o^2(G) = \lceil n(G)/2 \rceil$ and $\alpha(G) = \lfloor n(G)/2 \rfloor$ and thus (1) is not valid in these cases.

For all other graphs inequality (1) follows directly from Brooks' Theorem and Corollary 6. ■

Lemma 9 (Hansberg, Meierling, Volkmann [10]). *Let $k \ge 1$ be an integer. If G is a connected graph with $\delta(G) \le k-1$ and $\Delta(G) \le k$, then*

$$k\alpha(G) \ge n(G).$$

Theorem 10. *Let $k \ge 1$ be an integer. If G is a connected r-partite graph with $\Delta(G) \ge k$, then*

$$\gamma_o^k(G) \le \frac{\alpha(G)}{r}((r-1)r + k - 1).$$

Proof. Assume that $k = 1$. Since G is connected and $\Delta(G) \ge 1$, we note that $|L_1(G)| = 0$. Applying Theorem 1, and using the fact that $r\alpha(G) \ge n(G)$, we receive the desired inequality immediately.

Assume next that $k \ge 2$. Since G is connected and $G - L_k(G)$ is not empty, every component Q of $G[L_k(G)]$ fufills $\delta(Q) \le k-2$ and $\Delta(Q) \le k-1$. Hence Lemma 9 implies $(k-1)\alpha(Q) \ge n(Q)$. If $Q_1, Q_2, \ldots, Q_t$ are the components of $G[L_k(G)]$, we therefore deduce that

$$\alpha(G) \ge \alpha(G[L_k(G)]) = \sum_{i=1}^{t} \alpha(Q_i) \ge \frac{|L_k(G)|}{k-1}.$$

Combining $n(G) \le r\alpha(G)$ with Theorem 1, we receive the desired inequality as follows:

$$\begin{aligned} \gamma_o^k(G) &\le \frac{(r-1)n(G) + |L_k(G)|}{r} \\ &\le \frac{(r-1)r\alpha(G) + (k-1)\alpha(G)}{r} \\ &= \frac{\alpha(G)}{r}((r-1)r + k - 1). \end{aligned}$$

■

The case $r = 2$ in Theorem 10 leads to the next result.

Corollary 11. *Let $k \geq 1$ be an integer. If G is a connected bipartite graph with $\Delta(G) \geq k$, then*

$$\gamma_o^k(G) \leq \frac{(k+1)\alpha(G)}{2}.$$

Using Observation 3, we obtain the following known bounds on the 2-domination number.

Corollary 12 (Fujisawa, Hansberg, Kubo, Saito, Sugita, Volkmann [9] 2008)**.** *If G is a connected bipartite graph of order at least* 3, *then*

$$\gamma_2(G) \leq \frac{3\alpha(G)}{2}.$$

Corollary 13 (Blidia, Chellali, Favaron [1] 2005)**.** *If T is a tree of order at least* 3, *then*

$$\gamma_2(T) \leq \frac{3\alpha(T)}{2}.$$

Theorem 14 (Favaron, Hansberg, Volkmann [6] 2008)**.** *Let G be a graph. If $r \geq 1$ is an integer, then there is a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ of $V(G)$ such that*

$$|N_G(u) \cap V_i| \leq \frac{d_G(u)}{r} \tag{2}$$

for each $i \in \{1, 2, \ldots, r\}$ and each $u \in V_i$.

Theorem 15. *Let $k \geq 1$ be an integer. If G is a graph of order n with minimum degree $\delta \geq k$, then*

$$\gamma_o^k(G) \leq \frac{k+1}{k+2}n, \tag{3}$$

and the bound given in (3) *is best possible.*

Proof. Choose $r = k+2$ in Theorem 14, and let $V_1, V_2, \ldots, V_r$ be a partition of $V(G)$ as in Theorem 14 such that $|V_1| \geq |V_2| \geq \cdots \geq |V_r|$. If $D = V_2 \cup V_3 \cup \cdots \cup V_r$, then it follows from (2) and the hypothesis that $\delta \geq k$ for each $v \in V_1 = V(G) - D$ that

$$\begin{aligned} |N_G(v) \cap D| &\geq \left\lceil \frac{k+1}{k+2} d_G(v) \right\rceil \geq \left\lfloor \frac{d_G(v)}{k+2} \right\rfloor + k \\ &\geq |N_G(v) \cap V_1| + k = |N_G(v) - D| + k. \end{aligned}$$

Thus D is a global offensive k-alliance of G such that $|D| \leq (k+1)n/(k+2)$, and (3) is proved.

Let H be a connected graph, and let $G_k = H \circ K_{k+1}$. Then it is easy to see that $\gamma_o^k(G_k) = (k+1)n(G_k)/(k+2)$, and therefore (3) is the best possible. ■

Corollary 16 (Favaron, Fricke, Goddard, Hedetniemi, Hedetniemi, Kristiansen, Laskar, Skaggs [5] 2004)**.** *Let G be graph of order n and minimum degree δ.*
If $\delta \geq 1$, then $\gamma_o^1(G) \leq 2n/3$.
If $\delta \geq 2$, then $\gamma_o^2(G) \leq 3n/4$.

In the case that $\delta \geq k+2$, we obtain the following bound, improving the bound of Theorem 15.

Theorem 17. *Let $k \geq 2$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq k+2$. Then*

$$\gamma_o^k(G) \leq \frac{k}{k+1}n. \tag{4}$$

Proof. Choose $r = k+1$ in Theorem 14, and let $V_1, V_2, \ldots, V_r$ be a partition of $V(G)$ as in Theorem 14 such that $|V_1| \geq |V_2| \geq \cdots \geq |V_r|$. If $D = V_2 \cup V_3 \cup \cdots \cup V_r$, then it follows from (2) and the hypothesis $\delta \geq k+2$ for each $v \in V_1 = V(G) - D$ that

$$\begin{aligned} |N_G(v) \cap D| &\geq \left\lceil \frac{k}{k+1} d_G(v) \right\rceil \geq \left\lfloor \frac{d_G(v)}{k+1} \right\rfloor + k \\ &\geq |N_G(v) \cap V_1| + k = |N_G(v) - D| + k. \end{aligned}$$

Thus D is a global offensive k-alliance of G such that $|D| \leq kn/(k+1)$, and (4) is proved. ■

Theorem 18. *Let $k \geq 1$ be an integer, and let G be a connected non-complete graph such that $\delta(G) \geq k$ and $\gamma_o^k(G) = (\Delta(G)-1)\alpha(G)$. Then $\Delta(G) \leq k+2$, $\Delta(G) - \delta(G) \leq 1$ and if $k \geq 2$, then $\delta(G) \leq k+1$.*

Proof. Because of $\chi(G)\alpha(G) \geq n(G)$, Corollary 5 and the hypothesis imply that

$$(\Delta(G)-1)\alpha(G) = \gamma_o^k(G) \leq \frac{(\chi(G)-1)n(G)}{\chi(G)} \leq (\chi(G)-1)\alpha(G).$$

Since G is neither a complete graph nor a cycle of odd length, it follows from Brooks' Theorem that $\Delta(G) = \chi(G)$, $\chi(G)\alpha(G) = n(G)$ and

$$\gamma_o^k(G) = \frac{(\chi(G)-1)n(G)}{\chi(G)} = \frac{(\Delta(G)-1)n(G)}{\Delta(G)}. \tag{5}$$

If we suppose on the contrary that $\Delta(G) \geq k+3$, then it follows from (5) and Theorem 15 that

$$\frac{\Delta(G)-1}{\Delta(G)}n(G) = \gamma_o^k(G) \leq \frac{k+1}{k+2}n(G) \leq \frac{\Delta(G)-2}{\Delta(G)-1}n(G).$$

This contradiction shows that $\Delta(G) \leq k+2$.

If we suppose on the contrary that $\Delta(G) - \delta(G) \geq 2$, then we deduce that $\delta(G) = k$ and $\Delta(G) = k+2 = \chi(G)$. Since $\chi(G)\alpha(G) = n(G)$, there exists a partition of $V(G)$ in $\chi = \chi(G)$ colour classes $U_1, U_2, \ldots, U_\chi$ such that $|U_1| = |U_2| = \cdots = |U_\chi| = \alpha(G)$. Let v be a vertex of minimum degree $\delta(G) = k$, and assume, without loss of generality, that $v \in U_1$. As $d_G(v) = k$ and $\chi(G) = k+2$, there exists a colour class U_j with $2 \leq j \leq \chi$ such that v is not adjacent to any vertex in U_j. Therefore $U_j \cup \{v\}$ is an independent set. This is a contradiction to the fact that $|U_j| = \alpha(G)$, and the desired inequality $\Delta(G) - \delta(G) \leq 1$ is proved.

Next assume that $k \geq 2$, and suppose on the contrary that $\delta(G) \geq k+2$. Then $k \leq \Delta(G) - 2$ and (5) and Theorem 17 lead to the contradiction

$$\frac{\Delta(G)-1}{\Delta(G)}n(G) = \gamma_o^k(G) \leq \frac{k}{k+1}n(G) \leq \frac{\Delta(G)-2}{\Delta(G)-1}n(G).$$

Thus $\delta(G) \leq k \leq \delta(G)+1$ when $k \geq 2$, and the proof of Theorem 18 is complete. ■

Example 19. 1. Let $H_1, H_2, \ldots, H_t$ be $t \geq 2$ copies of the complete graph K_{k+1}, and let $u_i, v_i \in E(H_i)$ for $1 \leq i \leq t$. Define the graph G as the disjoint union $H_1 \cup H_2 \cup \cdots \cup H_t$ together with the edge set $\{v_1u_2, v_2u_3, \ldots, v_{t-1}u_t\}$. Then it is easy to verify that $\Delta(G) = k+1$, $\delta(G) = k$, $\alpha(G) = t$, $\gamma_o^k(G) = tk$ and thus $\gamma_o^k(G) = (\Delta(G)-1)\alpha(G)$.

2. Let F_1 and F_2 be 2 copies of the complete graph K_{k+1} with the vertex sets $V(F_1) = \{x_1, x_2, \ldots, x_{k+1}\}$ and $V(F_2) = \{y_1, y_2, \ldots, y_{k+1}\}$. Define the graph H as the disjoint union $F_1 \cup F_2$ together wit the edge set $\{x_1y_1, x_2y_2, \ldots, x_ky_k\}$. If $H_1, H_2, \ldots, H_t$ are $t \geq 2$ copies of H, then let

u_{2i-1} and u_{2i} be the vertices of degree k in H_i for all $i \in \{1, 2, \ldots, t\}$. Define the graph G as the disjoint union $H_1 \cup H_2 \cup \cdots \cup H_t$ together with the edge set $\{u_2u_3, u_4u_5, \ldots, u_{2t}u_1\}$. Then G is a $(k+1)$-regular graph with $\alpha(G) = 2t$, $\gamma_o^k(G) = 2kt$ and thus $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$.

3. Let $k \geq 2$, and let F_1 and F_2 be 2 copies of the complete graph K_k such that $V(F_1) = \{x_1, x_2, \ldots, x_k\}$ and $V(F_2) = \{y_1, y_2, \ldots, y_k\}$. Define the graph H as the disjoint union $F_1 \cup F_2$ together wit the edge set $\{x_1y_1, x_2y_2, \ldots, x_{k-1}y_{k-1}\}$. If $H_1, H_2, \ldots, H_t$ are $t \geq 2$ copies of H, then let u_{2i-1} and u_{2i} be the vertices of degree $k-1$ in H_i for all $i \in \{1, 2, \ldots, t\}$. Define the graph G as the disjoint union $H_1 \cup H_2 \cup \cdots \cup H_t$ together with the edge set $\{u_2u_3, u_4u_5, \ldots, u_{2t}u_1\}$. Then G is a k-regular graph with $\alpha(G) = 2t$, $\gamma_o^k(G) = 2(k-1)t$ and thus $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$.

4. Let H_1 and H_2 be 2 copies of the complete graph K_{k+2}, and let $x \in E(H_1)$ and $y \in E(H_2)$. Define the graph G' as the disjoint union $H_1 \cup H_2$ together with the edge xy. Then $\Delta(G') = k+2$, $\delta(G') = k+1$, $\alpha(G') = 2$, $\gamma_o^k(G') = 2(k+1)$ and thus $\gamma_o^k(G') = (\Delta(G') - 1)\alpha(G')$.

These four examples show that $\Delta = k+1$ and $\delta = k$, $\Delta = \delta = k+1$, $\Delta = \delta = k$ as well as $\Delta = k+2$ and $\delta = k+1$ in Theorem 18 are possible.

Theorem 20. *If G is a graph and k an integer such that $1 \leq k \leq \delta(G) - 1$, then*

$$\gamma_o^{k+1}(G) \leq \frac{\gamma_o^k(G) + n(G)}{2}.$$

Proof. Let S be a $\gamma_o^k(G)$-set, and let A be the set of isolated vertices in the subgraph induced by the vertex set $V(G) - S$. Then the subgraph induced by $V(G) - (S \cup A)$ contains no isolated vertices. If D is a minimum dominating set of $G[V(G) - (S \cup A)]$, then the well-known inequality of Ore [12] implies

$$|D| \leq \frac{|V(G) - (S \cup A)|}{2} \leq \frac{|V(G) - S|}{2} = \frac{n(G) - \gamma_o^k(G)}{2}.$$

Since $\delta(G) \geq k+1$, every vertex of A has at least $k+1$ neighbors in S, and therefore $D \cup S$ is a global offensive $(k+1)$-alliance of G. Thus we obtain the desired bound as follows:

$$\gamma_o^{k+1}(G) \leq |S \cup D| \leq \gamma_o^k(G) + \frac{n(G) - \gamma_o^k(G)}{2} = \frac{\gamma_o^k(G) + n(G)}{2}.$$ ■

The graphs G of even order and without isolated vertices with $\gamma(G) = n/2$ have been characterized independently by Payan and Xuong [13] and Fink, Jacobson, Kinch and Roberts [8].

Theorem 21 (Payan, Xuong [13] 1982 and Fink, Jacobson, Kinch, Roberts [8] 1985)**.** *Let G be a graph of even order n without isolated vertices. Then $\gamma(G) = n/2$ if and only if each component of G is either a cycle C_4 or the corona of a connected graph.*

A graph is P_4-free if and only if it contains no induced subgraph isomorphic to the path P_4 of order four. A graph is (K_4-e)-free if and only if it contains no induced subgraph isomorphic to the graph K_4-e, where e is an arbitrary edge of the complete graph K_4. The graph $\overline{G}$ denotes the complement of the graph G. Next we give a characterization of some special graphs attaining equality in Theorem 20.

Theorem 22. *Let G be a connected P_4-free graph such that $\overline{G}$ is $(K_4 - e)$-free. If k is an integer with $1 \leq k \leq \delta(G) - 1$, then $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$ if and only if*

1. *$G = K_{k+3}$ or*
2. *$\overline{G} = H \cup 2K_{1,1}$ such that $n(H) = k + 2$ and all components of H are isomorphic to $K_{1,1}$, to $K_{3,3}$, to $K_{3,4}$ or to $K_{4,4}$ or*
3. *$G = (Q_1 \cup Q_2) + F$, where Q_1, Q_2 and F are three pairwise disjoint graphs such that $1 \leq |V(F)| \leq k+1$, $\alpha(F) \leq 2$, and Q_1 and Q_2 are cliques with $|V(Q_1)| = |V(Q_2)| = k + 3 - |V(F)|$ such that*
   $|V(F)| \leq 2$ or
   $\alpha(F) = 1$ and $|V(F)| = k + 1$ or
   $\alpha(F) = 2$ and $F = K_{k+1} - M$, where M is a matching of F or
   $\alpha(F) = 2$ and $F = K_k - M$, where M is a perfect matching of F or
   $\alpha(F) = 2$ and $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ with $k \geq 3t + 3$ and all components of $\overline{F}$ are isomorphic to $K_{t+2,t+2}$, to $K_{t+2,t+3}$ or to $K_{t+3,t+3}$.

Proof. Assume that $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$. Following the same notation as used in the proof of Theorem 20, we obtain $|D| = \frac{|V(G)-S|}{2}$, and we observe that $S \cup D$ is a $\gamma_o^{k+1}(G)$-set. It follows that $G[V(G) - S]$ has no isolated vertices and so by Theorem 21, each component of $G[V(G) - S]$ is either a cycle C_4 or the corona of some connected graph. Using the hypothesis that G is P_4-free, we deduce that each component of $G[V(G)-S]$ is isomorphic to K_2 or to C_4. Since $\overline{G}$ is (K_4-e)-free, there remain exactly the three cases that $G[V(G)-S]$ is isomorphic to K_2, to C_4 or to $2K_2$.

Case 1. First assume that $G[V(G)-S]=K_2$. Suppose that G has an independent set Q of size at least two. Then the hypothesis $\delta(G)\geq k+1$ implies that $V(G)-Q$ is a global offensive $(k+1)$-alliance of G of size $n-|Q|<|S\cup D|=n-1$, a contradiction. Therefore $\alpha(G)=1$ and thus $G=K_{k+3}$.

Case 2. Second assume that $G[V(G)-S]$ is a cycle $C_4=x_0x_1x_2x_3x_0$. In the following the indices of the vertices x_i are taken modulo 4. Recall that $S\cup D$ is a $\gamma_o^{k+1}(G)$-set, and D contains two vertices of the cycle C_4. Clearly, since S is a $\gamma_o^k(G)$-set, every vertex of the cycle C_4 has degree at least $k+4$. Suppose that $d_G(x_i)\geq k+5$ for an $i\in\{0,1,2,3\}$. Then $\{x_{i+2}\}\cup S$ is a global offensive $(k+1)$-alliance of G of size $|S|+1<|S\cup D|=|S|+2$, a contradiction. Thus $d_G(x_i)=k+4$ for every $i\in\{0,1,2,3\}$. Now if Q is an $\alpha(G)$-set, then $|Q|\leq 2$, for otherwise the hypothesis $\delta(G)\geq k+1$ implies that $V(G)-Q$ is a global offensive $(k+1)$-alliance of G of size $|V(G)-Q|<|S\cup D|=n(G)-2$, a contradiction too. Since there are two non-adjacent vertices on the cycle C_4 and G is P_4-free, it follows that every vertex of S has at least three neighbors on the cycle C_4.

Subcase 2.1. Assume that $\alpha(G[S])=1$. Then the subgraph induced by S is complete and $|S|\geq k+2$. If $|S|=k+2$, then we observe that every vertex of S has exactly four neighbours on the cycle C_4. Thus, in each case, we deduce that $d_G(y)\geq k+5$ for every $y\in S$. But then for any subset W of S of size three, the set $V(G)-W$ is a global offensive $(k+1)$-alliance of G of size less than $|S\cup D|$, a contradiction.

Subcase 2.2. Assume that $\alpha(G[S])=2$. Suppose that there exists a vertex $w\in S$ with at least $k+1$ neighbors in S. Then, since $|N(w)\cap V(C_4)|\geq 3$, say $\{x_0,x_1,x_2\}\subseteq N(w)\cap V(C_4)$, we observe that $(S-\{w\})\cup\{x_0,x_2\}$ is a global offensive $(k+1)$-alliance of G of size $|S|+1<|S\cup D|$, a contradiction. Thus every vertex of S has at most k neighbors in S.

Let $S=X\cup Y$ such that every vertex of X has exactly three and every vertex of Y exactly 4 neighbors on C_4. We shall show that $X=\emptyset$. If $X\neq\emptyset$, then let $S_{x_i}\subseteq X$ be the set of vertices such that each vertex of S_{x_i} is not adjacent to x_{i+2} for $i\in\{0,1,2,3\}$. Because of $\alpha(G)=2$, we observe that the set $S_{x_i} \cup \{x_i\}$ induces a complete graph for each $i \in \{0,1,2,3\}$. In additon, since G is P_4-free it is straightforward to verify that all vertices of $X \cup C_4$ are adjacent to all vertices of Y and that $S_{x_i} \cup S_{x_{i+1}} \cup \{x_i, x_{i+1}\}$ induces a complete graph for each $i \in \{0,1,2,3\}$. Now assume, without loss of generality, that $S_{x_0} \neq \emptyset$, and let $w \in S_{x_0}$. On the one hand we have seen above that $d_G(w) \leq k+3$. On the other hand, we observe that $d_G(w) = d_G(x_0)$. But since $d_G(x_0) = k+4$, we have a contradiction.

Hence we have shown that $X = \emptyset$, and this leads to $|S| = k+2$. If we define $H = \overline{G[S]}$, then $\omega(H) = 2$, $\delta(H) \geq 1$ and $\Delta(H) \leq 4$. Since H is also P_4-free, H does not contain an induced cycle of odd length. Using $\omega(H) = 2$, we deduce that H is a bipartite graph. Now let H_i be a component of H. If H_i is not a complete bipartite graph, then H_i contains a P_4, a contradiction. Thus the components of H consists of $K_{1,1}$, $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, $K_{2,2}$, $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$.

If $K_{1,2}$ is a component of H, then $V(G) - V(K_{1,2})$ is a global offensive $(k+1)$-alliance of G of size $n-3$, a contradiction.

If $K_{1,3}$ is a component of H with a leaf u, then $(V(G) - V(K_{1,3})) \cup \{u\}$ is a global offensive $(k+1)$-alliance of G of size $n-3$, a contradiction.

If $K_{1,4}$ is a component of H and u, v are two leaves of this star, then $(V(G) - V(K_{1,3})) \cup \{u, v\}$ is a global offensive $(k+1)$-alliance of G of size $n-3$, a contradiction.

If $K_{2,2}$ is a component of H, then $V(G) - V(K_{2,2})$ is a global offensive $(k+1)$-alliance of G of size $n-4$, a contradiction.

Next let $K_{2,3}$ be a component of H with the bipartition $\{v_1, v_2, v_3\}$ and $\{u_1, u_2\}$. Then $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k+1)$-alliance of G of size $n-3$, a contradiction.

Finally, let $K_{2,4}$ be a component of H with the bipartition $\{v_1, v_2, v_3, v_4\}$ and $\{u_1, u_2\}$. Then $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k+1)$-alliance of G of size $n-3$, a contradiction.

Case 3. Third assume that $G[V(G) - S] = 2K_2$. Let $2K_2 = J_1 \cup J_2 = J$ such that $V(J_1) = \{u_1, u_2\}$ and $V(J_2) = \{u_3, u_4\}$. If $\alpha(G) \geq 3$, then we obtain the contradiction $\gamma_o^{k+1}(G) \leq n-3$. Thus $\alpha(G) = 2$. Since S is a $\gamma_o^k(G)$-set, every vertex of J has degree at least $k+2$. Suppose that $d_G(u_1) \geq k+3$ and $d_G(u_2) \geq k+3$. Then $\{u_3\} \cup S$ is a global offensive $(k+1)$-alliance of G of size $|S|+1 < |S \cup D| = |S|+2$, a contradiction. Thus J_1 contains at least one vertex of degree $k+2$, and for reason of symmetry, also J_2 contains a vertex of degree $k+2$. Since $\alpha(G) = 2$, every vertex of

S has at least two neighbors in J_1 or in J_2. Now let $x \in S$. If x has two neighbors in J_i and one neighbor in J_{3-i} for $i = 1, 2$, then the hypothesis that G is P_4-free implies that x is adjacent to each vertex of J. Consequently, S can be partioned in three subsets S_1, S_2 and A such that all vertices of S_1 are adjacent to all vertices of J_1 and there is no edge between S_1 and J_2, all vertices of S_2 are adjacent to all vertices of J_2 and there is no edge between S_2 and J_1, all vertices of A are adjacent to all vertices of J. Since G is P_4-free, it follows that there is no edge between S_1 and S_2, and that all vertices of S_i are adjacent to all vertices of A for $i = 1, 2$. Furthermore, $\alpha(G) = 2$ shows that $G[S_1]$ and $G[S_2]$ are cliques. Altogether we see that $d_G(u_i) = k + 2$ for each $i \in \{1, 2, 3, 4\}$ and therefore $|S_1| + |A| = |S_2| + |A| = k + 1$. It follows that $|S_1| = |S_2|$ and $|S| + |A| = 2k + 2$. Since G is connected, we deduce that $|A| \geq 1$ and so $1 \leq |A| \leq k + 1$. If we define $F = G[A]$ and $Q_i = G[S_i \cup V(J_i)]$ for $i = 1, 2$, then we derive the desired structure, since $\alpha(G[A]) \leq 2$.

Assume that $|V(F)| \geq 3$ and $\alpha(F) = 1$. If x_1, x_2, x_3 are three arbitrary vertices in F, then let $S_0 = V(G) - \{x_1, x_2, x_3\}$. If $d_G(x_i) \geq k + 5$ for each $i = 1, 2, 3$, then S_0 is a global offensive $(k+1)$-alliance of G, a contradiction. Otherwise, we have $n - 1 = d_G(x_i) \leq k + 4$ for at least one $i \in \{1, 2, 3\}$ and so $n \leq k + 5$ and thus $|V(F)| = k + 1$.

Assume next that $|V(F)| \geq 3$ and $\alpha(F) = 2$. As we have seen in Case 2, all components of $\overline{F}$ are complete bipartite graphs.

Subcase 3.1. Assume that $K_{1,1}$ is the greatest component of $\overline{F}$. Let u and v be the two vertices of the complete bipartite graph $K_{1,1}$. If $n \geq k+7$, then let w be a further vertex in F, and it is easy to verify that $V(G)-\{u, v, w\}$ is a global offensive $(k+1)$-alliance of G of size $n-3$, a contradiction. If $n = k+6$ and there exists a vertex w in F of degree $k + 5$, then $V(G) - \{u, v, w\}$ is a global offensive $(k + 1)$-alliance of G of size $n - 3$, a contradiction.

Subcase 3.2. Assume that $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ and $\overline{F}$ contains a component $K_{p,q}$ with $1 \leq p \leq q$ and $p+q \geq 3$. Let $\{v_1, v_2, \ldots, v_q\}$ and $\{u_1, u_2, \ldots, u_p\}$ be a partition of $K_{p,q}$.

If $K_{1,s} \subseteq \overline{F}$ with $s \geq t + 4$, then $\delta(G) \leq k$, a contradiction to $\delta(G) \geq k + 1$. Thus $q \leq t + 3$.

If $q \leq t + 1$ or $q = t + 2$ and $p \leq t + 1$, then it is easy to see that $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k + 1)$-alliance of G of size $n - 3$, a contradiction.

Conversely, if $G = K_{k+3}$, then obviously $\gamma_o^k(G) = k+1$, $\gamma_o^{k+1}(G) = k+2$ and so $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$.

Now let $\overline{G} = H \cup 2K_{1,1}$ such that $n(H) = k+2$ and the components of H are complete bipartite graphs $K_{1,1}$, $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$. Thus $k+1 \leq d_G(z) \leq k+4$ for every $z \in V(G)$, and G contains a cycle C on four vertices, where each vertex of C has degree $k+4$. Clearly, $V(H)$ is a global offensive k-alliance of G and so $\gamma_o^k(G) \leq n(G)-4$. If D is a $\gamma_o^k(G)$-set of size $|D| \leq n(G)-5$, then, since $\alpha(G) = 2$, the induced subgraph $G[V(G)-D]$ contains a vertex x of degree at least two. This leads to the contradiction $|N_G(x) \cap D| \leq k+1 < |N_G(x) - D| + k$. Hence we have shown that $\gamma_o^k(G) = n(G) - 4$.

Now let us prove that $\gamma_o^{k+1}(G) = n(G)-2$. Clearly, $\gamma_o^{k+1}(G) \geq \gamma_o^k(G) \geq n(G)-4$. Let D be a $\gamma_o^{k+1}(G)$-set. First, assume that $\gamma_o^{k+1}(G) = n(G)-4$. Then, since $n(G) = k+6$ and $\alpha(G) = 2$, the induced subgraph $G[V(G)-D]$ is isomorphic to $2K_{1,1}$, say ab and cd, and every vertex of $V(G)-D$ is adjacent to all vertices of D. Since $d_G(x) = k+3$ for every $x \in \{a,b,c,d\}$ it follows that a,b,c,d lie in one component C_4 of H, a contradiction. Second, assume that $\gamma_o^{k+1}(G) = n(G)-3$. Since every vertex has degree at most $k+4$, no vertex of $V(G)-D$ has two neighbors in $V(G)-D$. Moreover, since $\alpha(G) = 2$, $G[V(G)-D]$ is formed by two adjacent vertices x,y plus an isolated vertex w. Since w has degree at least two in $\overline{G}$, the vertices w,x,y lie in one component in H and so belong to $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$. Thus each of x and y has at least two non-neighbors in D and hence $|N(x) \cap D| \leq k+1$, a contradiction to the fact D is a $\gamma_o^{k+1}(G)$-set. Thus $|D| \geq n(G)-2$ and the equality follows from the fact that $V(G)$ minus any two non-adjacent vertices of C is a global offensive $(k+1)$-alliance of G. Therefore $\gamma_o^{k+1}(G) = n(G)-2 = (\gamma_o^k(G) + n(G))/2$.

Finally, let $G = (Q_1 \cup Q_2) + F$, where Q_1, Q_2 and F are three pairwise disjoint graphs such that $1 \leq |V(F)| \leq k+1$, $\alpha(F) \leq 2$, and Q_1 and Q_2 are cliques with $|V(Q_1)| = |V(Q_2)| = k+3-|V(F)|$ such that $|V(F)| \leq 2$ or
$\alpha(F) = 1$ and $|V(F)| = k+1$ or
$\alpha(F) = 2$ and $F = K_{k+1} - M$, where M is matching of F or
$\alpha(F) = 2$ and $F = K_k - M$, where M is a perfect matching of F or
$\alpha(F) = 2$ and $|V(F)| = k+1-t$ for $0 \leq t \leq k-2$ with $k \geq 3t+3$ and all components of $\overline{F}$ are isomorphic to $K_{t+2,t+2}$, to $K_{t+2,t+3}$ or to $K_{t+3,t+3}$.

Let D be a global offensive $(k+1)$-alliance of G. Since each vertex of Q_i has degree $k+2$, the set $V(G)-D$ contains at most one vertex of Q_i for every $i = 1,2$. Moreover, if $(V(G)-D) \cap V(Q_i) \neq \emptyset$, then $V(F) \subseteq D$.

Now suppose that $\gamma_o^{k+1}(G) \leq n-3$, and assume, without loss of generality, that $V(G) - D = \{u, v, w\}$. Then as noted above $V(Q_1) \cup V(Q_2) \subseteq D$, and hence the vertices u, v, w belong to $V(F)$. It follows that $|V(F)| \geq 3$.

Obviously, we obtain a contradiction when $\alpha(F) = 1$ and $|V(F)| = k+1$.

Assume next that $\alpha(F) = 2$. This implies that at least two vertices of $V(G) - D$ are adjacent in G.

First assume that $F = K_k - M$, where M is a perfect matching of F. Note that every vertex of $V(F)$ has degree $k+4$. Since M is perfect, $\{u, v, w\}$ induces either a path P_3 or a clique K_3 with center vertex, say v, in G. But then v has a non-neighbor in D for which it is matched in M, and so v has exaclty $k+2$ neighbors in D against two in $V(G) - D$, a contradiction.

Second assume that $F = K_{k+1} - M$, where M is a matching of F. Note that $n = k+5$ and $|D| = k+2$. As above, $\{u, v, w\}$ induces either a path P_3 or a clique K_3 with center vertex, say v, in G. But then v has at most $k+2$ neighbors in D against two in $V(G) - D$, a contradiction.

Assume now that $\alpha(F) = 2$ and $|V(F)| = k+1-t$ for $0 \leq t \leq k-2$ with $k \geq 3t+3$ and all components of $\overline{F}$ are isomorphic to $K_{t+2,t+2}$, to $K_{t+2,t+3}$ or to $K_{t+3,t+3}$. Note that in this case $n = k+5+t$ and so $|D| = n-3 = k+2+t$. Assume, without loss of generality, that u and v are adjacent in G. This leads to $|N_G(u) \cap D| \leq (k+5+t) - (t+2+2) = k+1$, a contradiction to the assumption that D is a global offensive $(k+1)$-alliance of G.

Altogether, we have shown that $\gamma_o^{k+1}(G) = n-2$. Finally, it is a simple matter to obtain $\gamma_o^k(G) = n-4$, and the proof of Theorem 22 is complete. ■

3. Lower Bounds

Our aim in this section is to give lower bounds on the global offensive k-alliance number of a graph in terms of its order n, minimum degree δ and maximum degree Δ.

Theorem 23. *Let k be a positive integer. If G is a graph of order n, minimum degree δ and maximum degree Δ, then*

$$\gamma_o^k(G) \geq \frac{n(\delta+k)}{2\Delta+\delta+k}. \tag{6}$$

Proof. If S is any $\gamma_o^k(G)$-set, then

$$\Delta\gamma_o^k(G) \;=\; \Delta|S| \geq \sum_{v\in S} d_G(v) \geq \sum_{v\in V(G)-S} \frac{d_G(v)+k}{2}$$

$$\geq \ |V(G)-S|\frac{\delta+k}{2} = (n-\gamma_o^k(G))\frac{\delta+k}{2}\,.$$

This leads to

$$\gamma_o^k(G)(2\Delta+\delta+k) \geq n(\delta+k),$$

and (6) is proved. ■

Theorem 24. *Let $k \geq 1$ be an integer, and let G be a graph of order n, minimum degree δ and maximum degree Δ. If δ is even and k odd or δ odd and k even, then*

$$\gamma_o^k(G) \geq \frac{n(\delta+k+1)}{2\Delta+\delta+k+1}. \tag{7}$$

Proof. If S is any $\gamma_o^k(G)$-set, then

$$\begin{aligned}\Delta\gamma_o^k(G) &= \Delta|S| \geq \sum_{v\in S} d_G(v)\\ &\geq \sum_{v\in V(G)-S,\ d_G(v)=\delta} \frac{d_G(v)+k+1}{2} + \sum_{v\in V(G)-S,\ d_G(v)>\delta} \frac{d_G(v)+k}{2}\\ &\geq |V(G)-S|\frac{\delta+k+1}{2} = (n-\gamma_o^k(G))\frac{\delta+k+1}{2}.\end{aligned}$$

This leads to

$$\gamma_o^k(G)(2\Delta+\delta+k+1) \geq n(\delta+k+1),$$

and (7) is proved. ■

Example 25. Let G be a k-regular bipartite graph of order n with the partite sets X and Y. Then

$$\gamma_0^k(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta+k)}{2\Delta+\delta+k}$$

and

$$\gamma_0^{k-1}(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta+(k-1)+1)}{2\Delta+\delta+(k-1)+1}$$

for $k \geq 2$. This family of graphs demonstrate that the bounds in Theorems 23 and 24 are best possible.

References

[1] M. Blidia, M. Chellali and O. Favaron, *Independence and 2-domination in trees*, Australas. J. Combin. **33** (2005) 317–327.

[2] M. Blidia, M. Chellali and L. Volkmann, *Some bounds on the p-domintion number in trees*, Discrete Math. **306** (2006) 2031–2037.

[3] R.L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc. **37** (1941) 194–197.

[4] M. Chellali, *Offensive alliances in bipartite graphs*, J. Combin. Math. Combin. Comput., to appear.

[5] O. Favaron, G. Fricke, W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, R.C. Laskar and D.R. Skaggs, *Offensive alliances in graphs*, Dicuss. Math. Graph Theory **24** (2004) 263–275.

[6] O. Favaron, A. Hansberg and L. Volkmann, *On k-domination and minimum degree in graphs*, J. Graph Theory **57** (2008) 33–40.

[7] H. Fernau, J.A. Rodríguez and J.M. Sigarreta, *Offensive r-alliance in graphs*, Discrete Appl. Math. **157** (2009) 177–182.

[8] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985) 287–293.

[9] J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita and L. Volkmann, *Independence and 2-domination in bipartite graphs*, Australas. J. Combin. **40** (2008) 265–268.

[10] A. Hansberg, D. Meierling and L. Volkmann, *Independence and p-domination in graphs*, submitted.

[11] P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi, *Alliances in graphs*, J. Combin. Math. Combin. Comput. **48** (2004) 157–177.

[12] O. Ore, Theory of Graphs (Amer. Math. Soc. Colloq. Publ. **38**, 1962).

[13] C. Payan and N.H. Xuong, *Domination-balanced graphs*, J. Graph Theory **6** (1982) 23–32.

[14] K. H. Shafique and R.D. Dutton, *Maximum alliance-free and minimum alliance-cover sets*, Congr. Numer. **162** (2003) 139–146.

[15] K. H. Shafique and R.D. Dutton, *A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets*, J. Combin. Math. Combin. Comput. **56** (2006) 139–145.

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