Discussiones Mathematicae
Graph Theory 29 (2009) 563–572

DECOMPOSITIONS OF NEARLY COMPLETE DIGRAPHS INTO t ISOMORPHIC PARTS

Mariusz Meszka and Zdzisław Skupień

Faculty of Applied Mathematics
AGH University of Science and Technology
al. Mickiewicza 30, 30–059 Kraków, Poland

e-mail: meszka@agh.edu.pl, skupien@agh.edu.pl

Abstract

An arc decomposition of the complete digraph $\mathcal{D}K_n$ into t isomorphic subdigraphs is generalized to the case where the numerical divisibility condition is not satisfied. Two sets of nearly tth parts are constructively proved to be nonempty. These are the floor tth class $(\mathcal{D}K_n - R)/t$ and the ceiling tth class $(\mathcal{D}K_n + S)/t$, where R and S comprise (possibly copies of) arcs whose number is the smallest possible. The existence of cyclically 1-generated decompositions of $\mathcal{D}K_n$ into cycles $\vec{C}_{n-1}$ and into paths $\vec{P}_n$ is characterized.

Keywords: decomposition, cyclically 1-generated, remainder, surplus, universal part.

2000 Mathematics Subject Classification: 05C70, 05C20.

1. Introduction

Let t be an integer, $t \geq 2$. We focus our considerations on decompositions into t isomorphic parts. One of the most significant results in the graph decomposition theory is that a tth part of (or one tth of) the complete digraph exists whenever the size of the digraph is divisible by t. In case $t = 2$ parts are *halves* and they are called *self-complementary digraphs*; their existence is proved by Read [15]. The relevant result for any t is proved in Harary *et al.* [8]. However, the related problem of characterizing tth parts remains open if the order of the complete digraph is large enough.

Given the complete digraph $\mathcal{D}K_n$ on n vertices (with $n(n-1)$ arcs), the numerical divisibility condition $t|n(n-1)$ is also known [8] to ensure that self-converse tth parts of $\mathcal{D}K_n$ exist. Our first aim is to extend this classical result to the case where parts are to be self-converse oriented graphs. Secondly, if the numerical divisibility condition is not satisfied, we consider tth parts of a corresponding nearly complete digraph obtained from $\mathcal{D}K_n$ either by adding a *surplus* S or by deleting a *remainder* R. Then S comprises copies of arcs and R is a subset of arcs, both S and R are to have cardinalities as small as possible, $|R| = n(n-1) \bmod t$ and $|S| = (t - |R|) \bmod t$. Thus R is a set and S is possibly a multiset. Following Skupień [16], the classes of such tth parts are denoted by $\lceil \mathcal{D}K_n/t \rceil_S := (\mathcal{D}K_n + S)/t$ and $\lfloor \mathcal{D}K_n/t \rfloor_R := (\mathcal{D}K_n - R)/t$, and are called the *ceiling tth class* and the *floor tth class*, respectively. Call elements of those classes (also if $S = \emptyset = R$) to be *near-tth parts* of $\mathcal{D}K_n$; more precisely, these are *ceiling-S tth parts* and *floor-R tth parts*, respectively.

The proof of theorem on divisibility of $\mathcal{D}K_n$ by t in Harary *et al.* [8] gives the following result.

Proposition 1. *$\mathcal{D}K_n/2$ contains a self-converse oriented graph, e.g. the transitive tournament T_n.*

For $t = 3 \leq n$, $R = \emptyset = S$ unless $n \equiv 2 \pmod 3$ and then $|S| = 1$, $|R| = 2$ and there are five configurations of R, which we call *admissible*, three of them being self-converse. In [11, 12] we have proved the following three theorems on third parts.

Theorem 2 [11]**.** *For each $n \geq 3$ and any admissible and self-converse R, the floor third class $\lfloor \mathcal{D}K_n/3 \rfloor_R$ contains a self-converse oriented graph unless either $n = 3$ or possibly $n = 8$ and R induces a path $\vec{P}_3$.*

A computer has not found such a member in case where $n = 8$ and R induces $\vec{P}_3$.

Theorem 3 [11]**.** *If $k \in \mathbb{N}$, $n = 3k - 1 \neq 5$, and $t = 3$, then $|S| = 1$ and the ceiling third class $\lceil \mathcal{D}K_n/3 \rceil_S$ contains a self-converse oriented graph.*

Theorem 4 [12]**.** *For $n = 5$, there is no ceiling third part of the complete digraph $\mathcal{D}K_5$ which could be a self-converse oriented graph.*

The main result of the paper provides a complete solution to the related existence problem for $t \geq 4$. It turns out that the problem we solve is not mentioned among unsolved problems listed in Harary and Robinson [7].

Theorem 5. *For every $n \geq 2$ and every $t \geq 2$ there exist a remainder R and surplus S (both of the smallest possible cardinality) such that both the floor class $\lfloor \mathcal{D}K_n/t \rfloor_R$ and the ceiling class $\lceil \mathcal{D}K_n/t \rceil_S$ contain self-converse digraphs. If neither $n = 5$ and $t = 3$ in case of the ceiling class nor $n = t = 3$ then the digraphs can be required to be self-converse oriented graphs.*

2. Notation and Terminology

We use standard notation and terminology of graph theory [4, 5] unless otherwise stated.

Digraphs are loopless and without multiple arcs. Multidigraphs may have multiple arcs, loops being forbidden. A digraph without 2-cycle $\mathcal{D}K_2$ $(= \vec{C}_2)$ is called an *oriented graph*.

The ordered pair (v_1, v_2) of vertices v_1 and v_2 (or the symbol $v_1 \to v_2$) denotes the arc which goes from the *tail* v_1 to the *head* v_2. The *converse* of a multidigraph is obtained by reversal of each arc. A multidigraph is called *self-converse* if it is isomorphic to its converse.

The symbol $\cup$ when applied to multidigraphs stands for the *vertex-disjoint* union. Moreover, given a digraph D, the symbol $D + A'$ denotes the spanning supermultidigraph of D with the arc set $A(D) \cup A'$, where A' is a set of (possibly copies of) arcs and $A' \cap A(D) = \emptyset$. Similarly, $D - A'$ denotes the spanning subdigraph obtained from D by removal of A', where A' is to be a subset of $A(D)$. We write $D \pm A' = D \pm a$ if $A' = \{a\}$ and a is an arc.

By a *decomposition* of a multidigraph D we mean a family of arc-disjoint submultidigraphs of D which include all arcs of D. Those substructures are called *elements of a decomposition*. By an *H-decomposition* we mean a decomposition of D into t elements all isomorphic to H; then we write $H|D$ or $t|D$. The isomorphism class of those t pairwise isomorphic elements of a decomposition is called a *tth part* of D.

There are two non-self-converse digraphs of size two each, namely,

$P^{\vee} = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_3, v_2)\})$, a *gutter*,

$P^{\wedge} = (\{v_1, v_2, v_3\}, \{(v_2, v_1), (v_2, v_3)\})$, a *roof*.

Note that $\mathcal{D}K_3/3$ comprises three digraphs, none of which is a self-converse oriented graph.

(1) $$\mathcal{D}K_3/3 = \{\vec{C}_2,\ P^{\vee},\ P^{\wedge}\}.$$

A decomposition of D is called to be 1-*generated* if there is a permutation γ of $V(D)$ which *generates* the decomposition from any single decomposition element H in the sense that, for $j = 0, \ldots, t-1$, the image of H under the jth iteration $(\gamma)^j$ of γ, denoted by $(\gamma)^j H$, is one of t decomposition parts, where $(\gamma)^0 = \mathrm{id}$. Call γ to be a *placement-generating permutation* for H. If, moreover, γ is a cyclic permutation then the decomposition is called *cyclically* 1-*generated* (cf. cyclic decomposition in Chartrand and Lesniak [5], see also Bosák [4] for the equivalent notion of a decomposition according to a cyclic group).

Proposition 6. *Each decomposition of the complete digraph $\mathcal{D}K_n$ into two isomorphic halves is* 1-*generated.*

Proof. Note that these halves are self-complementary digraphs. The result follows from the known characterization of complementing permutations for those halves, cf. Bosák [4, Ch. 14]. ∎

Given a self-converse multidigraph D on n vertices, we use the symbol φ $(= \varphi_n)$ to denote a *conversing permutation*, that is, a permutation of $V(D)$ such that φD is the converse multidigraph of D.

3. Cyclic Decompositions Into n Parts

Bermond and Faber prove [3] that the complete digraph $\mathcal{D}K_n$ is decomposable into cycles $\vec{C}_{n-1}$ of length $n-1$. It can be noted that $\vec{C}_{n-1}$-decomposition of $\mathcal{D}K_n$, presented in [3] as well as in [1], is not 1-generated. We are going to improve this result by characterizing cyclically 1-generated decompositions of $\mathcal{D}K_n$ into $(n-1)$-cycles. Namely, a cyclically 1-generated $\vec{C}_{n-1}$-decomposition exists precisely if n is odd. Additionally, a cyclically 1-generated decomposition of $\mathcal{D}K_n$ into hamiltonian paths for even n follows from the widely known construction presented in Berge [2, p. 232] and also in Lucas [10, Ch. 6] (who attributes this result to Walecki) by passing on from K_n to $\mathcal{D}K_n$. It is worth noting that just this cyclically 1-generated $\vec{P}_n$-decomposition of $\mathcal{D}K_n$ is presented in [10, Remark on p. 176] in terms of designing a set of single file walks for n children so that each child once is the first, once the last, and no ordered pair of neighbours in a file is repeated among the files in the set. We prove that this decomposition exists precisely if n is even (Theorem 7). In either case cyclically 1-generated decomposition plays a crucial role in the proof of the main result since those decompositions enable a recursive construction in proofs of Lemma 8 and Theorem 5.

Let $V(\mathcal{D}K_n) = \mathbb{Z}_n$, the cyclic group of order n. Let W_0 be a sequence of (possibly repeating) vertices of the digraph $\mathcal{D}K_n$, say $W_0 = <x_1, x_2, \ldots, x_k>$. In what follows we use the convention that W_0 refers to the walk whose subsequence of vertices is W_0. Moreover, the symbol $<W_0>$ stands for the graph induced by the arc set of the walk W_0.

Definition 1. Assume that $n \geq 3$. Define the vertex sequence, which depends on the parity of n and is denoted by $W_0(n)$ or W_0, as follows.

(i) For odd $n \geq 3$, $W_0 = <0, \frac{n-3}{2}, \ldots, \frac{n+1}{2}, 0>$, which represents a cycle in which $\frac{n-1}{2}$ is the only vertex which is omitted. If $n = 3$ then $W_0 := <0, 2, 0>$. If $n \geq 5$, we assume that the cycle W_0 comprises the following arcs, where k stands for an integer:

$$\begin{array}{llll}
(2) & k \to \frac{n-3}{2} - k, & & 0 \leq k \leq \frac{n-5}{4}, \\
(3) & \frac{n-3}{2} - k \to k+1, & & 0 \leq k \leq \frac{n-7}{4}, \\
(4) & \frac{n+3}{2} + k \to n-1-k, & & 0 \leq k \leq \frac{n-7}{4}, \\
(5) & n-1-k \to \frac{n+1}{2} + k, & & 0 \leq k \leq \frac{n-5}{4},
\end{array}$$

and also two arcs $\frac{n+1}{2} \to 0$, $\lfloor \frac{n-1}{4} \rfloor \to \lfloor \frac{3n+1}{4} \rfloor$.

$$\text{Hence} \qquad W_0 = \begin{cases} <0,1,4,3,0>, & n=5, \\ <0,3,1,2,7,6,8,5,0>, & n=9, \\ <0,2,1,5,6,4,0>, & n=7, \\ <0,4,1,3,2,8,9,7,10,6,0>, & n=11. \end{cases}$$

Figure 1 ($n = 7$, 9) shows the difference between cases $n \equiv 1, 3 \pmod 4$.

(ii) For even $n \geq 4$, $W_0 = < 0, 1, n-1, \ldots, \frac{n}{2} >$, which represents a hamiltonian path of $\mathcal{D}K_n$. It is assumed that the path includes the following arcs:

$$k \xrightarrow{(6)} n-k \xrightarrow{(7)} k+1, \qquad 1 \leq k \leq \frac{n-2}{2},$$

and the initial arc $0 \to 1$, see Figure 2 wherein $n = 8$.

Define the length of the arc (i, j) to be $(j - i) \bmod n$.

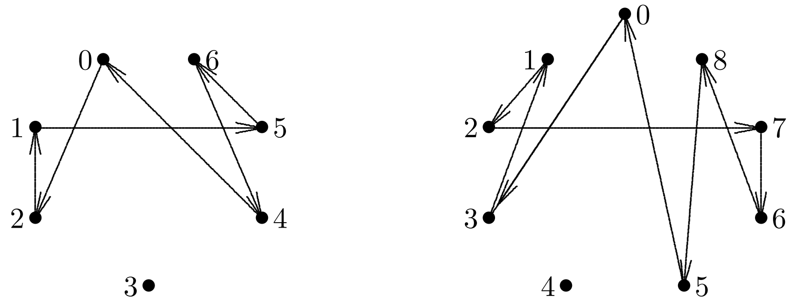

Figure 1. $n = 7, 9$

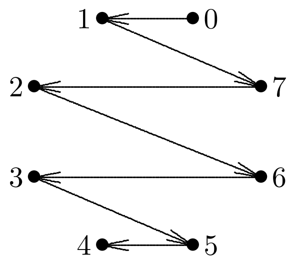

Figure 2. $n = 8$

Theorem 7. *For $n \geq 4$, the cyclic permutation $\gamma_0 := [0, 1, 2, \ldots, n-1]$ is a placement-generating permutation for the self-converse oriented graph $<W_0>$ in $\mathcal{D}K_n$, that is, $\mathcal{D}K_n$ has a cyclically 1-generated $<W_0>$-decomposition into n parts which are either the cycle $\vec{C}_{n-1}$ or path $\vec{P}_n$ according as n is odd or even. Moreover, if a 1-generated $\vec{C}_{n-1}$-decomposition [1-generated $\vec{P}_n$-decomposition] of $\mathcal{D}K_n$ exists then n is odd [n is even].*

Proof. Using the list of arcs of W_0 in Definition 1 one can see that arcs of W_0 have mutually distinct lengths. Namely, if $n \geq 4$ is odd and $<W_0> = \vec{C}_{n-1}$, the lengths are $\frac{n-3}{2} - 2k$ for arcs listed in (2), $\frac{n-5}{2} - 2k$ in (4), and similarly $\frac{n+5}{2} + 2k$ for arcs in (3), and $\frac{n+3}{2} + 2k$ in (5), for increasing k starting at $k = 0$. Two additional arcs $\frac{n+1}{2} \to 0$, $\lfloor \frac{n-1}{4} \rfloor \to \lfloor \frac{3n+1}{4} \rfloor$ have lengths $\frac{n-1}{2}$, $\frac{n+1}{2}$, respectively. Analogously, if n is even and $<W_0> = \vec{P}_n$, the lengths are even, $n - 2k$, for arcs with label (6) and odd, $2k + 1$, with label (7), where $k = 1, 2, \ldots, \frac{n-2}{2}$. The initial arc $0 \to 1$ has length 1.

Note that a noncyclic permutation of n vertices cannot generate n elements of a decomposition because then there is an edge which has less than

n different images under iterations of the permutation. Suppose that a cyclic permutation generates a decomposition of $\mathcal{D}K_n$ into n subdigraphs. It is easy to see that an $(n-1)$-cycle if n is even, as well as a path on n vertices for odd n, with all their arcs of different lengths do not exist. Namely, otherwise the sum of lengths would be 0 modulo n for the cycle (i.e., for even n) and non-zero for the path (i.e., for odd n). However, just the opposite is true. Namely, the sum of lengths is $s = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ and therefore $s \not\equiv 0 \pmod{n}$ if n is even but $s \equiv 0 \pmod{n}$ if n is odd. ■

Note that none of third parts of $\mathcal{D}K_3$ is a self-converse oriented graph, cf. (1), and therefore the case $n = 3$ is excluded in Theorem 7.

4. Decompositions Into t Parts

Harary *et al.* [8] prove that if the numerical divisibility condition $t|n(n-1)$ is satisfied then the complete digraph $\mathcal{D}K_n$ is decomposable into t isomorphic self-converse parts. If $t|n$ or $t|(n-1)$, we extend this result to the case where parts are to be self-converse oriented graphs and a decomposition is to be 1-generated.

Lemma 8. *For every $n \geq 2$, if $t|n$ or $t|(n-1)$ then there exists a 1-generated decomposition of the complete digraph $\mathcal{D}K_n$ into t isomorphic self-converse parts which are oriented graphs unless $n = t = 3$.*

Proof. For $t = 2$ and $t = 3$, Lemma is proved in [12] and [11], respectively. Namely, if $t = 2$, one of halves of $\mathcal{D}K_n$ with any n is the transitive tournament (see Propositions 1 and 6). For $t = 3$ and $n = 3t$ or $n = 3t + 1$, decompositions of $\mathcal{D}K_n$ constructed in [11] can be seen to be 1-generated. More precisely, oriented graphs D_i for $i = 4, 6, 7, 9$ constructed in the proof of Lemma 1.1 in [11] as well as an oriented graph D_n in the proof of Theorem 1 in [11] are required third parts of a decomposition, existence of which is a part of the assertion of Theorem 2. If $t = n$ (> 3), the result follows immediately from the above Theorem 7.

Consider the case $n = kt$, $k \in \mathbb{N}$, and $t \geq 4$. We proceed by induction on k. Theorem 7 can be viewed as the first step for $k = 1$, namely, the self-converse oriented graph $<W_0> \in \mathcal{D}K_t/t$ and the vertex set $V(<W_0>) = \{0, 1, \dots, t-1\}$. Assume now that Lemma is true for $k-1$, that is, for a fixed admissible self-converse oriented graph $D_{n-t} \in \mathcal{D}K_{n-t}/t$ where $V(D_{n-t}) = \{t, t+1, \dots, n-1\}$. We construct a self-converse oriented graph $D_n \in$

$\mathcal{D}K_n/t$. Let $V(D_n) = V(D_{n-t}) \cup V(<W_0>)$. The construction of D_n takes the advantage of the structure of $<W_0>$. Namely, one can easily find a conversing permutation φ_t of $<W_0>$ in $\mathcal{D}K_t$, which moves the vertex 0 into $\lceil \frac{t}{2} \rceil$ (cf. Definition 1). Then we define D_n to be the following digraph.

$$D_n = <W_0> \cup D_{n-t} + \Big\{(0,i), \Big(i, \Big\lceil \frac{t}{2} \Big\rceil\Big) : t \le i \le n-1\Big\}.$$

One can see that D_n is a self-converse oriented graph with conversing permutation $\varphi_n = (\varphi_t, \varphi_{n-t}) := \varphi_t \, \varphi_{n-t}$, where φ_{n-t} is a conversing permutation of D_{n-t}. Moreover, $D_n \in \mathcal{D}K_n/t$ with placement-generating permutation $\gamma_n = (\gamma_0, \gamma_{n-t})$, where γ_{n-t} is a placement-generating permutation of D_{n-t} in $\mathcal{D}K_{n-t}$ (see Theorem 7 for γ_0).

In the case $n = kt+1$, $k \in \mathbb{N}$, note first that the self-converse oriented graph $D'_{t+1} = <W_0> \cup [t] + \{(0,t), (t, \lceil \frac{t}{2} \rceil)\}$, where $[t] := K_1$ with $V(K_1) = \{t\}$, is the tth part of $\mathcal{D}K_{t+1}$ with placement-generating permutation $\gamma_{t+1} = (\gamma_0, (t))$ and conversing permutation $\varphi_{t+1} = (\varphi_t, (t))$. For $n = kt+1$ where $t \ge 4$ and $k \ge 2$, we construct a self-converse oriented graph $D'_n \in \mathcal{D}K_n/t$ analogously to the case above, using in the induction step digraphs D'_{t+1} and D_{n-t-1}. ∎

In [13] we have proved the following result on decompositions of the complete digraph into nonhamiltonian paths, which is useful in proving Theorem 5.

Theorem 9 [13]**.** *For any $n \ge 3$, the complete digraph $\mathcal{D}K_n$ is decomposable into paths of arbitrarily prescribed lengths ($\le n-2$) provided that the lengths sum up to the size $n\,(n-1)$ of $\mathcal{D}K_n$.*

Now we are ready to prove the main result of the paper.

***Proof of Theorem* 5.** If $t \le 3$, Theorem is true by Proposition 1 for $t = 2$ and by (1) and Theorems 2, 3, 4 for $t = 3$. It is so, too, if $t = n$, by Theorem 7. The result is easily seen for $n \le 3$ and $t > 3$. In particular, S can be chosen so that just $\vec{P}_3$ (the only possible candidate) is the ceiling-S tth part of $\mathcal{D}K_3$ for $t = 4$, 5.

Consider the case $t > n \ge 4$. Note that $\lfloor n(n-1)/t \rfloor \le n-2$ whence, in case of the floor class, the result is true by Theorem 9, elements of a decomposition as well as $<R>$ are oriented paths. In case of the ceiling class, if $t = n+1$ then $\lceil n(n-1)/t \rceil = n-1$, $|S| = n-1$ and, by Theorem 7, the result is true. For $t \ge n+2$, $\lceil n(n-1)/t \rceil \le n-2$ and analogously the result is true by Theorem 9, i.e., elements of a decomposition and $<S>$ are paths of prescribed lengths.

It remains to consider the case $4 \leq t < n$. Let $r = n \bmod t$. Then $r \leq n - t$ and $(n(n-1) \bmod t) = (r(r-1) \bmod t)$. Applying similar induction as in the proof of Lemma 8 one can easily construct required near-tth parts of $\mathcal{D}K_n$. Namely, in the induction step we use self converse oriented graphs D_t (see proof of Lemma 8 for a construction) and D_{n-t} (construction follows from the case $t > n \geq 2$ above and by induction) which are tth and corresponding near-tth parts of $\mathcal{D}K_t$ and $\mathcal{D}K_{n-t}$, respectively. ∎

5. Concluding Remarks

Note that the above relatively short proof of the main result is based on Theorem 9 (the proof of which is nontrivial) and on characterizations of special cyclically 1-generated decompositions, see Theorem 7.

It is an open problem to determine all possible cyclically 1-generated nth parts of $\mathcal{D}K_n$, which are self-converse oriented graphs. Similar problem concerns tth parts as considered in Lemma 8 above.

Another open problem is related to the notion of a tth universal floor part, say F, of $\mathcal{D}K_n$. The meaning of 'universal' is that packings of t copies of F into $\mathcal{D}K_n$ should leave remainders R of all possible shapes. The open problem is motivated by a conjecture of the second author, stated several years ago (see [17]), that universal floor parts of complete (undirected) graphs exist. Supporting results in [18, 9] cover infinitely many pairs (n, t). Moreover, Plantholt's deep theorem [14] on the chromatic index is equivalent to the truth of the conjecture for $t = n - 1$ with n being odd. On the other hand, there is no universal third part of $\mathcal{D}K_5$, see [6].

Acknowledgement

The research was supported by the Ministry of Science and Information Society Technologies of the Republic of Poland, Grant No. 1 P03A 037 27.

References

[1] B. Alspach, H. Gavlas, M. Šajna and H. Verrall, *Cycle decopositions IV: complete directed graphs and fixed length directed cycles*, J. Combin. Theory (A) **103** (2003) 165–208.

[2] C. Berge, Graphs and Hypergraphs (North-Holland, 1973).

[3] J.C. Bermond and V. Faber, *Decomposition of the complete directed graph into k-circuits*, J. Combin. Theory (B) **21** (1976) 146–155.

[4] J. Bosák, Decompositions of Graphs (Dordrecht, Kluwer, 1990, [Slovak:] Bratislava, Veda, 1986).

[5] G. Chartrand and L. Lesniak, Graphs and Digraphs (Chapman & Hall, 1996).

[6] A. Fortuna and Z. Skupień, *On nearly third parts of complete digraphs and complete* 2*-graphs*, manuscript.

[7] F. Harary and R.W. Robinson, *Isomorphic factorizations X: Unsolved problems*, J. Graph Theory **9** (1985) 67–86.

[8] F. Harary, R.W. Robinson and N.C. Wormald, *Isomorphic factorisation V: Directed graphs*, Mathematika **25** (1978) 279–285.

[9] A. Kędzior and Z. Skupień, *Universal sixth parts of a complete graph exist*, manuscript.

[10] E. Lucas, Récréations Mathématiques, vol. II (Paris, Gauthier-Villars, 1883).

[11] M. Meszka and Z. Skupień, *Self-converse and oriented graphs among the third parts of nearly complete digraphs*, Combinatorica **18** (1998) 413–424.

[12] M. Meszka and Z. Skupień, *On some third parts of nearly complete digraphs*, Discrete Math. **212** (2000) 129–139.

[13] M. Meszka and Z. Skupień, *Decompositions of a complete multidigraph into nonhamiltonian paths*, J. Graph Theory **51** (2006) 82–91.

[14] M. Plantholt, *The chromatic index of graphs with a spanning star*, J. Graph Theory **5** (1981) 45–53.

[15] R.C. Read, *On the number of self-complementary graphs and digraphs*, J. London Math. Soc. **38** (1963) 99–104.

[16] Z. Skupień, *The complete graph t-packings and t-coverings*, Graphs Combin. **9** (1993) 353–363.

[17] Z. Skupień, *Clique parts independent of remainders*, Discuss. Math. Graph Theory **22** (2002) 361.

[18] Z. Skupień, *Universal fractional parts of a complete graph*, manuscript.

Received 8 May 2008
Revised 22 September 2008
Accepted 13 October 2008