THE SET CHROMATIC NUMBER OF A GRAPH

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Abstract

For a nontrivial connected graph $G$, let $c : V(G) \rightarrow \mathbb{N}$ be a vertex coloring of $G$ where adjacent vertices may be colored the same. For a vertex $v$ of $G$, the neighborhood color set $NC(v)$ is the set of colors of the neighbors of $v$. The coloring $c$ is called a set coloring if $NC(u) \neq NC(v)$ for every pair $u, v$ of adjacent vertices of $G$. The minimum number of colors required of such a coloring is called the set chromatic number $\chi_s(G)$ of $G$. The set chromatic numbers of some well-known classes of graphs are determined and several bounds are established for the set chromatic number of a graph in terms of other graphical parameters.

Keywords: neighbor-distinguishing coloring, set coloring, neighborhood color set.

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1. Introduction

Many methods have been introduced in graph theory to distinguish all of the vertices of a graph or to distinguish every two adjacent vertices in a graph. Several of these methods involve graph colorings or graph labelings. In particular, with a given edge coloring $c$ of $G$, each vertex of $G$ can be labeled with the set of colors of its incident edges. If distinct vertices have distinct labels, then $c$ is a vertex-distinguishing edge coloring (see [2, 4]); while if every two adjacent vertices have distinct labels, then $c$ is a neighbor-distinguishing edge coloring (see [1]).

If all of the vertices of a graph $G$ of order $n$ are distinguished as a result of a vertex coloring of $G$, then of course $n$ colors are needed to accomplish this. On the other hand, if the goal is only to distinguish every two adjacent vertices in $G$ by a vertex coloring, then this can be accomplished by means of a proper coloring of $G$ and the minimum number of colors needed to do this is the chromatic number $\chi(G)$ of $G$. There are, however, other methods that can be used to distinguish every two adjacent vertices in $G$ by means of vertex colorings which may require fewer than $\chi(G)$ colors.

For a nontrivial connected graph $G$, let $c : V(G) \rightarrow \mathbb{N}$ be a vertex coloring of $G$ where adjacent vertices may be assigned the same color. For a set $S \subseteq V(G)$, define the set $c(S)$ of colors assigned to the vertices of $S$ by

$$c(S) = \{c(v) : v \in S\}.$$ 

For a vertex $v$ in a graph $G$, let $N(v)$ be the neighborhood of $v$ (the set of all vertices adjacent to $v$ in $G$). The neighborhood color set $NC(v) = c(N(v))$ is the set of colors of the neighbors of $v$. The coloring $c$ is called set neighbor-distinguishing or simply a set coloring if $NC(u) \neq NC(v)$ for every pair $u, v$ of adjacent vertices of $G$. The minimum number of colors required of such a coloring is called the set chromatic number of $G$ and is denoted by $\chi_s(G)$. We refer to the book [3] for graph theory notation and terminology not described in this paper.

For a graph $G$ with chromatic number $k$, let $c$ be a proper $k$-coloring of $G$. Suppose that $u$ and $v$ are adjacent vertices of $G$. Since $c(u) \in NC(v)$ and $c(u) \notin NC(u)$, it follows that $NC(u) \neq NC(v)$. Hence every proper $k$-coloring of $G$ is also a set $k$-coloring of $G$. Therefore, for every graph $G$,

$$\chi_s(G) \leq \chi(G).$$
Observe that if $G$ is a connected graph of order $n$, then $\chi_s(G) = 1$ if and only if $\chi(G) = 1$ (in this case $G = K_1$) and $\chi_s(G) = n$ if and only if $\chi(G) = n$ (in this case $G = K_n$). Thus if $G$ is a nontrivial connected graph of order $n$ that is not complete, then

\[(2) \quad 2 \leq \chi_s(G) \leq n - 1.\]

To illustrate these concepts, we consider the graph $G = C_5 + K_1$ (the wheel of order 6). The chromatic number of $G$ is $\chi(G) = 4$. In fact, the set chromatic number of $G$ is $\chi_s(G) = 3$. Figure 1 shows a set 3-coloring of $G$ and so $\chi_s(G) \leq 3$. We now show that $\chi_s(G) \geq 3$. Suppose that there is a set 2-coloring $c$ of $G$ using the colors 1 and 2. Consider a triangle in $G$ induced by three vertices $v_1, v_2, v_3$ of $G$. Since at least two of these three vertices are colored the same, we may assume that two of these vertices are assigned the color 1. Thus $NC(v_i) = \{1\}$ or $NC(v_i) = \{1, 2\}$ for each $i$ ($1 \leq i \leq 3$). This implies, however, that there are two adjacent vertices having the same neighborhood color set, which contradicts our assumption that $c$ is a set coloring. Thus $\chi_s(G) = 3$, as claimed.

![Figure 1. A set coloring of a graph.](image)

The following observation will be useful to us.

**Observation 1.1.** If $u$ and $v$ are two adjacent vertices in a graph $G$ such that $N(u) - \{v\} = N(v) - \{u\}$, then $c(u) \neq c(v)$ for every set coloring $c$ of $G$. Furthermore, if $S = N(u) - \{v\} = N(v) - \{u\}$, then $\{c(u), c(v)\} \nsubseteq c(S)$. 
2. The Set Chromatic Numbers of Some Classes of Graphs

Since every nonempty bipartite graph has chromatic number 2, the following is an immediate consequence of (1) and (2).

Observation 2.1. If $G$ is a nonempty bipartite graph, then $\chi_s(G) = 2$.

In fact, if $G$ is a nonempty graph, then $\chi_s(G) = 2$ if and only if $G$ is bipartite, as we show next. We may restrict our attention to connected graphs.

Proposition 2.2. If $G$ is a connected graph with $\chi(G) \geq 3$, then $\chi_s(G) \geq 3$.

Proof. Assume, to the contrary, that there exists a connected graph $G$ with $\chi(G) \geq 3$ for which there exists a set 2-coloring $c : V(G) \rightarrow \{1, 2\}$. Since $\chi(G) \geq 3$, it follows that $G$ contains an odd cycle $C : v_1, v_2, \ldots, v_\ell, v_1$, where $\ell \geq 3$ is an odd integer.

Consider the (cyclic) color sequence

\[ s : c(v_1), c(v_2), \ldots, c(v_\ell), c(v_1). \]

By a block of $s$, we mean a maximal subsequence of $s$ consisting of terms of the same color. First, we claim that $s$ cannot contain a block with an even number of terms; for suppose, without loss of generality, that $c(v_\ell) = 2$, $c(v_i) = 1$ for $1 \leq i \leq a$, where $a$ is an even integer with $2 \leq a \leq \ell - 1$, and $c(v_{a+1}) = 2$. Thus $\text{NC}(v_i) \in \{\{1\}, \{1, 2\}\}$ for $1 \leq i \leq a$. Since $\text{NC}(v_1) = \{1, 2\}$ and $c$ is a set coloring, it follows that

\[ \text{NC}(v_i) = \begin{cases} \{1\} & \text{if } i \text{ is even}, \\ \{1, 2\} & \text{if } i \text{ is odd} \end{cases} \]

for $1 \leq i \leq a$. However, this implies that $\text{NC}(v_a) = \{1\}$, which is impossible since $c(v_{a+1}) = 2$.

Hence either

(i) $c(v_i) = 1$ for all $i$ ($1 \leq i \leq \ell$) or

(ii) $s$ contains an even number of blocks each of which has an odd number of terms.

If (i) occurs, then $\text{NC}(v_i) \in \{\{1\}, \{1, 2\}\}$ for $1 \leq i \leq \ell$. Since $\ell$ is odd, there is an integer $j$ ($1 \leq j \leq \ell$) such that $\text{NC}(v_j) = \text{NC}(v_{j+1})$, which is impossible. If (ii) occurs, then $\ell$ is even, which is also impossible. \qed
The following three corollaries are immediate consequences of (1), Observation 2.1, and Proposition 2.2.

**Corollary 2.3.** A nonempty graph $G$ has set chromatic number 2 if and only if $G$ is bipartite.

**Corollary 2.4.** If $G$ is a 3-chromatic graph, then $\chi_s(G) = 3$.

**Corollary 2.5.** For each integer $n \geq 3$, $\chi_s(C_n) = \chi(C_n)$.

We have seen that $\chi_s(K_n) = n$ for $n \geq 1$. We now determine the set chromatic number of a class of graphs that are related to $K_n$. For a graph $H$, its corona $\text{cor}(H)$ is that graph obtained by adding a pendant edge at each vertex of $H$. For an integer $n \geq 2$ and an integer $t (0 \leq t \leq n)$, let $G_{n,t}$ denote the graph of order $n + t$ obtained from $K_n$ with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by adding $t$ new vertices $u_1, u_2, \ldots, u_t$ (if $t \geq 1$) and joining each $u_i$ to $v_i$ for $1 \leq i \leq t$. Therefore, $G_{n,0} = K_n$ while $G_{n,n} = \text{cor}(K_n)$. We show that $\chi_s(G_{n,t}) = n$ for all $t (0 \leq t \leq n)$. It is convenient to introduce some notation. For each integer $k$, let

$$N_k = \{1, 2, \ldots, k\}.$$

**Proposition 2.6.** For $n \geq 2$ and $0 \leq t \leq n$, $\chi_s(G_{n,t}) = n$.

**Proof.** The result follows immediately if $n = 2$ or $t = 0$, so we assume that $n \geq 3$ and $1 \leq t \leq n$. Since $\chi(G_{n,t}) = n$, we have $\chi_s(G_{n,t}) \leq n$ by (1). Suppose that $\chi_s(G_{n,t}) = k \leq n - 1$ and let there be given a set $k$-coloring of $G_{n,t}$ using the colors in $N_k$. Permuting colors if necessary, we can obtain a set $k$-coloring $c : V(G_{n,t}) \rightarrow N_k$ such that $c(V(K_n)) = N_\ell$ for some $\ell \leq k$.

Let $X$ be the subset of $V(K_n)$ such that for every $x \in X$ there exists a vertex $y \in X - \{x\}$ for which $c(y) = c(x)$. Since $c$ uses at most $n - 1$ colors, $|X| \geq 2$ and, furthermore, since each of the vertices in $V(K_n) - X$ receives a unique color, $n - |X| + 1 \leq \ell$. For each $x \in X$, either

(i) $\text{NC}(x) = N_\ell$ or
(ii) $\text{NC}(x) = N_\ell \cup \{c(u)\}$ if $u$ is the end-vertex adjacent to $x$ and $c(u) \notin N_\ell$.

Since at most one of the $|X|$ vertices can have the neighborhood color set $N_\ell$, at least $|X| - 1$ colors not in $N_\ell$ are needed to color the end-vertices so that the vertices in $X$ have distinct neighborhood color sets, that is,
\[ k \geq \ell + |X| - 1 \geq n, \]

which is a contradiction. Therefore, \( \chi_s(G_{n,t}) = n \). ■

We now determine the set chromatic number of every complete multipartite graph.

**Proposition 2.7.** For every complete \( k \)-partite graph \( G \), \( \chi_s(G) = k \).

**Proof.** By (1), \( \chi_s(G) \leq k \). Assume that the statement is false. Then there is a smallest positive integer \( k \) for which there exists a complete \( k \)-partite graph \( G \) with \( \chi_s(G) \leq k - 1 \). Necessarily, \( k \geq 4 \). Suppose that the partite sets of \( G \) are \( V_1, V_2, \ldots, V_k \). Let there be given a set \((k-1)\)-coloring \( c : V(G) \rightarrow \mathbb{N}_{k-1} \) of \( G \). We claim that for each partite set \( V_i \) \((1 \leq i \leq k)\) the coloring \( c_i = c|_{V_i} \) is a set coloring of \( G - V_i \), which is a complete \((k-1)\)-partite graph. In order to see that this is the case, let \( u \) and \( v \) be adjacent vertices in \( G - V_i \). In \( G \) we have \( \text{NC}_c(u) \neq \text{NC}_c(v) \). Since

\[ \text{NC}_c(u) = \text{NC}_{c_i}(u) \cup c(V_i) \quad \text{and} \quad \text{NC}_c(v) = \text{NC}_{c_i}(v) \cup c(V_i), \]

it follows that \( \text{NC}_{c_i}(u) \neq \text{NC}_{c_i}(v) \). This implies that the coloring \( c_i \) of \( G - V_i \) is a set coloring, as claimed. Since \( \chi_s(G - V_i) = k - 1 \), it follows that \( c(V(G) - V_i) = \mathbb{N}_{k-1} \). Thus \( \text{NC}_c(x) = \mathbb{N}_{k-1} \) for every vertex \( x \) of \( V_i \). Since the partite set \( V_i \) was chosen arbitrarily, \( \text{NC}_c(x) = \mathbb{N}_{k-1} \) for every vertex \( x \) of \( G \), which is impossible. ■

By Proposition 2.7, the complete \( k \)-partite graph \( K_{1,1,\ldots,1,n-(k-1)} \) has set chromatic number \( k \), giving the following result.

**Corollary 2.8.** For each pair \( k,n \) of integers with \( 2 \leq k \leq n \), there is a connected graph \( G \) of order \( n \) with \( \chi_s(G) = k \).

It is well known that the chromatic number of a graph \( G \) is at least as large as its clique number \( \omega(G) \), which is the largest order of a clique (a complete subgraph) in \( G \). The following observation will be useful to us.

**Observation 2.9.** Let \( G \) be a graph of order \( n \geq 2 \). Then \( \chi(G) = n - 1 \) if and only if \( \omega(G) = n - 1 \).
Proposition 2.10. For a connected graph $G$ of order $n \geq 3$,
\[ \chi_s(G) = n - 1 \quad \text{if and only if} \quad \chi(G) = n - 1. \]

Proof. If $\chi_s(G) = n - 1$, then $G \neq K_n$ and so the result immediately follows by (1). For the converse, assume that $\chi(G) = n - 1$. Then by Observation 2.9, $\omega(G) = n - 1$ and so $G$ is obtained from $K_{n-1}$ by adding a new vertex $u$ and joining $u$ to some (but not all) vertices of $K_{n-1}$. Assume, to the contrary, that $\chi_s(G) = k \leq n - 2$ and let there be given a set $k$-coloring of $G$ using the colors in $\mathbb{N}_k$. Permuting the colors if necessary, we can obtain a set $k$-coloring $c : V(G) \rightarrow \mathbb{N}_k$ such that $c(V(K_{n-1})) = \mathbb{N}_k$, where $1 \leq k \leq k$. Since $\ell < n - 1$, some vertices in $K_{n-1}$ are colored the same. Let $X \subseteq V(K_{n-1})$ such that for each $x \in X$, there exists a vertex $y \in X - \{x\}$ such that $c(y) = c(x)$. Hence $|X| \geq 2$. Since each of the remaining $n - 1 - |X|$ vertices in $K_{n-1}$ receives a unique color, it follows that $n - |X| \leq \ell$. For each $x \in X$, either

(i) $NC(x) = N_{\ell}$ or
(ii) $NC(x) = N_{\ell} \cup \{c(u)\}$ if $x \in N(u)$ and $c(u) \notin N_{\ell}$.

This implies that $|X| \leq 2$. Hence $|X| = 2$ and so $\ell = n - 2$. Then $k = \ell + 1$ (since $c(u) \notin N_{\ell}$) and

\[ n - 2 = \ell = k - 1 \leq n - 3, \]

which is impossible. ■

By Proposition 2.10 and its proof, a connected graph $G$ of order $n \geq 3$ has $\chi_s(G) = n - 1$ if and only if $G = (K_{n-1-k} \cup K_1) + K_k$ for some integer $k$ with $1 \leq k \leq n - 2$.

Corollary 2.11. If $G$ is a connected graph of order $n$ such that $\chi(G) \in \{1, 2, 3, n - 1, n\}$, then $\chi_s(G) = \chi(G)$.

3. Lower Bounds for the Set Chromatic Number

We have already observed that $\chi_s(G) \leq \chi(G)$ for every graph $G$. There is also a lower bound for the set chromatic number of a graph in terms of its chromatic number.
Proposition 3.1. For every graph $G$,

$$\chi_s(G) \geq \left\lceil \log_2(\chi(G) + 1) \right\rceil.$$ 

**Proof.** Since this is true if $1 \leq \chi(G) \leq 3$, we may assume that $\chi(G) \geq 4$. Let $\chi_s(G) = k$ and let there be given a set $k$-coloring of $G$ using the colors in $\mathbb{N}_k$. Thus $\text{NC}(x) \subseteq \mathbb{N}_k$ for every vertex $x$ of $G$. Since $\text{NC}(u) \neq \text{NC}(v)$ for every two adjacent vertices $u$ and $v$ of $G$, it follows that $\text{NC}(x)$ can be considered as a color for each $x \in V(G)$, that is, the coloring $c$ of $G$ defined by $c(x) = \text{NC}(x)$ for $x \in V(G)$ is a proper coloring of $G$. Since there are $2^k - 1$ nonempty subsets of $\mathbb{N}_k$, it follows that $c$ uses at most $2^k - 1$ colors. Thus $\chi(G) \leq 2^k - 1$ or $\chi(G) + 1 \leq 2^k$. Thus $\chi_s(G) = k \geq \left\lceil \log_2(\chi(G) + 1) \right\rceil$, as desired. 

By Corollary 2.11, the lower bound for the set chromatic number of a graph $G$ in Proposition 3.1 is sharp if $\chi(G) \in \{1, 2\}$. If $\chi(G) = 3$, then $\chi_s(G) = 3 > \left\lceil \log_2(3 + 1) \right\rceil = 2$ and so this bound is not sharp in this case.

The Grötzsch graph $G^*$ of Figure 2 is known to have chromatic number 4. A set 3-coloring of $G^*$ is also given in Figure 2 and so $\chi_s(G^*) \leq 3$. By Proposition 2.2, $\chi_s(G^*) \geq 3$. Thus $\chi_s(G^*) = 3$. Since $\left\lceil \log_2(\chi(G^*) + 1) \right\rceil = \left\lceil \log_2 5 \right\rceil = 3$, the lower bound for $\chi_s(G^*)$ is attained in this case.

![Figure 2. A set 3-coloring of the Grötzsch graph.](image)

While $\chi(G) \geq \omega(G)$ for every graph $G$, the clique number is not a lower bound for the set chromatic number of a graph.

Proposition 3.2. For every graph $G$,

(3) $$\chi_s(G) \geq 1 + \left\lfloor \log_2 \omega(G) \right\rfloor.$$
Proof. If \( \omega(G) = 2 \), then \( \chi_s(G) \geq 2 \); while if \( \omega(G) = 3 \), then \( \chi_s(G) \geq 3 \). Thus we may assume that \( \omega(G) = \omega \geq 4 \). Let \( H \) be a clique of order \( \omega \) in \( G \) with \( V(H) = \{ v_1, v_2, \ldots, v_\omega \} \). Suppose that \( \chi_s(G) = k \) and let \( c: V(G) \to \mathbb{N}_k \) be a set \( k \)-coloring of \( G \). We consider two cases, according to whether there are two vertices in \( V(H) \) colored the same or no two vertices in \( V(H) \) are assigned the same color.

Case 1. There are two vertices in \( V(H) \) colored the same, say \( c(v_1) = c(v_2) = 1 \). Then \( 1 \in NC(v_i) \) for \( 1 \leq i \leq \omega \). Since there are exactly \( 2^{k-1} \) subsets of \( \mathbb{N}_k \) containing 1, it follows that \( \omega \leq 2^{k-1} \) and so \( k - 1 \geq \log_2 \omega \). Therefore, (3) holds.

Case 2. No two vertices in \( V(H) \) are colored the same. Then \( \omega \) distinct colors are used for the vertices in \( V(H) \) and so \( \omega \leq k \). Since \( \omega \geq 4 \), it follows that

\[
  k \geq \omega > 1 + \lfloor \log_2 \omega(G) \rfloor
\]

Again, (3) holds.

The lower bound for the set chromatic number of a graph in Proposition 3.2 is sharp. To see this, we construct a connected graph \( G \) with \( \omega(G) = 2^{k-1} \) and \( \chi_s(G) = k \) for each integer \( k \geq 2 \). We start with the complete graph \( H = K_{2^{k-1}} \) of order \( 2^{k-1} \), where \( V(H) = \{ v_1, v_2, \ldots, v_{2^{k-1}} \} \). Let \( S_1, S_2, \ldots, S_{2^{k-1}} \) be the \( 2^{k-1} \) subsets of \( \mathbb{N}_{k-1} \), where \( S_1 = \emptyset \). For each integer \( i \) with \( 2 \leq i \leq 2^{k-1} \), we add \( |S_i| \) pendant edges at the vertex \( v_i \), obtaining the connected graph \( G \) with \( \omega(G) = 2^{k-1} \). It remains to show that \( \chi_s(G) = k \). By Proposition 3.2, \( \chi_s(G) \geq k \). Define a \( k \)-coloring of \( G \) by assigning

(i) the color \( k \) to each vertex of \( H \) and
(ii) the colors in \( S_i \) to the \( |S_i| \) end-vertices adjacent to \( v_i \) for \( 2 \leq i \leq 2^{k-1} \).

Figure 3 shows the graph \( G \) for \( k = 4 \) and the corresponding 4-coloring. Thus \( NC(v_i) = S_i \cup \{ k \} \) for \( 1 \leq i \leq 2^{k-1} \). Hence \( |NC(v_i)| \geq 2 \) for \( 2 \leq i \leq 2^{k-1} \) and \( |NC(x)| = 1 \) for each end-vertex \( x \) of \( G \). This implies that every two adjacent vertices in \( G \) have different neighborhood color sets. Consequently, \( c \) is a set \( k \)-coloring of \( G \) and so \( \chi_s(G) = k \).
4. Vertex or Edge Deletions and the Set Chromatic Number

For the graph $G$ of Figure 4(a), $\chi(G) = \omega(G) = 4$. By Proposition 3.2, $\chi_s(G) \geq 1 + \lceil \log_2 \omega(G) \rceil = 3$. The set 3-coloring of $G$ in Figure 4(b) shows that $\chi_s(G) = 3$. The graph $G - x_2$ is shown in Figure 4(c) together with a set 4-coloring. Observe that the graph $G - x_2$ is isomorphic to the graph $G_{4,3}$ described prior to Proposition 2.6. In fact, $\chi_s(G - x_2) = \chi_s(G_{4,3}) = 4$ by Proposition 2.6.

Figure 4. A set 3-coloring of a graph $G$ and a set 4-coloring of $G - x_2$. 

The preceding example shows that it is possible for a graph $G$ to contain a vertex $v$ such that the set chromatic number of $G - v$ is greater than the
set chromatic number of $G$. If $G = C_5$, then $\chi_s(G - v) = 2 = \chi_s(G) - 1$ for every vertex $v$ of $G$. If $G = C_5 + K_1$ where $v$ is the central vertex of $G$, then $\chi_s(G - v) = 3 = \chi_s(G)$. Therefore, for each $i \in \{-1, 0, 1\}$, there exists a graph $G$ containing a vertex $v$ such that $\chi_s(G - v) = \chi_s(G) + i$.

In fact, $\chi_s(G - v)$ can exceed $\chi_s(G)$ by more than 1. Prior to showing this, we introduce additional notation. For integers $a$ and $b$ with $a < b$, let

$$[a..b] = \{x \in \mathbb{Z} : a \leq x \leq b\}.$$  

In particular, $[1..b] = \mathbb{N}_b$.

Let $G$ be a graph of order $n = 11$ and clique number $\omega(G) = 8$ constructed from $K_8$ with $V(K_8) = \{v_1, v_2, \ldots, v_8\}$ by adding three pairwise nonadjacent vertices $u_1, u_2, u_3$ and joining $v_i$ and $u_j$ as follows: Let $S_1 = \emptyset$, $S_2 = \{1\}$, $S_3 = \{2\}$, $S_4 = \{1, 2\}$, and $S_i = S_{i-4} \cup \{3\}$ for $5 \leq i \leq 8$. For $1 \leq i \leq 8$ and $1 \leq j \leq 3$, $v_i u_j \in E(G)$ if and only if $j \in S_i$ (see Figure 5). By Proposition 3.2, $\chi_s(G) \geq 1 + \lceil \log_2 8 \rceil = 4$, while the coloring $c_1 : V(G) \to \mathbb{N}_4$ of $G$ defined by

$$c_1(v) = \begin{cases} i & \text{if } v = u_i (1 \leq i \leq 3), \\ 4 & \text{otherwise} \end{cases}$$

is a set 4-coloring. Therefore, $\chi_s(G) = 4$.

![Graph G with $\chi_s(G - u_3) = \chi_s(G) + 3$.](image)

**Figure 5.** A graph $G$ with $\chi_s(G - u_3) = \chi_s(G) + 3$.  

For the graph $G$ of Figure 5, let $H = G - u_3$. We claim that $\chi_s(H) = 7$. First observe that the coloring $c_2 : V(H) \to \mathbb{N}_7$ of $H$ defined by
is a set 7-coloring and so \( \chi_s(H) \leq 7 \). Assume, to the contrary, that there exists a set \( \ell \)-coloring of \( H \) using the colors in \( \mathbb{N}_\ell \) for some \( \ell \leq 6 \). Permuting the colors if necessary, we can obtain a set \( \ell \)-coloring \( c_3 : V(H) \to \mathbb{N}_\ell \) of \( H \) such that \( c_3(V(K_8)) = \mathbb{N}_\ell' \) for some integer \( \ell' \) with \( 1 \leq \ell' \leq \ell \). Since \( \ell' < 8 \), some vertices in \( K_8 \) are colored the same. Let \( X \) be the subset of \( V(K_8) \) such that for each \( x \in X \), there exists a vertex \( y \in X - \{x\} \) with \( c_3(y) = c_3(x) \). Since each vertex of \( V(K_8) - X \) receives a unique color and at least one additional color is used for the vertices in \( X \), it follows that \( (8 - |X|) + 1 = 9 - |X| \leq \ell' \leq 6 \) and so \( |X| \geq 3 > 2 \).

The remaining \( \ell - \ell' \) colors are used for the two vertices \( u_1 \) and \( u_2 \), implying that \( \ell - \ell' \leq 2 \). Also, since each vertex \( x \in X \) must have a unique neighborhood color set containing \( \mathbb{N}_\ell' \) as a subset, the set NC(\( x \)) - \( \mathbb{N}_\ell' \) is a unique subset of \( [\ell' + 1, \ell] \). Therefore, \( 2 < |X| \leq 2^{\ell - \ell'} \leq 2^2 \), implying that \( \ell - \ell' = 2 \) and so \( \ell' \leq 4 \). However, since \( |X| \leq 4 \),

\[
5 \leq 9 - |X| \leq \ell' \leq 4,
\]

which is impossible.

Therefore, \( \chi_s(H) = 7 \), as claimed, and so \( \chi_s(G - u_3) = \chi_s(G) + 3 \). In fact, \( \chi_s(G - u_i) = \chi_s(G) + 3 \) for each vertex \( u_i \) (\( 1 \leq i \leq 3 \)). Observe for the graph \( G \) of Figure 5 that \( \deg_{G} u_i = 4 \) for each \( i \) (\( 1 \leq i \leq 3 \)). In general, we have the following result.

**Theorem 4.1.** If \( v \) is a vertex of a graph \( G \), then

\[
\chi_s(G) - 1 \leq \chi_s(G - v) \leq \chi_s(G) + \deg v.
\]

**Proof.** First, we verify the lower bound for \( \chi_s(G - v) \). Suppose that \( \chi_s(G - v) = k \). Let \( c_1 : V(G - v) \to \mathbb{N}_k \) be a set \( k \)-coloring of \( G - v \). Then the coloring \( c'_1 \) of \( G \) defined by

\[
c'_1(x) = \begin{cases} 
c_1(x) & \text{if } x \neq v, \\
k + 1 & \text{if } x = v
\end{cases}
\]
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is a set coloring of $G$ using $k + 1$ colors. Therefore, $\chi_s(G) \leq k + 1 = \chi_s(G - v) + 1$.

Next, we show that $\chi_s(G - v) \leq \chi_s(G) + \deg v$. Suppose that $\chi_s(G) = \ell$ and $\deg v = d$, where $N(v) = \{v_1, v_2, \ldots, v_d\}$. Let $c_2 : V(G) \to N_\ell$ be a set $\ell$-coloring of $G$. Then the coloring $c'_2$ of $G - v$ defined by

$$c'_2(x) = \begin{cases} c_2(x) & \text{if } x \notin N(v), \\ \ell + i & \text{if } x = v_i \ (1 \leq i \leq d) \end{cases}$$

is a set coloring of $G - v$, using at most $\ell + d$ colors. Therefore, $\chi_s(G - v) \leq \ell + d = \chi_s(G) + \deg v$.

We have already seen that the lower bound for $\chi(G - v)$ given in Theorem 4.1 is sharp. To see that the upper bound in Theorem 4.1 is sharp, let $n = 2k \geq 4$. We construct a graph $G$ of order $2n$ from $K_n$ with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by adding $n$ new vertices $u_1, u_2, \ldots, u_{n-1}, w$ and joining

(i) $u_i$ to $v_i$ for $1 \leq i \leq n - 1$ and
(ii) $w$ to $v_i$ for $k + 1 \leq i \leq n - 1$.

Hence $\deg w = k - 1$ and, furthermore, $G - w$ is isomorphic to the graph $G_{n,n-1}$ described prior to Proposition 2.6. Since $\chi_s(G - w) = n = 2k$, it follows by Theorem 4.1 that

$$\chi_s(G) \geq \chi_s(G - w) - \deg w = k + 1.$$ 

Furthermore, since the coloring $c : V(G) \to N_{k+1}$ defined by

$$c(v) = \begin{cases} i & \text{if } v \in \{u_i, u_{k+1}\} \ (1 \leq i \leq k - 1), \\ k & \text{if } v \in \{u_k, w\}, \\ k + 1 & \text{otherwise} \end{cases}$$

is a set $(k + 1)$-coloring of $G$, it follows that $\chi_s(G) \leq k + 1$ and so $\chi_s(G) = k + 1$. Consequently,

$$\chi_s(G - w) = \chi_s(G) + \deg w,$$

establishing the sharpness of the upper bound in Theorem 4.1.

We now consider how the set chromatic number of a connected graph $G$ is affected by deleting an edge from $G$. Consider the connected graph $G$ of Figure 6(a) and the three edges $e_{-1} = v_1v_2$, $e_0 = u_2u_3$, and $e_1 = u_4v_5$
in $G$. For the three graphs $G - e_i$ for $i \in \{-1, 0, 1\}$, observe that $\omega(G) = \omega(G - e_0) = \omega(G - e_1) = 5$ and $\omega(G - e_{-1}) = 4$. Hence $\chi_s(H) \geq 4$ for $H \in \{G, G - e_0, G - e_1\}$ and $\chi_s(G - e_{-1}) \geq 3$ by Proposition 3.2. The colorings given in Figure 6 show that $\chi_s(G) = \chi_s(G - e_0) = 4$ and $\chi_s(G - e_{-1}) = 3$.

We now show that $\chi_s(G - e_1) = 5$. Since $\chi(G - e_1) = 5$, it suffices to verify that $\chi_s(G - e_1) \neq 4$. Assume, to the contrary, that $c$ is a set 4-coloring of $G - e_1$. For the graph $F = (G - e_1) - e_0$, it was shown in Proposition 2.6 that $\chi_s(F) = 5$, that is, $c$ is not a set coloring of $F$. Note that $NC_{G-e_1}(x) = NC_F(x)$ for every $x \in V(G - e_1) - \{u_2, u_3\}$ and so we may assume that $NC_F(v_2) = NC_F(u_2) = \{c(v_2)\} = \{1\}$.

However, this implies that $NC_{G-e_1}(v_1) = NC_{G-e_1}(v_2) = \{1\}$, contradicting the fact that $c$ is a set coloring of $G - e_1$. Therefore, $\chi_s(G - e_1) = 5$. Hence for $-1 \leq i \leq 1$,

$$\chi_s(G - e_i) = \chi_s(G) + i.$$

![Figure 6](image-url)

Figure 6. Graphs $G$ and $G - e_i$ with $\chi_s(G - e_i) = \chi_s(G) + i$ for $i \in \{-1, 0\}$. 
Next, we show that for every graph $G$ and an edge $e$ in $G$, the difference between $\chi_s(G)$ and $\chi_s(G - e)$ cannot exceed 2.

**Proposition 4.2.** If $e$ is an edge of a graph $G$, then

$$|\chi_s(G) - \chi_s(G - e)| \leq 2.$$ 

**Proof.** Let $e = uv$. First, we verify that $\chi_s(G) - \chi_s(G - e) \leq 2$. Suppose that $\chi_s(G - e) = k$ and let $c_1 : V(G - e) \rightarrow \mathbb{N}_k$ be a set $k$-coloring of $G - e$. Then observe that the coloring $c'_1$ of $G$ defined by

$$c'_1(x) = \begin{cases} 
    k + 1 & \text{if } x = u, \\
    k + 2 & \text{if } x = v, \\
    c_1(x) & \text{otherwise}
\end{cases}$$

is a set coloring of $G$ that using at most $k + 2$ colors. Therefore, $\chi_s(G) \leq k + 2 = \chi_s(G - e) + 2$ and so $\chi_s(G) - \chi_s(G - e) \leq 2$. To verify that $\chi_s(G - e) - \chi_s(G) \leq 2$, suppose that $\chi_s(G) = \ell$ and consider a set $\ell$-coloring $c_2 : V(G) \rightarrow \mathbb{N}_\ell$ of $G$. Then the coloring $c'_2$ defined by

$$c'_2(x) = \begin{cases} 
    \ell + 1 & \text{if } x = u, \\
    \ell + 2 & \text{if } x = v, \\
    c_2(x) & \text{otherwise}
\end{cases}$$

is a set coloring of $G - e$ using at most $\ell + 2$ colors. Thus, $\chi_s(G - e) \leq \ell + 2 = \chi_s(G) + 2$. 

We are unaware of a graph $G$ and an edge $e$ of $G$ such that $|\chi_s(G) - \chi_s(G - e)| = 2$. Nevertheless, we conclude by presenting a sufficient condition that $|\chi_s(G) - \chi_s(G - e)| \leq 1$ for an edge $e = uv$ that is not a bridge in a graph $G$ in terms of the distance between $u$ and $v$ in $G$. For a vertex $v$ in a graph $G$, let $N_G[v] = N_G(v) \cup \{v\}$ be the closed neighborhood of $v$ in $G$.

**Proposition 4.3.** If $e = uv$ is an edge of a graph $G$ that is not a bridge such that $d_{G - e}(u, v) \geq 4$, then

$$|\chi_s(G) - \chi_s(G - e)| \leq 1.$$
Proof. We first verify that \( \chi_s(G) - \chi_s(G-e) \leq 1 \). Suppose that \( \chi_s(G-e) = k \) and let \( c_1 : V(G-e) \to \mathbb{N}_k \) be a set \( k \)-coloring of \( G-e \). We show that the coloring \( c'_1 \) defined by

\[
c'_1(x) = \begin{cases} 
c_1(x) & \text{if } x \neq u, \\
k + 1 & \text{if } x = u
\end{cases}
\]

is a set coloring of \( G \) that uses at most \( k+1 \) colors. Observe that \( NC_{c'_1}(x) = NC_{c_1}(x) \) for every \( x \in V(G) - N_G[u] \), while \( k+1 \in NC_{c'_1}(x) \) for every \( x \in N_G(u) \). Let \( x, y \) be a pair of adjacent vertices in \( G \). If \( \{x, y\} \subseteq N_G(u) \), then \( NC_{c'_1}(x) \neq NC_{c'_1}(y) \). Hence we may assume that \( \{x, y\} \subseteq N_G(u) \).

Note that \( v \notin \{x, y\} \) since \( d_{G-e}(u, v) \geq 2 \). Thus, \( \{x, y\} \subseteq N_{G-e}(u) \). Since \( NC_{c_1}(x) \neq NC_{c_1}(y) \) and \( c_1(u) \in NC_{c_1}(x) \cap NC_{c_1}(y) \), there exists a color \( i^* \in \mathbb{N}_k - \{c_1(u)\} \) that belongs to exactly one of \( NC_{c_1}(x) \) and \( NC_{c_1}(y) \), say \( i^* \in NC_{c_1}(x) - NC_{c_1}(y) \). Hence \( c'_1 \) is a set coloring of \( G \) and so \( \chi_s(G) \leq k + 1 = \chi_s(G-e) + 1 \).

To verify that \( \chi_s(G-e) \leq \chi_s(G) + 1 \), suppose that \( \chi_s(G) = \ell \) and let \( c_2 : V(G) \to \mathbb{N}_\ell \) be a set \( \ell \)-coloring of \( G \). We show that the coloring \( c'_2 \) defined by

\[
c'_2(x) = \begin{cases} 
c_2(x) & \text{if } x \notin \{u, v\}, \\
\ell + 1 & \text{if } x \in \{u, v\}
\end{cases}
\]

is a set coloring of \( G-e \) using at most \( \ell + 1 \) colors. Observe that \( NC_{c'_2}(x) = NC_{c_2}(x) \) for every \( x \in V(G) - (N_{G-e}[u] \cup N_{G-e}[v]) \), while \( \ell + 1 \in NC_{c'_2}(x) \) for every \( x \in N_{G-e}(u) \cup N_{G-e}(v) \). Suppose that \( x, y \) is a pair of adjacent vertices in \( G-e \). If \( \{x, y\} \subseteq N_{G-e}(u) \cup N_{G-e}(v) \), then \( NC_{c'_2}(x) \neq NC_{c'_2}(y) \). On the other hand, since \( d_{G-e}(u, v) \geq 4 \), no vertex in \( N_{G-e}(u) \) is adjacent to a vertex in \( N_{G-e}(v) \). Hence if \( \{x, y\} \subseteq N_{G-e}(u) \cup N_{G-e}(v) \), then either \( \{x, y\} \subseteq N_{G-e}(u) \) or \( \{x, y\} \subseteq N_{G-e}(v) \), say the former. Since \( NC_{c_2}(x) \neq NC_{c_2}(y) \) and \( c_2(u) \in NC_{c_2}(x) \cap NC_{c_2}(y) \), there is a color \( j^* \in \mathbb{N}_\ell - \{c_2(u)\} \) that belongs to exactly one of \( NC_{c_2}(x) \) and \( NC_{c_2}(y) \), say \( j^* \in NC_{c_2}(x) - NC_{c_2}(y) \). Then \( j^* \in NC_{c'_2}(x) - NC_{c'_2}(y) \). Therefore, \( c'_2 \) is a set coloring of \( G-e \) and so \( \chi_s(G-e) \leq \ell + 1 = \chi_s(G) + 1 \).

According to the proof of Proposition 4.3, if there is a graph \( G \) with an edge \( e = uv \) having the property that \( |\chi_s(G) - \chi_s(G-e)| = 2 \), then \( d_{G-e}(u, v) \leq 3 \). In particular, if \( \chi_s(G) - \chi_s(G-e) = 2 \), then \( d_{G-e}(u, v) = 2 \).
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References


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