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NOTE

PACKING OF NONUNIFORM HYPERGRAPHS — PRODUCT AND SUM OF SIZES CONDITIONS

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Abstract

Hypergraphs $H_1, \ldots, H_N$ of order n are mutually packable if one can find their edge disjoint copies in the complete hypergraph of order n. We prove that two hypergraphs are mutually packable if the product of their sizes satisfies some upper bound. Moreover we show that an arbitrary set of the hypergraphs is mutually packable if the sum of their sizes is sufficiently small.

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1. INTRODUCTION

In [3] Pilśniak and Woźniak stated the following problem: *are two hypergraphs of order n and size at most $n/2$ packable*? They solved this problem in the case when the two hypergraphs are isomorphic. We solve the general case in a slightly stronger form (Corrolary 1). Let us first introduce the notation we shall use.

By a (*nonuniform*) *hypergraph* we mean a pair (V, E), where V is a finite set and $E \subseteq 2^V - \{\emptyset, V\}$. The elements of V are called *vertices* and the elements of E are called *edges* of the hypergraph. A hypergraph is k-*uniform* if all its edges are k-element sets. We exclude the sets V and $\emptyset$ as edges, for convenience. The *loops* are edges which have just one element. By an *order* of a hypergraph we mean the number $|V|$ and by its *size* the number $|E|$. All hypergraphs in every theorem of this paper have the same order which will be denoted by n. The hypergraph $(V, \emptyset)$ is called *empty*. The hypergraph $(V, 2^V - \{\emptyset, V\})$ is called *complete* and denoted by $K_{(n)}$, where $n = |V|$. The hypergraph (W, F) is called a *subhypergraph* of the hypergraph (V, E), if $W \subseteq V$ and $F \subseteq E$. The hypergraphs (V, E) and (W, F) are *isomorphic* if there exists a bijection $f : V \to W$ such that $e \in E$ if and only if $f(e) \in F$. If hypergraphs H and G are isomorphic, we write $H \cong G$. If H is a hypergraph and $v \in V(H)$ then by $H - v$ we mean the hypergraph $(V(H) - \{v\}, E')$, where $E' = \{f - \{v\} : f \in E(H), f \neq \{v\}\}$.

Definition 1. Let H_1, H_2 be hypergraphs of the same order. A bijection $\sigma : V(H_1) \to V(H_2)$ such that $(\forall_{e \in E(H_1)}) \sigma(e) \notin E(H_2)$ we shall call a *packing* of H_1 and H_2.

Definition 2. The hypergraphs $H_1, \ldots, H_N$ of order n are *mutually packable* if there exist edge disjoint subhypergraphs $G_1, \ldots, G_N$ of $K_{(n)}$ such that $G_i \cong H_i$ for $i = 1, \ldots, N$.

Let us notice that in the case $N = 2$ hypergraphs H_1, H_2 of the same order are mutually packable if and only if there exists a packing of them.

The fundamental results on packing of graphs were obtained by Sauer and Spencer [5], Burns and Schuster [1, 2] and Schuster [6]. More results for graphs can be find in Woźniak [7, 8, 9] and Yap [10]. Some generalizations of results of Sauer and Spencer for uniform hypergraphs were obtained by Rödl, Ruciński and Taraz [4].

Our theorems can be viewed as generalizations for hypergraphs of those presented in [5] by Sauer and Spencer for graphs.

2. THEOREMS

We shall restrict ourselves to hypergraphs without edges of size greater than $n/2$. It is only small loss of generality. To see this we construct for a given hypergraph $H = (V, E)$ an auxiliary hypergraph $\tilde{H}$ in the following way. Let $E_1 = \{e \in E : |e| \leqslant n/2\}$ and $E_2 = \{e \in E : |e| > n/2\}$. We set $\tilde{H} = (\tilde{V}, \tilde{E})$ where $\tilde{V} = V$ and $\tilde{E} = E_1 \cup \{V - e : e \in E_2\}$. Let us emphasize that we do not consider multiple edges. The hypergraph $\tilde{H}$ has at most as many edges as H and its edges are of size not greater than $n/2$.

Lemma 1. *If $\tilde{H}, \tilde{G}$ are mutually packable then H, G are mutually packable.*

Proof. Let σ be any packing of $\tilde{H}$ and $\tilde{G}$. We claim that σ is also a packing of H and G.

Indeed, let e be an edge of the hypergraph H.

If $|e| \leqslant n/2$ then $e \in E(\tilde{H})$. It implies $\sigma(e) \notin E(\tilde{G})$ because σ is a packing of $\tilde{H}$ and $\tilde{G}$. Suppose that $\sigma(e) \in E(G)$. Then $\sigma(e) \in E(\tilde{G})$ because $|\sigma(e)| \leqslant n/2$. This contradiction shows that $\sigma(e) \notin E(G)$ as claimed.

If $|e| > n/2$ then $V(H) - e \in E(\tilde{H})$ so $\sigma(V(H) - e) \notin E(\tilde{G})$ but $\sigma(V(H) - e) = V(G) - \sigma(e)$ and we obtain $\sigma(e) \notin E(G)$. ∎

The complete hypergraph $K_{(n)}$ has relatively few small and big edges. One can expect that if we forbid our hypergraphs to have such edges we shall be able to pack hypergraphs of greater sizes. The following theorem shows that it is in fact so. To prove it we use the same method like Sauer and Spencer in [5] and Pilśniak and Woźniak in [3].

Theorem 1. *Let H_1, H_2 be hypergraphs of order n without edges of sizes smaller than k and greater than $n-k$, for some $1 \leqslant k \leqslant \lfloor n/2 \rfloor$, such that $|E(H_1)||E(H_2)| < \binom{n}{k}$. Then H_1, H_2 are mutually packable.*

Proof. By Lemma 1 we can assume that our hypergraphs do not have edges of size greater than $n/2$. We count all bijections from $V(H_1)$ onto $V(H_2)$ which are not packings. Let us denote by X the set of such bijections. For $e \in E(H_1)$ and $f \in E(H_2)$ let $X_{ef} = \{\sigma : \sigma(e) = f\}$, that is X_{ef} is the set of bijections which map the edge e onto the edge f. We have $X = \bigcup_{e \in E(H_1), f \in E(H_2)} X_{ef}$. If $|e| \neq |f|$ then $|X_{ef}| = 0$. If $|e| = |f| = i$ then $|X_{ef}| = i!(n-i)!$. Let m_i, l_i, $i = k, \ldots, \lfloor n/2 \rfloor$, denote the numbers of i-element edges of the hypergraphs H_1, H_2, respectively.

$$
\begin{aligned}
|X| &= \Big| \bigcup_{e \in E(H_1), f \in E(H_2)} X_{ef} \Big| \leqslant \sum_{e,f} |X_{ef}| = \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{e,f \\ |e|=|f|=i}} |X_{ef}| = \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{e,f \\ |e|=|f|=i}} i!(n-i)! \\
&= \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} m_i l_i i!(n-i)! \leqslant k!(n-k)! \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} m_i l_i \leqslant k!(n-k)! \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} m_i \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} l_i \\
&= k!(n-k)! |E(H_1)||E(H_2)| < n!.
\end{aligned}
$$

It means that there is a bijection from $V(H_1)$ onto $V(H_2)$ which is a packing. ∎

The theorem shown above is sharp, at least for n and k such that there exists a $(k-1)-(n,k,1)$-design that is a k-uniform hypergraph of order n such that each $(k-1)$-element subset of the set of its vertices is contained in exactly one edge. Indeed, let $\mathbf{B^k}$ be such a design and let $\mathbf{A^k}$ be a k-uniform hypergraph whose edges are k-element subsets containing a fixed $(k-1)$-element set (which is called the center of $\mathbf{A^k}$). Let σ be any bijection from $V(\mathbf{B^k})$ onto $V(\mathbf{A^k})$ and let C be the subset of $V(\mathbf{B^k})$ which is mapped onto center of $\mathbf{A^k}$ by σ. The set C is contained in some edge e of $\mathbf{B^k}$. This edge contains a vertex v which is not an element of C. No matter how σ maps v, the image $\sigma(e)$ of e is the edge of $\mathbf{A^k}$. Thus the hypergraphs $\mathbf{B^k}$ and $\mathbf{A^k}$ are not mutually packable. The sizes of the considered hypergraphs are $|E(\mathbf{B^k})| = \binom{n}{k-1}/k$ and $|E(\mathbf{A^k})| = n-k+1$. The product of these numbers is exactly $\binom{n}{k}$.

Corollary 1. *Let $H_1, \ldots, H_N$ be hypergraphs of order n without edges of sizes smaller than k and greater than $n-k$, for some $1 \leqslant k \leqslant \lfloor n/2 \rfloor$, such that $\sum_{i=1}^{N} |E(H_i)| \leqslant 2\sqrt{\binom{n}{k}} - 1$. Then $H_1, \ldots, H_N$ are mutually packable.*

Proof. We apply induction on N.

If $N = 2$ our thesis follows immediately from Theorem 1 as the product of two positive numbers which sum is bounded by $2L$ is the greatest when both of them are L.

Let $N > 2$. The hypergraphs $H_1, \ldots, H_{N-1}$ satisfy the induction hypothesis so they are mutually packable. Thus there are edge disjoint subhypergraphs $G_1, \ldots, G_{N-1}$ of the complete hypergraph $K_{(n)}$ such that $G_i \cong H_i$ for $i = 1, \ldots, N-1$. Let us define a hypergraph $H = (V(K_{(n)}), \bigcup_{i=1}^{N-1} E(G_i))$. We have $|E(H)| = \sum_{i=1}^{N-1} |E(H_i)|$. Therefore $|E(H)| + |E(H_N)| \leqslant 2\sqrt{\binom{n}{k}} - 1$ so H and H_N are mutually packable. Due to the definition of the hypergraph H the hypergraphs $H_1, \ldots, H_N$ are mutually packable. ∎

We give also the following theorem which is a bit stronger for the most general case i.e., $k = 1$.

Theorem 2. *If $H_1, \ldots, H_N$ are the hypergraphs of order n such that $\sum_{i=1}^{N} |E(H_i)| \leqslant n$, then $H_1, \ldots, H_N$ are mutually packable.*

Proof. We prove the case $N = 2$ first. The theorem will follow by an inductive argument as in the proof of Corollary 1. Again we consider only hypergraphs without edges of size greater than $n/2$.

Let us notice that if none of the hypergraphs H_i has a loop and $n > 4$ then they satisfy the hypothesis of the Corollary 1 so they are mutually packable. In the case $n = 2$ at least one of our hypergraphs is empty or both of them have only one edge which is a loop. In both cases they are mutually packable. A similar reasoning shows that the theorem is true for the cases when $n = 3, 4$ and our hypergraphs do not have loops. Thus we can assume that at least one of the hypergraphs has a loop and $n \geqslant 3$.

We apply induction on n.

If $n = 3$, we have two hypergraphs with at most three edges which are loops because we consider hypergraphs with edges of size not greater than $n/2$ only. Therefore these hypergraphs are packable.

Assume that $n > 3$ and H_1 has a loop $\{v\}$. We can find a vertex $w \in V(H_2)$ such that $\{w\} \notin E(H_2)$ because otherwise we would have $|E(H_1)| + |E(H_2)| \geqslant n+1$. Let us consider hypergraphs $H_1 - v$ and $H_2 - w$. They have $n-1$ vertices and $|E(H_1 - v)| + |E(H_2 - w)| = |E(H_1)| - 1 + |E(H_2)| \leqslant n-1$ so they satisfy the induction hypothesis. Therefore there exists a packing σ of $H_1 - v$ and $H_2 - w$. We define a bijection $\sigma' : V(H_1) \to V(H_2)$ in the following way:

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \neq v, \\ w & \text{if } x = v. \end{cases}$$

We claim that σ' is a packing of H_1 and H_2. Indeed, let e be an edge of the hypergraph H_1. If $v \notin e$ then $e \in E(H_1 - v)$ so $\sigma(e) \notin E(H_2 - w)$ but in this case $\sigma'(e) = \sigma(e)$ and thus $\sigma'(e) \notin E(H_2)$. If $v \in e$ then $\sigma'(e) = \sigma(e - \{v\}) \cup \{w\}$. Therefore $\sigma'(e) \notin E(H_2)$ because otherwise $\sigma(e - \{v\}) \in E(H_2 - w)$, a contradiction. ∎

This theorem is best possible since a collection of 1-uniform hypergraphs is mutually packable if and only if the sum of their sizes is not greater than n.

Let $f_k(n)$ be the least integer such that there are k-uniform hypergraphs H_1 and H_2 of order n which are not mutually packable and $|E(H_1)| + |E(H_2)| = f_k(n)$.

Our Corollary 1 implies that $f_k(n) \geqslant 2\sqrt{\binom{n}{k}}$. We have $f_1(n) = n + 1$ and it was shown by Sauer and Spencer [5] that $f_2(n) = \lceil 3n/2 \rceil - 1$.

The construction of hypergraphs $\mathbf{A^2}$ and $\mathbf{B^2}$ succeeding Theorem 1 gives, for n even, an $(n-1)$-edge star and an $n/2$-edge matching. This is an extremal pair of graphs in this case, i.e., $|E(\mathbf{A^2})| + |E(\mathbf{B^2})| = f_2(n)$. We have also $|E(\mathbf{A^1})| + |E(\mathbf{B^1})| = f_1(n)$, for any n.

Finding the exact value of $f_k(n)$, for $k \geqslant 3$, is an open problem. We conjecture that for an arbitrary k our hypergraphs $\mathbf{A^k}$ and $\mathbf{B^k}$ form an extremal pair of hypergraphs for this problem, i.e., $|E(\mathbf{A^k})| + |E(\mathbf{B^k})| = f_k(n)$, provided that a $(k-1)-(n,k,1)$-design exists.

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