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ON NORMAL PARTITIONS IN CUBIC GRAPHS

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Abstract

A *normal partition* of the edges of a cubic graph is a partition into *trails* (no repeated edge) such that each vertex is the end vertex of exactly one trail of the partition. We investigate this notion and give some results and problems.

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1. Introduction and Notations

Following Bondy [1], a *walk* in a graph G is sequence $W := v_0e_1v_1 \dots e_kv_k$, where $v_0, v_1, \dots, v_k$ are vertices of G, and $e_1, e_2, \dots, e_k$ are edges of G and v_{i-1} and v_i are the ends of e_i, $1 \le i \le k$. v_0 and v_k are the *end vertices* and e_1 and e_k are the *end edges* of this walk, while $v_1, \dots, v_{k-1}$ are the *internal vertices* and $e_2, \dots, e_{k-1}$ are the *internal edges*. The *length* $l(W)$ of W is the number of edges (namely k). W is *odd* whenever k is odd and *even* otherwise. The walk W is a *trail* if its edges $e_1, e_2, \dots, e_k$ are distincts and a *path* if its vertices $v_0, v_1, \dots, v_k$ are distincts. If $W := v_0e_1v_1 \dots e_kv_k$, is a walk of G $W' := v_ie_{i+1} \dots e_jv_j$ $(0 \le i \le j \le k)$ is a *subwalk* of W (*subtrails* and *subpaths* are defined analogously). If v is an internal vertex of a walk W with ends x and y, $W(x, v)$ and $W(v, y)$ are the subwalks of W obtained in cutting W in v. Conversely if W_1 and W_2 have a common end v, the *concatenation* of these two walks *on* v gives rise to a new walk (denoted by $W_1 + W_2$) with v as an internal vertex. When no confusion, is possible, it will be convenient to omit the edges in the description of a walk, that is $W := v_0e_1v_1 \dots e_kv_k$ will be shorten in $W := v_0v_1 \dots v_k$.

Let $G = (V, E)$ be a cubic graph (loops and multiple edges are allowed) and let $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$ be a partition of $E(G)$ into trails. Every vertex $v \in V(G)$ is either an end vertex three times in the partition and we shall say that v is an *eccentric* vertex, or an end vertex exactly once, and we shall say that v is a *normal* vertex. To each vertex, we can associate a set $E_{\mathcal{T}}(v)$ containing the end vertices of the unique trail with v as an internal vertex, when such a trail exists in $\mathcal{T}$. When v is eccentric we obviously have $E_{\mathcal{T}}(v) = \emptyset$. It must be clear that we can have $v \in E_{\mathcal{T}}(v)$ since we consider a partition of trails.

Definition 1.1. A partition $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$ of $E(G)$ into trails is *normal* when every vertex is normal.

When $\mathcal{T}$ is a normal partition, we can associate to each vertex the unique edge with end v which is the end edge of a trail of $\mathcal{T}$. We shall denote this edge by $e_{\mathcal{T}}(v)$ and it will be convenient to say that $e_{\mathcal{T}}(v)$ is the *marked* edge associated to v. When it will be neces sary to illustrate our purpose by a figure the marked edge associated to a vertex will be figurate by a $\vdash$ close to this vertex.

Definition 1.2. A partition $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$ of $E(G)$ into trails is *odd* when every trail in $\mathcal{T}$ is odd.

Definition 1.3. A partition $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$ of $E(G)$ where each trail is a path will be called a *path partition*.

Definition 1.4. A partition $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ of $V(G)$ into paths is a *perfect path partition* when every vertex of G is contained in $\mathcal{P}$ (let us note that $k \geq \frac{n}{2}$). A perfect matching is thus a perfect path partition where each path has length 1.

When $F \subseteq E(G)$, $V(F)$ is the set of vertices which are incident with some edges of F and $G - F$ is the graph obtained from G in deleting the edges of F. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2. Elementary Properties

Proposition 2.1. *Let G be a cubic graph. Then we can find a normal partition of $E(G)$ within a linear time.*

Proof. We can easily obtain a partition $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ of $E(G)$ into trails via a greedy algorithm. If every vertex is normal then $\mathcal{T}$ is normal and we are done. If v is an eccentric vertex then v is the end vertex of two distinct trails T_1 and T_2. Let T' be the trail obtained by concatenation of T_1 and T_2 on v. v is an internal vertex of T' and $T - \{T_1, T_2\} + T'$ is a partition of $E(G)$ into trails with one eccentric vertex less (namely v). This operation can be repeated as long as the current partition into trails has an eccentric vertex and we end with a normal partition in at most $O(n)$ steps. ■

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Proof. Assume that $\mathcal{T}$ is normal, then every vertex is the end of exactly one trail. Hence $|\mathcal{T}| = \frac{n}{2}$.

Conversely, let $\mathcal{T}$ be a partition of the edge set of G into trails. Assume that $|\mathcal{T}| = \frac{n}{2}$ and T is not normal. Then, performing the operation described in Proposition 2.1 on eccentric vertices leads to a normal partition $\mathcal{T}'$ such that $|\mathcal{T}'| < \frac{n}{2}$, since the concatenation of two trails on a vertex decreases the number of trails in the partition, a contradiction. ■

We shall denote by $n_{\mathcal{T}}(i)$ the number of trails of length i and by $\mu(T)$ the mean length of trails in a normal partition.

Proposition 2.3. *Let T be a normal partition of a cubic graph G on n vertices. Then*

- $\mu(\mathcal{T}) = 3$,
- $\sum_{i=1}^{i=n+1} (3 - i) n_{\mathcal{T}}^i = 0$.

Proof. $\mathcal{T}$ being normal, we have $|\mathcal{T}| = \frac{n}{2}$ by Proposition 2.2. Since $|E(G)| = \frac{3n}{2}$ we have obviously $\mu(\mathcal{T}) = 3$.

We have

$$\sum_{i=1}^{i=n+1} i \times n_{\mathcal{T}}(i) = \frac{3n}{2} = 3 \sum_{i=1}^{i=n+1} n_{\mathcal{T}}(i)$$

and hence

$$\sum_{i=1}^{i=n+1} (3-i)n_{\mathcal{T}}(i) = 0.$$

■

The *length* of a normal partition $\mathcal{T}$ (denoted by $l(T)$) is the length of the longest trail in $\mathcal{T}$.

Proposition 2.4. *A cubic graph G on n vertices has an hamiltonian path if and only if G has a normal partition $\mathcal{T}$ such that $l(\mathcal{T}) = n+1$.*

Proof. Assume that $P = v_1v_2\dots v_n$ is an hamiltonian path of G. We shall consider that v_i is joined to v_{i+1} by the edge e_i in P. Let w_1 (w_n, respectively) a vertex adjacent to v_1 (w_1, respectively) by the edge e'_1 (e'_n, respectively) not in $E(P)$($e'_1 \neq e'_n$). Let T_1 be the trail $w_1e'_1v_1e_1v_2e_2\dots e_{n-1}v_ne'_nw_n$. $E(G) - T_1$ is reduced to a matching of size $\frac{n-2}{2}$ and it can be easily checked that this matching together with T_1 is a normal partition of G of length $n+1$.

Conversely, let $\mathcal{T}$ be a normal partition of G of length $n+1$ and let $T_1 = w_1e_1v_1e_1v_2e_2\dots e_{n-1}v_ne_nw_n$ be a trail of maximum length in T. Since the only vertices which can appear twice in T_1 are precisely w_1 and w_n, $P = v_1v_2\dots v_n$ is an hamiltonian path of G. ■

Theorem 2.5. *Let G be a cubic graph having a perfect path partition $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$. Assume that the ends of P_i are x_i and y_i ($\forall i = 1, \dots, k$). Then G has a normal partitions $\mathcal{T} = \{T_1, T_2, \dots, T_{\frac{n}{2}}\}$ such that T_i is obtained from P_i in adding one edge incident to x_i and one edge incident to y_i ($\forall i = 1, \dots, k$).*

Proof. The subgraph of G obtained in deleting the edges of each P_i is a set of disjoint paths. Let us give an arbitrary orientation to these paths. We get a normal partition $\mathcal{T}$ in adding the outgoing edge incident to x_i and to y_i ($\forall i = 1, \dots, k$), the remaining edges being a set of trails of length 1 in $\mathcal{T}$. ■

Let $l_1, l_2, \dots, l_{\frac{n}{2}}$ be a set of integers ($l_i \geq 1$) such that

$$\sum_{i=1}^{\frac{n}{2}} l_i = \frac{3n}{2}$$

can we find a normal partition $\mathcal{T} = \{T_1, T_2, \dots, T_{\frac{n}{2}}\}$ where $l(T_i) = l_i \ \forall i = 1, \dots, \frac{n}{2}$? There is no complete answer in general, however, when G has an hamiltonian cycle we have the following result (see [2]):

Theorem 2.6. *Let G be a cubic hamiltonian graph. Let $l_1, l_2, \dots, l_{\frac{n}{2}}$ be a set of integers such that*

- $\sum_{i=1}^{\frac{n}{2}} l_i = \frac{3n}{2}$,
- $l_i \geq 1 \quad l_i \neq 2 \ \forall i = 1, \dots, \frac{n}{2}$.

Then G has a normal partition $\mathcal{T} = \{T_1, T_2, \dots, T_{\frac{n}{2}}\}$ where $l(T_i) = l_i \ \forall i = 1, \dots, \frac{n}{2}$.

Proof. Let $\lambda_i = l_i - 2$ and assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\frac{n}{2}}$. The first k values (for some $k \leq \frac{n}{2}$) are greater than 1, and the remaining values are -1, since $l_i \neq 2$ for all $i = 1, \dots, \frac{n}{2}$. We have

$$\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} (l_i - 2) = \sum_{i=1}^{k} l_i - 2k,$$

$$\sum_{i=1}^{k} l_i - 2k = \sum_{i=1}^{k} l_i - 2k + \sum_{j=k+1}^{\frac{n}{2}} l_j - \left(\frac{n}{2} - k + 1\right)$$

since $\sum_{i=1}^{k} l_i + \sum_{j=k+1}^{\frac{n}{2}} l_j = \frac{3n}{2}$ we get that

$$\sum_{i=1}^{k} \lambda_i = n - k + 1.$$

Let C be an hamiltonian cycle of G, we can thus arrange a set $\mathcal{P}$ of vertex disjoint paths P_i of length λ_i $(i = 1, \dots, k)$ along this cycle. $\mathcal{P}$ is a perfect path partition and, applying theorem 2.5 we have a normal partition of G as claimed. ∎

Let $\mathcal{T}$ be a normal partition of a cubic graph G and let v be any vertex of G. $E_{\mathcal{T}}(v)$ contains exactly two vertices, namely x and y and one of them, at least, must be distinct from v (we may assume that $v \neq x$). Let T_1 be the trail with ends x and y such that v is an internal vertex of T_1. Since $\mathcal{T}$ is normal, there is a trail T_2 ending in v (with the edge $e_{\mathcal{T}}(v)$).

If T_1' denotes the trail obtained by concatenation of $T_1(x, v)$ and T_2 on v, then $\mathcal{T} - \{T_1, T_2\} + T_1' + T_1(v, y)$ is a new normal partition of G. We shall say that the above operation is a *switch on* v. When $v \notin E_{\mathcal{T}}(v)$ two such switchings are allowed (see Figure 1), but when $v \in E_{\mathcal{T}}(v)$ only one switching is possible (see Figure 2). A switch on a vertex v (leading from a normal partition $\mathcal{T}$ to the normal partition $\mathcal{T}' = \mathcal{T} * v$) does not change the edge marked associated to w when $w \neq v$. That is $e_{\mathcal{T}}(w) = e_{\mathcal{T}'}(w)$. On the other hand, the sets $E_{\mathcal{T}'}(w)$ may have changed for vertices of T_1 and T_2. When $\mathcal{T}$ is a normal odd partition and when $\mathcal{T}' = \mathcal{T} * v$ remains to be an odd partition, the switch on v is said to be an *odd switch*. It is not difficult to see that, given a normal odd partition, an odd switch is always possible on every vertex.

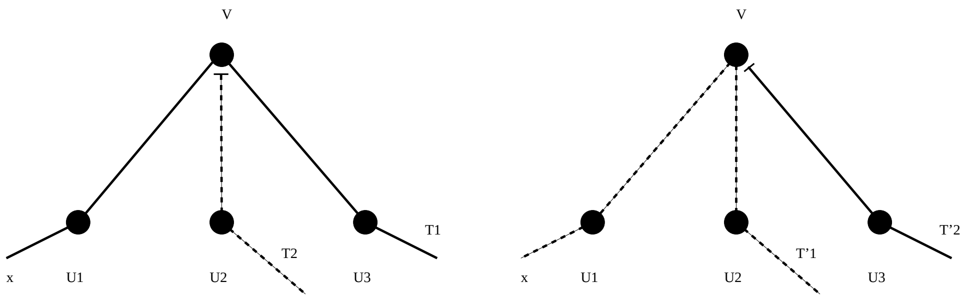


Figure 1. Switching on v with two distinct trails.

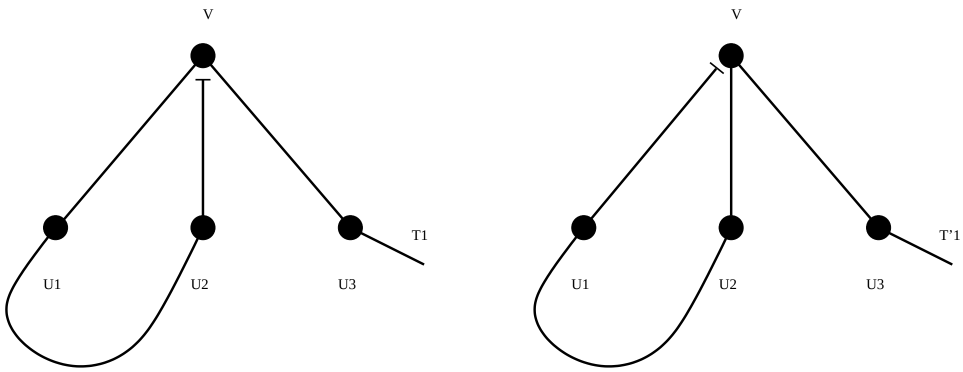


Figure 2. Switching on v with one trail.

We shall say that $\mathcal{T}$ and $\mathcal{T}'$ are *switching equivalent* (resp. *odd switching equivalent*) whenever $\mathcal{T}'$ can be obtained from $\mathcal{T}$ by a sequence of switchings (resp. odd switchings). The *switching class* (resp. *odd switching class*) of $\mathcal{T}$ is the set of normal partitions which are switching equivalent (resp. odd switching equivalent) to $\mathcal{T}$.

Theorem 2.7. *Let G be a cubic graph and let $\mathcal{T}$ and $\mathcal{T}'$ be any two normal (resp. odd) partitions. Then $\mathcal{T}'$ can be obtained from $\mathcal{T}$ by a sequence of (resp. odd) switchings of length at most $2n$.*

Proof. Let $A_{\mathcal{T}\mathcal{T}'} = \{v|\ v \in V(G) \quad e_{\mathcal{T}}(v) = e_{\mathcal{T}'}(v)\}$ and assume that $V(G) - A \neq \emptyset$. We want to pick a vertex in $V(G) - A$ and try to switch the normal partition $\mathcal{T}$ in this vertex (or $\mathcal{T}'$) in order to increase the size of A. We can suppose that $\mathcal{T}'$ is not in the switching class of $\mathcal{T}$ and, moreover, among the switching equivalent normal partitions of $\mathcal{T}$ and those of $\mathcal{T}'$, $A_{\mathcal{T}\mathcal{T}'}$ has maximum cardinality.

Let $v \in A_{\mathcal{T}\mathcal{T}'}$ and let u_1, u_2 and u_3 be its neighbors. Assume that $e_{\mathcal{T}}(v) = vu_1$ and $e_{\mathcal{T}'}(v) = vu_2$. Recall that in both partitions a switch (resp. odd switch) is always possible on v.

Consider first a possible switch (resp. odd switch) on v in $\mathcal{T}$, if $e_{\mathcal{T}*v} = vu_2$ then $A_{\mathcal{T}*v,\mathcal{T}'} = A_{\mathcal{T},\mathcal{T}'} \cup \{v\}$, a contradiction. If in switching (resp. odd switching) $\mathcal{T}'$ on v we have $e_{\mathcal{T}'*v} = vu_1$ then $A_{\mathcal{T},\mathcal{T}'*v} = A_{\mathcal{T},\mathcal{T}'} \cup \{v\}$, a contradiction. Finally, if $e_{\mathcal{T}*v} \neq vu_2$ and $e_{\mathcal{T}'*v} \neq vu_1$ that means that $e_{\mathcal{T}*v} = vu_3$ and $e_{\mathcal{T}'*v} = vu_3$, thus $A_{\mathcal{T}*v,\mathcal{T}'*v} = A_{\mathcal{T},\mathcal{T}'} \cup \{v\}$, a contradiction.

Hence any two normal partitions are switching equivalent (resp. odd switching equivalent). In order to increase the size of $A_{\mathcal{T}\mathcal{T}'}$, we have seen that we eventually are obliged to proceed to two switchings on the same vertex (one with $\mathcal{T}$ and one with $\mathcal{T}'$). It is thus clear that we need at most $2n$ such switching on the road leading to $\mathcal{T}'$ from $\mathcal{T}$. ∎

Theorem 2.7 suggests a simulated annealing approach in order to search for a longest path in a cubic graph. We have got results in that direction in [3] when considering linear partitions (partitions of the edge set of a cubic graph into two forests of paths). Instead of using a switching on a vertex the elementary operation involved was a switching on an edge, but, in that case, it is not true that any two linear partitions are switching equivalent.

Theorem 2.8. *Let G be a cubic graph. Then G has an odd normal partition if and only if G has a perfect matching.*

Proof. Let M be a perfect matching in G. Then $G - M$ is a 2-factor of G. Let us give any orientation to the cycles of this 2-factor and for each vertex v let us denote the outgoing edge $o(v)$. For each edge $e = uv \in M$, let P_{uv} be the path of length 3 obtained in concatenating $o(u)$ uv and $o(v)$. Then $T = \{P_{uv} | uv \in M\}$ is a normal odd partition (of length 3) of G.

Conversely, let $\mathcal{T} = \{T_1, T_2, \ldots, T_{\frac{n}{2}}\}$ be a normal odd partition of G. For each trail $T_i \in T$ let us say that en edge $e = uv$ of T_i is *odd* whenever the subtrails of T_i obtained in deleting e have odd lengths (an *even* edge being defined in the obvious way). A vertex $v \in V(G)$ is internal in exactly one trail of $\mathcal{T}$. The edges of this trail being alternatively odd and even, v is incident to exactly one odd edge. Hence the odd edges so defined induce a perfect matching of G. ■

Given a set of edges $F = \{e_v | v \in V(G)\}$, where each vertex of $V(G)$ appears exactly once as the end of an edge of F. Under which condition can we say that this set of edges is the set of marked edges associated to a normal partition?

Theorem 2.9. *Let $F = \{e_v | v \in V(G)\}$ be a set of edges of G, where each vertex of $V(G)$ appears exactly once as the end of an edge of F. Then there exists a normal partition $\mathcal{T}$ such that $F = \{e_T(v) | v \in V(G)\}$ if and only if F is a transversal of the cycles of G.*

Proof. Let $\mathcal{T}$ be a normal partition, the set of marked edges $\{e_T(v) | v \in V(G)\}$ is obviously a transversal of the cycles of G, since $\mathcal{T}$ is partitioned into trails. Conversely, assume that $F = \{e_v | v \in V(G)\}$ is a transversal of the cycles of G. Then the spanning subgraph $G - F$ is a set of paths $\{P_1, P_2, \ldots, P_k\}$ (some of them being eventually reduced to a vertex). Let u_i and v_i be the end vertices of P_i $(1 \leq i \leq k)$ (when P_i is reduced to a single vertex, we have $u_i = v_i$). We add to each path P_i the edges of F which are incident to u_i and v_i and distinct from e_{u_i} and e_{v_i}. We get thus a set of trails $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ which partition the edge set. We claim that $\mathcal{T}$ is a normal partition. Indeed, let v be any vertex of G. v is contained in some path P_i of $G - F$ and T_i must contain the two edges incident to v and distinct from the unique edge associated to v in F. Hence v must be an internal vertex of T_i which implies that v is normal. ■

3. On Compatible Normal Partitions

Definition 3.1. Two partitions $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ and $\mathcal{T}' = \{T'_1, T'_2, \ldots, T'_k\}$ of $E(G)$ into trails are *compatible* when $e_T(v) \neq e_{T'}(v)$ for every vertex $v \in V(G)$.

Theorem 3.2 below was previously stated in [2]:

Theorem 3.2. *Let G be a cubic graph having a perfect matching M. Then G has 2 compatible normal odd partitions of length 3.*

Proof. Let us give an orientation to the 2-factor of $G - M$. We get a normal partition $\mathcal{T}$ with all paths of length 3 when the edges of M are concatenated with the outgoing edges of the 2-factor (see Theorem 4.9. If we change the orientations on each cycle of the 2-factor we obtain a second normal partition $\mathcal{T}'$. These two partitions are easily seen compatible. ■

Compatible perfect path double covers

When we can find two compatible normal path partitions in a cubic graph we have, in fact a particular $PPDC$ of its edge set.

Definition 3.3. A *Perfect Path Double Cover* ($PPDC$ for short) is a collection $\mathcal{P}$ of paths such that each edge of G belongs to exactly two members of $\mathcal{P}$ and each vertex occurs exactly twice as an end path of $\mathcal{P}$.

This notion has been introduced by Bondy (see [1]) who conjectured that every simple graph admits a $PPDC$. This conjecture was proved by Li [9]. When dealing with two compatible normal path partitions $\mathcal{P}$ and $\mathcal{P}'$ in a cubic graph, we have a particular $PPDC$. Indeed every edge belongs to exactly one path of $\mathcal{P}$ and one path of $\mathcal{P}'$ and every vertex occurs exactly once as an end vertex of a path in $\mathcal{P}$ and as an end vertex of a path in $\mathcal{P}'$. The qualifying adjective *compatible* says that the two end edges are distinct for each vertex.

As a refinement of the notion of $PPDC$ we can define a $CPPDC$ for a simple graph.

Definition 3.4. A *Compatible Perfect Path Double Cover* ($CPPDC$ for short) is a collection $\mathcal{P}$ of paths such that each edge of G belongs to exactly two members of $\mathcal{P}$ and each vertex occurs exactly twice as an end path of $\mathcal{P}$ and these two ends are distinct.

A natural question is thus to know which graphs admits a $CPPDC$. If we restrict ourself to connected graphs, we immediately can see that as soon as a graph as a pendant edge, a $CPPDC$ does not exist. We need thus to consider graphs with a certain connectivity condition. It can be proved that a simple minimal 2-edge connected graph admits a $CPPDC$.

And we propose as an open Problem.

Problem 3.5. Every 2-edge connected graph admits a $CPPDC$.

Remark 3.6. Assume that a connected graph G admits $CPPDC$. In doubling every edge e in e' and e'' (let G_2 the graph so obtained), this $CPPDC$ leads to an Euler tour of G_2. This Euler tour is compatible (in the sense given by Kotzig [8]) with the set of transitions defined by e' and e'' in each vertex.

4. On Three Compatible Normal Partitions

We shall say that G has 3 compatible normal partitions $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ whenever these partitions are pairwise compatibles.

Theorem 4.1. *A cubic graph G has three compatible normal partitions if and only if G has no loop.*

Proof. Let G be a cubic graph with three compatible normal partitions $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$. Assume that G contains a loop vv, let $w \neq v$ be the vertex adjacent to v then one of these normal partitions, say $\mathcal{T}$, would be such that $e_{\mathcal{T}}(v) = vw$. In that case vv would be the trail containing v as an internal vertex, impossible.

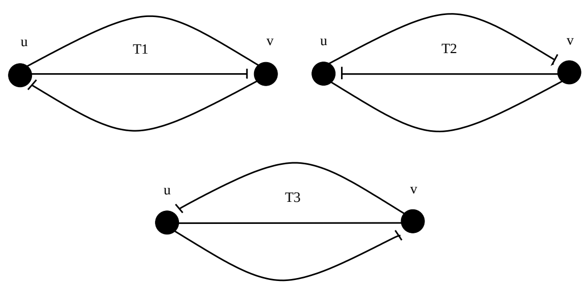


Figure 3. Cubic graph on 2 vertices with 3 compatible normal partitions.

Conversely, assume that G has no loop and G can not be provided with 3 compatible normal partitions. We can suppose that G has been chosen with the minimum number of vertices for that property. Figure 3 shows that G has certainly at least 4 vertices.

Claim 1. If u and v are joined by two edges e_1 and e_2, then there is a third vertex w adjacent to u and v.

Proof. Assume that u is adjacent to u' and v to v' with $u' \neq u$ and $v' \neq v$. Let G' be the cubic graph obtained from G in deleting u and v and joining u' and v' by a new edge. G' is obviously a cubic graph with no loop and $|V(G)| < |V(G')|$. We can thus find 3 compatible normal partitions $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$in G'.

The edge $u'v'$ of G' is contained into $T \in \mathcal{T}$, $T' \in \mathcal{T}'$ and $T'' \in \mathcal{T}''$. For convenience, T_1 and T_2 will the subtrails of T we have obtained in deleting $u'v'$, with u' an end of T_1 and v' an end of T_2. Following the same trick we get T'_1 and T'_2, T''_1 and T''_2 when considering T' and T''. It can be noticed that some of these subtrails may have length 0, which means that, following the cases, uv is the marked edge associated to u or (and) v in $\mathcal{T}$, $\mathcal{T}'$ or $\mathcal{T}''$.

Let $P_1 = T_1 + u'u$, $P_2 = T_2 + v'v e_1 u e_2 v$ and $\mathcal{Q} = \mathcal{T} - P + \{P_1, P_2\}$. We can easily check that $\mathcal{Q}$ is a normal partition of G where $e_{\mathcal{Q}}(x) = e_{\mathcal{T}}(x)\ \forall x \neq u, v$ and $e_{\mathcal{Q}}(u) = uu'$, $e_{\mathcal{Q}}(v) = e_2$.

In the same way, let $P'_1 = T'_1 + u'ue_2ve_1u$, $P'_2 = T'_2 + v'v$ and $\mathcal{Q}' = \mathcal{T}' - P' + \{P'_1, P'_2\}$. Then $e_{\mathcal{Q}'}(x) = e_{\mathcal{T}'}(x)\ \forall x \neq u, v$ and $e_{\mathcal{Q}'}(u) = e'_1$, $e_{\mathcal{Q}'}(v) = vv'$. Hence $\mathcal{Q}'$ is a normal partition compatible with $\mathcal{Q}$.

Finally, let $P''_1 = T''_1 + u'ue_1v$, $P''_2 = T'_2 + v've_2u$ and $\mathcal{Q}'' = \mathcal{T}'' - P'' + \{P''_1, P''_2\}$. Then $e_{\mathcal{Q}''}(x) = e_{\mathcal{T}''}(x)\ \forall x \neq u, v$ and $e_{\mathcal{Q}''}(u) = e'_2$, $e_{\mathcal{Q}''}(v) = e'_1$. Hence $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$ are 3 compatible normal partitions of G, a contradiction. ■

of T, T' and T'' obtained in deleting $u'u''$ (with u' an end of trails with subscript 1 and u'' an end of trails with subscript 2). If $R \in \mathcal{T}$, $R' \in \mathcal{T}'$ and $R'' \in \mathcal{T}''$ are the trails using $v'v''$, we can define also $R_1, R_2, R'_1, R'_2, R''_1$ and R''_2.

We are going to construct 3 normal partition $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$ of G in transforming locally $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ in such a way that $e_{\mathcal{Q}}(x) = e_{\mathcal{T}}(x)$ $e_{\mathcal{Q}'}(x) = e_{\mathcal{T}'}(x)$ and $e_{\mathcal{Q}''}(x) = e_{\mathcal{T}''}(x)$ $\forall x \neq u, v$. The verification of this point, left to the reader, is immediate.

Let $P''_1 = T''_1 + u'uu'' + T''_2$, $P''_2 = R''_1 + v'vv'' + R''_2$ and $P''_3 = uv$. $\mathcal{Q}''$ is then $\mathcal{T}'' - \{P'', R''\} + \{P''_1, P''_2, P''_3\}$. We can remark that we have subdivided P'' and R'' an we have add a trail of length one (uv). We have hence, $e_{\mathcal{Q}''}(u) = uv$ and $e_{\mathcal{Q}''}(v) = uv$.

It must be clear that we may have $T = R$ in $\mathcal{T}$, which means that $u'u''$ and $v'v''$ are contained in the same trail of $\mathcal{T}$. But we certainly have either $T_1 \neq R_1$ or $T_1 \neq R_2$ since R_1 and R_2 are two disjoint trails. Let us consider the following partitions of the edge set of G:

$$\mathcal{Q}_1 = \mathcal{T} - \{T_1, T_2\} + \{T_1 + u'uvv' + R_1, T_2 + u''u, R_2 + v''v\},$$
$$\mathcal{Q}_2 = \mathcal{T} - \{T_1, T_2\} + \{T_1 + u'uvv'' + R_2, T_2 + u''u, R_2 + v'v\},$$
$$\mathcal{Q}_3 = \mathcal{T} - \{T_1, T_2\} + \{T_1 + u'u, R_1 + v'vuu'' + T_2, R_2 + v''v\},$$
$$\mathcal{Q}_4 = \mathcal{T} - \{T_1, T_2\} + \{T_1 + u'u, R_1 + v'v, T_2 + u''uvv'' + R_2\}.$$

$\mathcal{Q}_1$ is a normal partition of G as soon as $T_1 \neq R_1$ and we can check, in that case, that $\mathcal{Q}_2$, $\mathcal{Q}_3$ and $\mathcal{Q}_4$ are normal partitions of G. In the same way, $\mathcal{Q}_2$ is a normal partition of G as soon as $T_1 \neq R_2$ and we can check, in that case, that $\mathcal{Q}_1$, $\mathcal{Q}_3$ and $\mathcal{Q}_4$ are normal partitions of G. $\mathcal{Q}_3$ is a normal partition of G as soon as $T_2 \neq R_1$ and, in that case, $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_4$ are normal partitions of G. $\mathcal{Q}_4$ is a normal partition of G as soon as $T_2 \neq R_2$ and, in that case, $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are normal partitions of G.

We can define analogously $\mathcal{Q}'_1$, $\mathcal{Q}'_2$, $\mathcal{Q}'_3$ and $\mathcal{Q}'_4$ when considering $\mathcal{T}'$.

We can check moreover that these normal partitions (when they are well defined) $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, $\mathcal{Q}_4$, $\mathcal{Q}'_1$, $\mathcal{Q}'_2$, $\mathcal{Q}'_3$ and $\mathcal{Q}'_4$ are compatible with $\mathcal{Q}''$ since

$$e_{\mathcal{Q}_i}(u) = uu' \text{ or } e_{\mathcal{Q}_i}(u) = uu'' \ i = 1, 2, 3, 4,$$
$$e_{\mathcal{Q}_i}(v) = vv' \text{ or } e_{\mathcal{Q}_i}(v) = vv'' \ i = 1, 2, 3, 4,$$
$$e_{\mathcal{Q}'_i}(u) = uu' \text{ or } e_{\mathcal{Q}'_i}(u) = uu'' \ i = 1, 2, 3, 4,$$
$$e_{\mathcal{Q}'_i}(v) = vv' \text{ or } e_{\mathcal{Q}'_i}(v) = vv'' \ i = 1, 2, 3, 4.$$

We can verify that in each case to be considered with $\mathcal{T}$ ($T_1 = R_1$ and $T_2 \neq R_2$, $T_2 = R_2$ and $T_1 \neq R_1$, $T_1 = R_2$ and $T_2 \neq R_1$, $T_2 = R_1$ and $T_1 \neq R_2$, T_1, T_2, R_1, R_2 all distinct) together with the similar cases for $\mathcal{T}'$ we can choose a normal partition $\mathcal{Q}$ in $\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4\}$ and a normal partition $\mathcal{Q}'$ in $\{\mathcal{Q}'_1, \mathcal{Q}'_2, \mathcal{Q}'_3, \mathcal{Q}'_4\}$ which are compatible and hence 3 normal partitions compatible $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$ for G, a contradiction. ■

Assume that u and v are joined by two edges in G, then, from Claim 1, there is unique new vertex w joined to u and v. This vertex is adjacent to $x \neq u, v$ which have itself a neighbor $z \neq u, v$. Le z' and z'' be the neighbors of z distinct from x. Then from Claim 2 $z' = z''$. But, in that case, z is joined to z' by two edges and the remaining neighbors of z and z' are distinct, a contradiction with Claim 1. Hence, we can assume that G has no multiple edge, but, in that case, every edge contradicts Claim 2, impossible. Hence G does not exist and the proof is complete. ■

Proposition 4.2. *Let G be a cubic graph having 3 compatible normal partitions. Then every edge $e \in E(G)$ verifies exactly one of the followings*

- *e is an internal edge in exactly one partition,*
- *e is an internal edge in exactly two partitions.*

Moreover, in the second case, the edge e itself is a trail of the third partition.

Proof. Let $e = xy$ be any edge of G and let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ three compatible normal partitions. If e is not an internal edge in $\mathcal{T}$, $\mathcal{T}'$ or $\mathcal{T}''$ then e is an end edge for a trail of $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$. In x or y we should have two partitions (say $\mathcal{T}$ and $\mathcal{T}'$) for which $e_{\mathcal{T}}(x) = e_{\mathcal{T}'}(x)$ ($e_{\mathcal{T}}(y) = e_{\mathcal{T}'}(y)$ respectively), a contradiction. If e is an internal edge in $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$. Let a and b the two other neighbors of x. We should have then

- $e_{\mathcal{T}}(x) = xa$ or xb,
- $e_{\mathcal{T}'}(x) = xa$ or xb,
- $e_{\mathcal{T}''}(x) = xa$ or xb,

which is impossible since the three partitions are compatible. Assume now that e is an internal edge of a trail in $\mathcal{T}$ and in $\mathcal{T}'$ and let a and b the two other neighbors of x. Up to the names of vertices we have

- $e_{\mathcal{T}}(x) = xa$,
- $e_{\mathcal{T}'}(x) = xb$.

From the third partition $\mathcal{T}''$, we must have $e_{\mathcal{T}''}(x) = xy$. In the same way we should obtain $e_{\mathcal{T}''}(y) = yx$. Hence the trail containing $e = xy$ is reduced to e, as claimed. ■

It can be noticed that whenever a cubic graph can be provided with 3 compatible normal partitions at least one edge is the internal edge in exactly one partition.

Proposition 4.3. *Let G be a cubic graph having 3 compatible normal partitions. Then at least one edge $e \in E(G)$ is the internal edge in exactly one partition.*

Proof. Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be three compatible normal partitions of G. The set of trails of length 1 in $\mathcal{T}$ is a matching of G which means that $\mathcal{T}$ has at most $\frac{n}{2}$ such trails. If each edge of G is the internal edge in exactly two partitions we must have

$$|E(G)| = n^1_{\mathcal{T}} + n^1_{\mathcal{T}'} + n^1_{\mathcal{T}''} \leq 3\frac{n}{2} = |E(G)|.$$

Hence the set of edges which are trails of length 1 in $\mathcal{T}$ is a perfect matching M of G. In that case, the marked edges associated to $\mathcal{T}$ is precisely this set M, which is not transversal of the cycles of G, a contradiction with Theorem 2.9. ■

Theorem 4.4. *Let G be a 3-edge colourable cubic graph . Then G has three compatible normal partitions $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ such that*

- *$\mathcal{T}$ is odd,*
- *$\mathcal{T}'$ has length 3,*
- *$\mathcal{T}''$ has length 4.*

Proof. We shall prove first this result for simple graphs. In [6], it is proved that, given a 3-edge colouring of G with α, β and γ then there exists a strong matching intersecting every cycle belonging to the 2-factor induced by the two colours (α and β). Assume that $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ is such 2-factor ($G - \mathcal{C}$ is a perfect matching) and let $F = \{u_iv_i \in C_i | \ 1 \leq i \leq k)$(minimal for the inclusion) be a strong matching intersecting this 2-factor.

For each u_iv_i, x_i is the vertex in the neighborhood of u_i which is not one of its neighbor on C_i while y_i is defined similarly for v_i (note that x_i

and y_i may be vertices of C_i or not). Let T_i be the trail obtained from C_i in adding the edge u_ix_i and considering that this trail ends with v_iu_i (Note that u_i is an internal vertex of T_i).

Let $\mathcal{T}$ be the trail partition containing every trail T_i $(1 \leq i \leq k)$ and all the edges of the perfect matching $G - \mathcal{C}$ which are not in some T_i. We can check that $\mathcal{T}$ is a normal odd partition for which the followings hold

- $e_{\mathcal{T}}(u_i) = u_iv_i$,
- $e_{\mathcal{T}}(x_i) = x_iu_i$,
- $e_{\mathcal{T}}(v)$ is the edge of $G - \mathcal{C}$.

We construct now $\mathcal{T}'$ in giving an orientation to each cycle of $\mathcal{C}$. This orientation is such that the successor of u_i is v_i. For each vertex v, $o(v)$ denotes the successor of v in that orientation and $p(v)$ its predecessor. As in Theorem 2.8 we get hence a normal partition $\mathcal{T}'$ where each trail is a path of length 3. Moreover $e_{\mathcal{T}'}(v) = vp(v)$.

Before constructing $\mathcal{T}''$, we construct $\mathcal{T}'''$ in using the reverse orientation on each cycle of $\mathcal{C}$. This normal partition of length 3 is such that $e_{\mathcal{T}'''}(v) = vo(v)$.

For each vertex $v \neq u_i$ $1 \leq i \leq k$ we have $e_{\mathcal{T}}(v) \neq e_{\mathcal{T}'}(v) \neq e_{\mathcal{T}'''}(v)$.

For $v = u_i$ $1 \leq i \leq k$, we have $e_{\mathcal{T}}(u_i) = u_iv_i$, $e_{\mathcal{T}'}(u_i) = u_ip(u_i))$ (where $p(u_i) \neq v_i$) and $e_{\mathcal{T}'''}(u_i) = u_iv_i$. Since $e_{\mathcal{T}}(u_i) = e_{\mathcal{T}'''}(u_i)$, $\mathcal{T}$ and $\mathcal{T}'$ are not compatible.

Our goal now is to proceed to switchings on $\mathcal{T}'''$ in each vertex u_i in order to get $\mathcal{T}''$ where these incompatibilities are dropped. For this purpose, the path of length 3 of $\mathcal{T}'''$ ending with v_iu_i is augmented with the edge $u_ip(u_i)$. We get hence of path of length 4 and, since F is a strong matching, we are sure that we cannot extend this path in the other direction. The path of $\mathcal{T}'''$ ending with $u_ip(u_i)$ is shorten in deleting the edge $u_ip(u_i)$, we get hence of path of length 2 ending with x_iu_i, and we are sure that this path cannot be shorten at the other end, since F is a strong matching. Let $\mathcal{T}''$ be the partition so obtained. $\mathcal{T}'''$ being normal and $\mathcal{T}''$ having the same number of trails $\mathcal{T}''$ is also normal by Proposition 2.2.

For each vertex $v \neq u_i$ $1 \leq i \leq k$, $e_{\mathcal{T}'''}(v) = e_{\mathcal{T}''}(v)$ and we have thus $e_{\mathcal{T}}(v) \neq e_{\mathcal{T}'}(v) \neq e_{\mathcal{T}''}(v)$. For $v = u_i$ $1 \leq i \leq k$, we have $e_{\mathcal{T}}(u_i) = u_iv_i$, $e_{\mathcal{T}'}(u_i) = u_ip(u_i)$ and $e_{\mathcal{T}''}(u_i) = u_ix_i$).

$\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ are thus compatible, $\mathcal{T}$ is odd, $\mathcal{T}'$ has length 3 and $\mathcal{T}''$ has length 4 as claimed. ∎

Theorem 4.5. *Let G be a cubic graph then the followings are equivalent*

(i) *G can be provided with 3 compatible normal partitions of length 3,*

(ii) *G can be provided with 3 compatible normal odd partitions where each edge is an internal edge in exactly one partition,*

(iii) *G is bipartite.*

Proof. Assume first that G can be provided with three compatible normal partitions of length 3, say $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$. Since the mean length of each partition is 3 (Proposition 2.3), each trail of each partition has length exactly 3. $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ are thus three normal odd partitions and from Proposition 4.2, each edge is the internal edge of one trail in exactly one partition. Conversely assume that G can be provided with 3 compatible normal odd partitions where each edge is an internal edge in exactly one partition, then from Proposition 4.2 there is no trail of length 1 in any of these partitions. Since the mean length of each partition is 3, that means that each trail in each partition has length exactly 3. Hence (i) $\equiv$ (ii).

We prove now that (i) $\equiv$ (iii). Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ three compatible normal partitions of length 3. Following the proof of Theorem 2.8 the internal edges of trails of $\mathcal{T}$ ($\mathcal{T}'$ and $\mathcal{T}''$ respectively) constitute a perfect matching (say M M' and M'' respectively).

Let $a_0a_1a_2a_3$ be a trail of $\mathcal{T}$ and let b_1 and b_2 the third neighbors of a_1 and a_2 respectively. By definition, we have $e_{\mathcal{T}}(a_1) = a_1b_1$ and $e_{\mathcal{T}}(a_2) = a_2b_2$.

Since a_0a_1 and a_2a_3 must be internal edges in a trail of $\mathcal{T}'$ or (exclusively) $\mathcal{T}''$, assume w.l.o.g. that a_0a_1 is an internal edge of a trail T'_1 of $\mathcal{T}'$. T'_1 does not use a_1a_2 otherwise $e_{\mathcal{T}'}(a_1) = a_1b_1$, a contradiction with $e_{\mathcal{T}}(a_1) = a_1b_1$ since $\mathcal{T}$ and $\mathcal{T}'$ are compatible. Hence T'_1 uses a_1b_1 and $e_{\mathcal{T}'}(a_1) = a_1a_2$.

Assume now that a_2a_3 is an internal edge of a trail T'_2 of $\mathcal{T}'$. Reasoning in the same way, we get that $e_{\mathcal{T}'}(a_2) = a_2a_1$. These two results leads to the fact that a_1a_2 must be a trail in $\mathcal{T}'$, which is impossible since each trail has length exactly 3.

Hence, whenever a_0a_1 is supposed to be an internal edge in a trail of $\mathcal{T}'$, we must have a_2a_3 as an internal edge in a trail of $\mathcal{T}''$. The two internal vertices of $a_0a_1a_2a_3$ can be thus distinguished, following the fact that the end edge of $\mathcal{T}$ to whom they are incident is internal in $\mathcal{T}'$ (say *red* vertices) or $\mathcal{T}''$ (say *blue* vertices). The same holds for each trail in $\mathcal{T}$ (and incidently for each partition $\mathcal{T}'$ and $\mathcal{T}''$). The edge a_1b_1 as end-edge of $\mathcal{T}$ cannot be an internal edge in T' since the trail of length 3 going through a_0a_1 ends with a_1b_1. Hence a_1b_1 is an internal edge in $\mathcal{T}''$ and b_1 is a blue vertices. Considering now a_0, this vertex is the internal vertex of a trail of length 3 of $\mathcal{T}$. Since $a_0a_1 \in M'$ and M' is a perfect matching, a_0 cannot be incident to an other internal edge of a trail in $\mathcal{T}'$ and a_0 must be a blue vertex. Hence a_1 is a red vertex and its neighbors are all blue vertices. Since we can perform this reasoning in each vertex, G is bipartite as claimed.

Conversely, assume that G is bipartite and let $V(G) = \{W, B\}$ be the bipartition of its vertex set. In the following, a vertex in W will be represented by a circle ($\circ$) while a vertex in B will be represented by a bullet ($\bullet$). From König's Theorem [7] G is a 3-edge colourable cubic graph . Let us consider a coloring of its edge set with three colors $\{\alpha, \beta, \gamma\}$. Let us denote by $\alpha \bullet \beta \circ \gamma$ a trail of length 3 which is obtained in considering an edge uv ($u \in B$ and $v \in W$) colored with β together with the edge colored α incident with u and the edge colored with γ incident with v. It can be easily checked that the set $\mathcal{T}$ of $\alpha \bullet \beta \circ \gamma$ trails of length 3 is a normal odd partition of length 3. We can define in the same way $\mathcal{T}'$ as the set of $\beta \bullet \gamma \circ \alpha$ trails of length 3 and $\mathcal{T}''$ as the set of $\gamma \bullet \alpha \circ \beta$ trails of length 3.

Hence $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ is a set of three normal odd partitions of length 3. We claim that these partitions are compatible. Indeed, let $v \in W$ be a vertex and u_1, u_2 and u_3 its neighbors. Assume that u_1v is colored with α, u_2v is colored with β and u_3v is colored with γ. Hence u_1v is internal in an $\gamma \bullet \alpha \circ \beta$ trail of $\mathcal{T}''$ and $e_{\mathcal{T}''}(v) = vu_3$. The edge u_2v is internal in an $\alpha \bullet \beta \circ \gamma$ trail of $\mathcal{T}$ and $e_{\mathcal{T}}(v) = vu_1$. The edge u_3v is internal in an $\beta \bullet \gamma \circ \alpha$ trail of $\mathcal{T}'$ and $e_{\mathcal{T}'}(v) = vu_2$. Since the same reasoning can be performed in each vertex of G, the three $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ partitions are compatible. ∎

Theorem 4.6. *Let G be a cubic graph with three compatible normal partitions $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ such that*

- *$\mathcal{T}$ has length* 3,
- *$\mathcal{T}'$ and $\mathcal{T}''$ are odd.*

Then G is a 3-edge colourable cubic graph.

Proof. Since $\mathcal{T}$ has length 3, every trail of $\mathcal{T}$ has length 3. Hence there is no edge which can be an internal edge of a trail of $\mathcal{T}'$ and a trail of $\mathcal{T}''$, since, by Proposition 4.2 such an edge would be a trail of length 1 in $\mathcal{T}$. The perfect matchings associated to $\mathcal{T}'$ and $\mathcal{T}''$ (see Theorem 2.8) are thus disjoint and induce an even 2-factor of G, which means that G is a 3-edge colourable cubic graph, as claimed. ∎

Proposition 4.7. *Let G be a cubic graph which can be provided with 3 compatible normal odd partitions then G', the graph obtained in replacing a vertex by a triangle, can also be provided with 3 compatible normal odd partitions.*

Proof. Let u be a vertex of G and v_1, v_2, v_3 its neighbors (not necessarily distinct). Assume that $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$is a set of 3 compatible normal odd partitions of G such that, $e_{\mathcal{T}}(u) = uv_1$, $e_{\mathcal{T}'}(u) = uv_2$ and $e_{\mathcal{T}''}(u) = uv_3$. Let T_1 and T_2 the two trails of $\mathcal{T}$ such that u is an end of T_1 and an internal vertex of T_2. T_1^1 ending in v_1, T_1^2 ending in v_2 and T_2^2 ending in v_3 denote the subtrails of T_1 and T_2 obtained in deleting u. We define similarly $T_1'^1$ ending in v_2, $T_1'^2$ ending in v_1 and $T_2'^2$ ending in v_3 when considering T_1' and T_2' in $\mathcal{T}'$ as well as $T_1''^1$ ending in v_3, $T_1''^2$ ending in v_2 and $T_2''^2$ ending in v_1 when considering T_1'' and T_2'' in $\mathcal{T}''$.

When we transform G in G' the vertex u is deleted and replaced by the triangle u_1, u_2, u_3 with u_i joined to v_i $(i = 1, 2, 3)$.

Let $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$ be defined in G' by

$$\mathcal{Q} = \mathcal{T} - \{T_1, T_2\} + \{T_1^1 + v_1u_1, T_1^2 + v_2u_2u_1u_3v_3 + T_2^2, u_2u_3\},$$

$$\mathcal{Q}' = \mathcal{T}' - \{T_1', T_2'\} + \{T_1'^1 + v_2u_2, T_1'^2 + v_1u_1u_2u_3v_3 + T_2'^2, u_1u_3\},$$

$$\mathcal{Q}'' = \mathcal{T}'' - \{T_1'', T_2''\} + \{T_1''^1 + v_3u_3, T_1''^2 + v_2u_2u_1u_3v_3 + T_2''^2, u_2u_1\}.$$

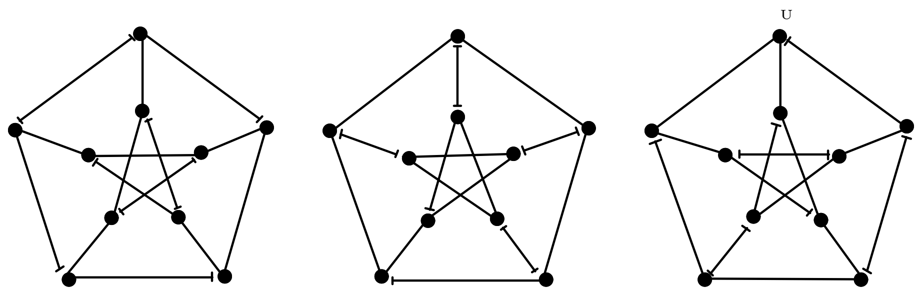
It is a routine matter to check that $\mathcal{Q}, \mathcal{Q}'$ and $\mathcal{Q}''$ are 3 compatible normal odd partitions. ■

It can be pointed out that cubic graphs with with 3 compatible normal odd partitions are bridgeless.

Proposition 4.8. *Let G be a cubic graph with 3 compatible normal odd partitions. Then G is bridgeless.*

Proof. Assume that xy is a bridge of G and let C be the connected component of $G - xy$ containing x. Since G has 3 compatible normal odd partitions, one of these partitions, say $\mathcal{T}$, is such that $e_{\mathcal{T}}(x) = xy$. The edges of C are thus partitioned into odd trails (namely the trace of $\mathcal{T}$ on C). We have

$$m = |E(C)| = \frac{3(|C| - 1) + 2}{2}$$



(a) (b) (c)

Figure 4. Normal odd partitions of the Petersen's graph.

and m is even whenever $|C| \equiv 3 \mod 4$ while m is odd whenever $|C| \equiv 1 \mod 4$. The trace of $\mathcal{T}$ on C is a set of $\frac{|C|-1}{2}$ trails and this number is odd when $|C| \equiv 3 \mod 4$ and even otherwise. Hence, when $|C| \equiv 3 \mod 4$ we must have an odd number of odd trails partitioning $E(C)$ but, in that case m is even and when $|C| \equiv 1 \mod 4$ we must have an even number of odd trails partitioning $E(C)$ but, in that case m is odd, contradiction. ∎

Fan and Raspaud [3] conjectured that any bridgeless cubic graph can be provided with three perfect matching with empty intersection.

Theorem 4.9. *Let G be a cubic graph with 3 compatible normal odd partitions. Then there exist 3 perfect matching M, M' and M'' such that $M \cap M' \cap M'' = \emptyset$.*

Proof. Following the proof of Theorem 2.8 the odd edges of trails of $\mathcal{T}$ ($\mathcal{T}'$ and $\mathcal{T}''$ respectively) constitute a perfect matching (say M M' and M'' respectively). Let v be any vertex and u_1, u_2 and u_3 its neighbors. $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ being compatible, we can suppose that $e_{\mathcal{T}}(v) = vu_1$, $e_{\mathcal{T}'}(v) = vu_2$ and $e_{\mathcal{T}''}(v) = vu_3$. vu_1 is an end edge of a trail in $\mathcal{T}$, this edge is not an odd edge in $\mathcal{T}$ and thus $vu_1 \notin M'$. In the same way $vu_2 \notin M'$ and $vu_3 \notin M''$. Hence, any edge incident to v is contained in at most two perfect matchings among M, M' and M''. Which means that $M \cap M' \cap M'' = \emptyset$. ∎

Theorem 4.9 above implies that the Fan Raspaud Conjecture is true for graphs with 3 compatible normal odd partitions. By the way, this conjecture seems to be originated independently by Jackson. Goddyn [4] indeed mentioned this problem proposed by Jackson for r-graphs (r-regular graphs with an even number of vertices such that all odd cuts have size at least r, as defined by Seymour [10]) in the proceedings of a joint summer research conference on graphs minors which dates back 1991. It seems difficult to characterize the class of cubic graphs with 3 compatible normal odd partitions. The Petersen's graph has this property (see Figure 4). In a forthcoming paper we prove that 3-edge colorable graphs also have this property as well as the *flower snarks*.

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