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KERNELS AND CYCLES' SUBDIVISIONS IN ARC-COLORED TOURNAMENTS

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Abstract

Let D be a digraph. D is said to be an m-colored digraph if the arcs of D are colored with m colors. A path P in D is called monochromatic if all of its arcs are colored alike. Let D be an m-colored digraph. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of D if it satisfies the following conditions: a) for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them; and b) for every vertex $x \in V(D) - N$ there is a vertex $n \in N$ such that there is an xn-monochromatic directed path in D.

In this paper we prove that if T is an arc-colored tournament which does not contain certain subdivisions of cycles then it possesses a kernel by monochromatic paths. These results generalize a well known sufficient condition for the existence of a kernel by monochromatic paths obtained by Shen Minggang in 1988 and another one obtained by Hahn *et al.* in 2004. Some open problems are proposed.

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1. General Concepts and Notation

Let D be a *digraph*. $V(D)$ and $A(D)$ will denote the set of vertices and arcs of D. A subdigraph H of D is a *spanning subdigraph* if $V(H) = V(D)$; if $S \subseteq V(D)$ is non empty then the *subdigraph* $D[S]$ *induced* by S is that digraph having vertex set S and whose arc set consists of all those arcs of D joining vertices of S. An arc $z_1z_2 \in A(D)$ is called *asymmetrical* (*symmetrical*) if $z_2z_1 \notin A(D)$ ($z_2z_1 \in A(D)$); the *asymmetrical part of* D (the *symmetrical part of* D) denoted by $Asym(D)$ ($Sym(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (symmetrical) arcs of D; D is called an *asymmetrical digraph* if $Asym(D) = D$. A digraph is called *semicomplete* if for every two distinct vertices u and v of D, at least one of the arcs (u, v) or (v, u) is present in D. A semicomplete asymmetrical digraph is called a *tournament*.

An arc $z_1z_2 \in A(D)$ will be called an S_1S_2-*arc* whenever $z_1 \in S_1 \subseteq V(D)$ and $z_2 \in S_2 \subseteq V(D)$. By $[z_1, z_2]_T$ we denote one of the two possible arcs between z_1 and z_2. For a directed walk W we will denote its length by $\ell(W)$. And if z_1, $z_2 \in V(W)$ then we denote by (z_1, W, z_2) the z_1z_2- directed walk contained in W. We will denote by C_n a *directed cycle* with length n and by T_3 the transitive tournament on 3 vertices. We call a *triangle* a transitive tournament of order 3 or a cycle of length 3. Let $C = (0, 1, \ldots, m, 0)$ be a directed cycle of D, a *pseudodiagonal* of C is an arc $f = (i, j) \in A(D) - A(C)$ such that $i \neq j$, $\{i, j\} \subseteq V(C)$ and $\ell(i, C, j) \leq \ell(C) - 1$. A *pole* of the cycle C is the terminal vertex y of a pseudodiagonal (x, y) of C. Throughout the paper all the paths and cycles considered are directed paths and directed cycles.

A set $S \subseteq V(D)$ is *independent* if $A(D[S]) = \emptyset$. For general concepts on digraphs we refer the reader to [1].

2. Introduction

2.1. Kernels

The concept of a kernel was first presented in [23] (under the name *solution*) in the context of Game Theory by Von Neumann and Morgenstern as an interesting solution for cooperative n-person games with general n. In [1] Berge presented it as follows: Suppose that n players wish to select a point x from a set X of *situations*. As the individual preferences might not be compatible, then it is necessary to introduce the *effective preference*: the situation a is effectively preferred to b if there is a set of players who prefer a to b and who are capable of enforcing their preference. Consider the digraph D with X as a set of vertices and such that $(b, a) \in F(D)$ if the situation a is effectively preferred to situation b. Von Neumann and Morgenstern suggested that the situations that are preferred must be the elements of a distinguished set S which is an independent (stable) set and such that for every situation y not in S there is a situation w in S which is effectively preferred to y. They called such a set S a *solution*.

The (solution) set defined in this problem is now well known as a kernel of a digraph. More formally, a *kernel* N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D. As the reader can see, not every digraph has a kernel and when a digraph contains a kernel, it may not be the only one. This simple observation compels us to ask for sufficient conditions for the existence of a kernel in a digraph. It is well known that if D is finite, the decision problem of the existence of a kernel in D is NP-complete for a general digraph (see [5] and [27]) and for a planar digraph with indegrees less than or equal to 2, outdegrees less than or equal to 2 and degrees less than or equal to 3 (see [8]). For any tighter constraints the problem is solvable in linear time.

A digraph D such that every induced subdigraph in D has a kernel is called a *kernel-perfect* digraph (or simply, a KP-*digraph*). The following sufficient conditions for a digraph to be a KP-digraph are known:

Theorem 1. *D is a kernel-perfect digraph if one of the following conditions holds:*

(i) *D has no cycles of odd length.*

(ii) *Every directed cycle of odd length in D has at least two symmetric arcs.*

(iii) *$Asym(D)$ is acyclic.*

(iv) *Every directed cycle of odd length in D has at least two consecutive poles.*

(v) *Every directed cycle in D has at least one symmetrical arc.*

These claims were proved respectively by Richardson [26], Duchet [6], Duchet and Meyniel [7], Galeana-Sánchez and Neumann-Lara [12], and by Berge and Duchet [2]. There are many applications of this concept in the context of game theory, logic and decision theory (see [1]), as well as several interesting related results (see also [2, 12] and [13, 28, 13, 3]). A selected bibliography can be found in [9], and we also recommend the survey [4].

2.2. Arc colored digraphs and antecedents

Now, let us consider a more realistic variation of the problem first presented by Von Neumann and Morgenstern. If each player can make their own choice then a natural model of this new problem will be an arc colored digraph in which colors and persons will be related by a bijective function. This simple modification to the initial problem brings an interesting generalization of the concept of kernel. In order to present it in a more formal way together with some previous results, first let us introduce some notation.

D is an *m-colored digraph* if the arcs of D are colored with m colors. Let D be an m-colored digraph. A directed path (or cycle) is called *monochromatic* if all of its arcs are colored alike and it is called *quasimonochromatic* if with at most one exception all of its arcs are colored alike. A subdigraph H of D is called a *k-colored digraph* if all of its arcs are colored with only k colors, in particular for $k = 2$ we say that H is *bicolor*. We will say that a subdigraph H of D is an *at most k-colored digraph* if all of its arcs are colored with at most k colors, in particular for $k = 2$ we say that H is at most bicolor; H will be called a *polychromatic digraph* if all of its arcs are colored with at least 3 colors.

A set $N \subseteq V(D)$ is said to be a *kernel by monochromatic paths* of D if it satisfies the following conditions: a) N is an *independent set by monochromatic paths*: for every pair of different vertices u and v in N there is no monochromatic path between them in D; and b) N is an *absorbing set by monochromatic paths*: for every vertex $x \in V(D) - N$ there is a vertex $n \in N$ such that there is an xn-monochromatic path in D. The concept of kernel by monochromatic paths is a generalization of the concept of kernel. Several results arise around this concept (see [15, 16, 17] and [30]) but the following contributions are the foundations of this work.

In [24], Sands *et al.* have proved that any 2-colored digraph has a kernel by monochromatic paths, in particular they proved that every 2-colored tournament T has a vertex v such that for any $x \in V(T) - \{v\}$ there is a monochromatic path from x to v (i.e., $\{v\}$ is a kernel by monochromatic paths of T). They also raised the following problem: Let T be an m-colored tournament such that every cycle of length 3 is a quasimonochromatic cycle. Must T have a kernel by monochromatic paths? This question is already answered for $m \geq 5$ by Shen Minggang in [25] where he proved that if T is an m-colored tournament such that every triangle (that is, a transitive tournament of order 3 or a cycle of length 3) is a quasimonochromatic subdigraph of T, then T has a kernel by monochromatic paths (**Theorem A**). He also proved that this hypothesis is tight for $m \geq 5$.

Later, in [11] Galeana-Sánchez proved that if every cycle of length at most 4 in an m-colored tournament is a quasimonochromatic cycle, then every induced subdigraph of T has a kernel by monochromatic paths (**Theorem B**).

Finally in [22] Hahn *et al.* proved the following generalization of the previously cited results of Shen Minggang and Galeana-Sánchez: For a tournament T, if every triangle is quasimonochromatic or for some $s \geq 4$ each cycle of length s is a quasimonochromatic cycle and no cycle of length less than s is a polychromatic cycle (colored with at least three colors), then T admits an absorbing vertex (**Theorem C**).

2.3. Description of results

In this paper we obtain two sufficient conditions for the existence of a kernel by monochromatic paths not only in an m-colored tournament, but in every induced subdigraph of T. The first one is a generalization of **Theorem C** (see [22]) and the second one is a generalization of **Theorem A**, Shen Minggang's result (see [25]). We also prove that our conditions are not implied by those known previously.

The spirit of our proofs arises from structural properties of arc colored tournaments (see Lemma 1) and these properties are deduced by working with previous results on kernels (see Theorem 1-v) on an new digraph associated with our original tournament, its closure.

Definition 1. For an m-colored digraph D, the *closure* of D, denoted by $C(D)$, is the multidigraph such that:

$$V(C(D)) = V(D),$$
$$A(C(D)) = A(D) \cup \{uv \mid \text{there is an } uv\text{-monochromatic path in } D\}.$$

Notice that by definition of $C(D)$ it holds that $N \subseteq V(D)$ is a kernel by monochromatic paths of D if and only if $N \subseteq V(C(D))$ is a kernel of $C(D)$. With this we can see that the closure of a digraph D relates in a very natural way kernels by monochromatic paths in this digraph with kernels in its closure. Now, notice that if certain properties which imply that D has a kernel also hold in the closure of D, then we can assert that $C(D)$ has a kernel and hence D has a kernel by monochromatic paths (by the definition of $C(D)$). In particular, if the closure of a digraph D satisfies some of the sufficient conditions in Theorem 1, then we get as an immediate application of this theorem that D has a kernel by monochromatic paths. This is an important point to mention because this is not the case with the following results: the sufficient conditions stated in our results hold in tournaments and not in its closure.

3. A Preliminary Lemma

The following Lemma gives us structural properties (i.e., existence and color properties) of certain subdigraphs of an arc colored tournament whose closure is not a KP-digraph and such that every C_3 is a quasimonochromatic cycle.

Lemma 1. *Let T be an m-colored tournament. If every $C_3 \subseteq T$ is a quasi-monochromatic cycle and $C(T)$ is not a* KP*-digraph then there exists a cycle $\gamma = (z_0, z_1, z_2 = 0, 1, 2, \ldots, p = z_0) \subseteq C(T)$ such that the following properties hold:*

(a) $\ell(\gamma) \geq 4$,

(b) $\gamma \subseteq T$,

(c) $(z_0, z_1) \in A(T)$ *with color* a, $(z_1, z_2) \in A(T)$ *with color* b *and there exists a* z_2z_0*-path* $\alpha = (z_2 = 0, 1, 2, \ldots, p = z_0)$ $(p \geq 2)$ *with color* c, $a \neq b$, $b \neq c$, $a \neq c$, *let* a*=red*, b*=blue*, c*=black,*

(d) $(z_2, z_0) \notin A(T)$ (*so* $(z_0, z_2) \in A(T)$),

(e) *There is no* z_1z_0*-monochromatic path in* T *and there is no* z_2z_1*-monochromatic path in* T,

(f) *Every arc between* z_1 *and an internal vertex in* α *is not black.*

Proof. Proceeding by contradiction, let us suppose that $C(T)$ is not a KP-digraph so a well known theorem by Berge and Duchet (see Theorem 1-v) asserts that there is a cycle $\Gamma \subseteq Asym(C(T))$. Let $\Gamma = (z_0, z_1, \ldots, z_{n-1}, z_n = z_0) \subseteq Asym(C(T))$ be a cycle with minimal length contained in $Asym(C(T))$. Through the following claims we will discover interesting color properties of this cycle and they will allow us to prove the lemma.

Claim 1. $\ell(\Gamma) = n \geq 3$.
Recall $\Gamma \subseteq Asym(C(T))$, so $\ell(\Gamma) = n \neq 2$.

Claim 2. $\Gamma \subseteq T$.
Suppose that there is an arc $(z_i, z_{i+1}) \in \Gamma - T$. Since T is a tournament we have that $(z_{i+1}, z_i) \in T$ and so $\{(z_i, z_{i+1}, (z_{i+1}, z_i)\} \subseteq Asym(C(T))$, a contradiction.

Claim 3. $(z_0, z_1) \in A(T)$ has color a, $(z_1, z_2) \in A(T)$ has color b, $a \neq b$.
Since Γ is not a monochromatic cycle (by the contrary: $(z_0, \Gamma, z_{n-1}) \subseteq Asym(C(T))$ is a monochromatic path, thus $(z_0, z_{n-1}) \in A(C(T))$ and hence $(z_{n-1}, z_0) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction), then there exist two consecutive arcs in Γ colored differently. Say $(z_0, z_1) \in A(\Gamma)$ is red and $(z_1, z_2) \in A(\Gamma)$ is blue.

Claim 4. For any $\{z_i, z_j\} \subset V(\Gamma)$ such that $j \notin \{i-1, i+1\}$ it holds that $\{(z_i, z_j), (z_j, z_i)\} \subseteq A(C(T))$. Let $\{z_i, z_j\} \subset V(\Gamma)$ be such that $j \notin \{i-1, i+1\}$. Since T is a tournament, $(z_i, z_j) \in A(T)$ or $(z_j, z_i) \in A(T)$, without loss of generality let $(z_i, z_j) \in A(T)$. Then $\Gamma' = (z_i, z_j, z_{j+1}, z_{j+2}, \ldots, z_{i-1}, z_i) \subseteq T$ is a cycle with $\ell(\Gamma') < \ell(\Gamma)$. Hence $\Gamma' \nsubseteq Asym(C(T))$ and so $(z_i, z_j) \in A(Sym(C(T))$.

Claim 5. $(z_2, z_0) \notin A(T)$.
If $(z_2, z_0) \in A(T)$ then there exists $C_3 = (z_0, z_1, z_2, z_0) \subseteq T$ and it is a quasimonochromatic cycle by hypothesis, so $(z_2, z_0) \in A(T)$ is red or blue. If $(z_2, z_0) \in A(T)$ is red then $(z_2, z_0, z_1) \subseteq T$ is a z_2z_1-monochromatic path and $(z_1, z_2) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction. If $(z_2, z_0) \in A(T)$ is blue, then $(z_1, z_2, z_0) \subseteq T$ is a z_1z_0-monochromatic path and $(z_0, z_1) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction again.

Now, by claims (2.3.) and (2.3.) there exist a z_2z_0-monochromatic path in T with length at least 2. Let $\alpha = (z_2 = 0, 1, 2, \ldots, p = z_0) \subseteq T$ be a z_2z_0-monochromatic path with minimal length ($p \geq 2$).

Claim 6. $z_1 \notin V(\alpha)$.
Otherwise $z_1 \in V(\alpha)$ and then (z_2, α, z_1) is a z_2z_1-monochromatic path in T so $(z_1, z_2) \in Sym(C(T))$, contradiction.

Claim 7. α is neither red nor blue.
If α is red then $\alpha \cup (z_0, z_1)$ is a z_2z_1-monochromatic path in T and $(z_2, z_1) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction. If α is blue then $(z_1, z_2) \cup \alpha \subseteq T$ is a z_1z_0-monochromatic path in T and $(z_1, z_0) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction again. Let α be black.

Consider $\gamma = (z_0, z_1, z_2) \cup \alpha$. Clearly γ satisfies the first four properties of our Lemma 1. Let us conclude with the following points.

Claim 8. There is no z_1z_0-monochromatic path in T and there is no z_2z_1-monochromatic path in T.
Notice that $\{(z_0, z_1), (z_1, z_2)\} \subseteq Asym(C(T))$.

Claim 9. Every arc between z_1 and an internal vertex in α is not black.
If there exists i, $1 \leq i \leq p - 1$ such that $(i, z_1) \in A(T)$ resp. $(z_1, i) \in A(T))$ is black then $(z_2 = 0, \alpha, i) \cup (i, z_1) \subseteq T$ (resp. $(z_1, i) \cup (i, \alpha, z_0))$ is a z_2z_1-monochromatic path in T (resp. is a z_1z_0-monochromatic path), a contradiction. ■

4. The Main Results

In order to present our main results we must introduce certain subdigraphs whose arc coloration in an arc colored digraph D will allow us to assert the existence of a kernel by monochromatic paths in D and in every induced subdigraph of this digraph.

4.1. Condition I

Definition 2. A (2, k-2)-*subdivision* of C_2-bicolor is defined to be a cycle of length k containing a monochromatic path of length k-2 and a monochromatic path of length 2.

In particular for $k = 3$ we get a bicolor C_3. The importance of this subdigraph in our condition is that its coloration is less restrictive than the quasimonochromatic coloration of C_s-cycles in Theorem C. We will use these subdigraphs in order to prove a more general sufficient condition than the condition in Theorem C.

Definition 3. Let T be an m-colored tournament. T has property PI_k if the following conditions hold for some fixed integer $k \geq 4$:

(a) Every $C_k \subseteq T$ is at most bicolor and is not a (2,k-2)-subdivision of C_2-bicolor, and

(b) every $C_t \subseteq T$ ($t < k$) is at most bicolor (it is not polychromatic).

Theorem 2. *Let T be an m-colored tournament. If T satisfies property PI_k for some integer $k \geq 4$ then $C(T)$ is a* KP-*digraph.*

Proof. To prove the theorem we proceed by contradiction. Suppose that $C(T)$ is not a KP-digraph, then by Lemma 1 there exists a cycle $\gamma = (z_0, z_1, z_2 = 0, 1, 2, \ldots, p = z_0)$ satisfying properties (a) to (f). The following assertions will allow us to obtain a contradiction.

Claim 1. $p > k - 2$.
From property (c) of Lemma 1 we have that γ is a 3-colored cycle. Then the assertion holds from the hypothesis of Theorem 2.

Claim 2. For each i with $p-(k-2) \geq i \geq 0$ we have: If $(z_1, i) \in A(T)$ then for every j with $p > i + j(k-2) \geq i$, it holds that $(z_1, i + j(k-2)) \in A(T)$.
Let i with $p - (k-2) \geq i \geq 0$ and assume that $(z_1, i) \in A(T)$. If there is some j, $p > i + j(k-2) \geq i$, such that $(z_1, i + j(k-2)) \notin A(T)$ then let $j_0 = \min\{j \mid p > i + j(k-2) \geq i \text{ and } (z_1, i + j(k-2)) \notin A(T)\}$. Since T is a tournament we have $(i + j_0(k-2), z_1) \in A(T)$ and it follows from the choice of j_0 that $(z_1, i + j_0(k-2) - (k-2)) \in A(T)$ (notice that $i + j_0(k-2) - (k-2) \geq i$ as $(z_1, i) \in A(T)$) then there exists $C_k = (z_1, i + j_0(k-2)-(k-2)) \cup (i+j_0(k-2)-(k-2), \alpha, i+j_0(k-2)) \cup (i+j_0(k-2), z_1) \subseteq T$ and it is an at most bicolor cycle by hypothesis, so $(z_1, i+j_0(k-2)-(k-2))$ and $(i+j_0(k-2), z_1)$ have the same color and they are not black (by Lemma 1-f), hence C_k is a (2, k-2)-subdivision of C_2-bicolor, a contradiction. We conclude that $(z_1, i + j(k-2)) \in A(T)$ for each j with $p > i + j(k-2) \geq i$.

Claim 3. For each i, $p \geq i > k-2$ we have: If $(i, z_1) \in A(T)$ then for every j, $p - (k-2) \geq i - j(k-2) > 0$, it holds that $(i - j(k-2), z_1) \in A(T)$.
Let i be such that $p \geq i > k-2$ and $(i, z_1) \in A(T)$. If there exists j, $p - (k-2) \geq i - j(k-2) > 0$, such that $(i - j(k-2), z_1) \notin A(T)$ then let $j_0 = \min\{j \mid p - (k-2) \geq i - j(k-2) > 0 \text{ and } (i - j(k-2), z_1) \notin A(T)\}$. As before we have that $(z_1, i - j_0(k-2)) \in A(T)$ (T is a tournament) then $(z_1, i - j_0(k-2) + (k-2) = i - (j_0 - 1)(k-2)) \in A(T)$ (as a consequence of Claim 2) since $p - (k-2) \geq i - j_0(k-2) \geq 0)$, contradicting the choice of j_0.

Now we conclude the proof by analyzing the following two cases.

Case A. $p = m(k-2)$, with $m \in \mathbb{N}$ and $m \geq 2$ (recall that $p > k-2$). $(z_1, z_2 = 0) \in A(T)$ so it follows from Claim 2 that $(z_1, p - (k-2)) \in A(T)$, then there exists $C_k = (z_1, p-(k-2)) \cup (p-(k-2), \alpha, p = z_0) \cup (z_0, z_1) \subseteq T$ and it is an at most bicolor cycle by hypothesis, then we have that $(z_1, p - (k-2)) \in A(T)$ is red (it is not black by Lemma 1-f) and so $C_k = (z_1, p-(k-2)) \cup (p-(k-2), \alpha, p = z_0) \cup (z_0, z_1) \subseteq T$ is a (2, k-2)-subdivision of C_2-bicolor, a contradiction.

Case B. $p = m(k-2)+r$, with $m, r \in \mathbb{N}$, $m \geq 1$ and $k-2 > r > 0$.

Claim 4. $(z_1, p-r) \in A(T)$ and it is red.
$(z_1, z_2 = 0) \in A(T)$ so $(z_1, m(k-2) = p-r) \in A(T)$ (by Claim 2). Then $C_t = (z_1, p-r) \cup (p-r, \alpha, p = z_0) \cup (z_0, z_1) \subseteq T$ is a cycle with length $t = r+2$ with $t < k$ (as $r < k-2$). It is an at most 2-colored cycle by hypothesis, hence $(z_1, p-r) \in A(T)$ is black or it is red and we conclude it is colored red because of Lemma 1-f.

Claim 5. $(r, z_1) \in A(T)$ and it is blue.
$(z_0, z_1) \in A(T)$ so by Claim 3 we have that $(p-m(k-2) = r, z_1) \in A(T)$. Then $C_t = (z_1, z_2 = 0) \cup (0, \alpha, p-m(k-2) = r) \cup (r, z_1) \subseteq T$ is a cycle with length $t = r+2 < k$ and it is an at most bicolor cycle by hypothesis, so $(r, z_1) \in A(T)$ is black or blue. As a consequence of Lemma 1-f we conclude $(r, z_1) \in A(T)$ is blue.

Claim 6. $(z_0, r) \in A(T)$.
If $(r, z_0) \in A(T)$ then there exists $C_s = (r, z_0, z_1, z_2 = 0) \cup (z_2 = 0, \alpha, r) \subseteq T$ a cycle of length $s = r+3 \leq k$) and it is an at least 3-colored cycle ($k \geq 4$ since $k-3 \geq r \geq 1$), a contradiction.

Then there exists the at least 3-colored cycle $C_q = (z_0 = p, r, z_1, p-r) \cup (p-r, \alpha, z_0) \subseteq T$ ($q \leq k$ as $r \leq k-3$), a contradiction again. ■

Corollary 1. *In combination with the Shen Minggang condition, Theorem* 2 *is a generalization of the Theorem C by Hahn et al.*

4.2. Condition II

Definition 4. A (1, 1, k-2)-C-*subdivision* of a 3-colored C_3 with colors 1, 2 and 3, is defined to be a cycle of length k having a monochromatic path of length k-2 colored 1, one arc colored 2 and one arc colored 3.

Definition 5. A (1, 1, k-2)-T-*subdivision* of a 3-colored T_3 with colors 1, 2 and 3, can be obtained from such T_3 by the substitution of one of its arcs with a path colored alike.

Notice that any of the arcs of T_3 can be replaced by a path, so depending on which arc is replaced, we can get three different 3-colored digraphs.

Definition 6. Let T be an m-colored tournament. We say that T satisfies property PII_k for some fixed integer $k \geq 3$ if

(a) There is no $(1, 1, t-2)$-C-subdivision in T, with $t \leq k$ and $t \geq 3$, and

(b) there is no $(1, 1, k-2)$-T-subdivision in T.

Theorem 3. *Let T be an m-colored tournament. If T satisfies property PII_k for some integer $k \geq 3$ then $C(T)$ is a* KP*-digraph.*

Proof. We proceed by contradiction. Suppose that $C(T)$ is not a KP-digraph, then by Lemma 1 there exists a cycle $\gamma = (z_0, z_1, z_2 = 0, 1, 2, \ldots, p = z_0)$ satisfying properties (a) to (f). The following claims will allow us to obtain a contradiction:

Claim 1. $p > k-2$.
By the contrary, if $p \leq k-2$ then $\gamma \subseteq T$ is a $(1, 1, p)$-subdivision of a 3-colored C_3 in T $(p+2 \leq k)$, a contradiction. So $p > k-2$.

Claim 2. For each i with $p-(k-2) > i \geq 0$ we have: If $[z_1, i]_T$ is colored a ($a \neq$ black, by Lemma 1-f) then for each j with $p > i + j(k-2) \geq i$, we have that $[z_1, i+j(k-2)]_T$ is also colored a.

Let i, $0 \leq i < p-(k-2)$, be a fixed integer and suppose that $[z_1, i]_T$ is colored a. Suppose, by contradiction, that there exists j' with $p > i + j'(k-2) \geq i$, such that $[z_1, i+j'(k-2)]_T$ is colored b, with $a \neq b$. Consider $j_0 = \min\{j' \mid p > i + j'(k-2) \geq i$ and such that $[z_1, i+j'(k-2)]_T$ is colored b, with $a \neq b\}$. Then $D[\{z_1, j_0, j_0-1\}]$ is a 3-colored triangle, a contradiction.

Claim 3. For each i with $p \geq i > k-2$ the following holds: If $[z_1, i]_T$ is colored a ($a \neq$ black) then for every j, $p-(k-2) \geq i - j(k-2) > 0$, we have that $[z_1, i-j(k-2)]_T$ is colored a.
Immediate from the previous claim.

Depending on the length of α we analyze the following two cases.

Case A. $p = m(k-2)$, $m \in \mathbb{N}$, $m \geq 2$ $(p \geq k-2)$.
$(z_1, z_2 = 0) \in A(T)$ is blue and $[z_1, p-(k-2)]_T$ is blue by Claim 2. If $(z_1, p-(k-2)) \in A(T)$ then $C_k = (p = z_0, z_1, p-(k-2)) \cup (p-(k-2), \alpha, p = z_0) \subseteq T$ is a $(1, 1, k-2)$-subdivision of a 3-colored C_3 in T. If $(p-(k-2), z_1) \in A(T)$ then $T_k = (p-(k-2), \alpha, p = z_0) \cup (p = z_0, z_1) \cup (p-(k-2), z_1) \subseteq T$ is a $(1, 1, k-2)$-subdivision of a 3-colored T_3 in T, in both cases we get a contradiction.

Case B. $p = m(k-2) + r$, $m, r \in \mathbb{N}$, $m \geq 1$, $1 \leq r < k-2$.

Claim 4. $(p-r, z_1) \in A(D)$ and it is blue: $(z_1, z_2 = 0) \in A(T)$ and it is blue, so $[z_1, m(k-2) = p-r]_T$ is also blue (by Claim 2). If $(z_1, p-r) \in A(T)$ then $C_t = (z_1, p-r) \cup (p-r, \alpha, p = z_0) \cup (z_0, z_1) \subseteq T$ $(t < k)$ is a (1,1,t-2)-subdivision of a 3-colored C_3 in T, a contradiction. Then $(p-r, z_1) \in A(D)$ and it is blue.

Let $h = (k-2) - r$, $A_1 = \{j(k-2) \mid m \geq j \geq 0\}$, $A_2 = \{p - j(k-2) \mid m \geq j \geq 0\}$ and $A = A_1 \cup A_2$. Notice that $A_1 \cap A_2 = \emptyset$, as $0 < r < k-2$.

Claim 5. For every $i \in A$ we have that $(z_1, i) \in A(T)$.
By contradiction let us define $f = \min\{u \in A \mid (u, z_1) \in A(T)\}$ $(f > 0$ because $(z_1, 0) \in A(T))$. Let $w \in A$ defined as follows: $f - h = w$ whenever $f \in A_1$ and $f - r = w$ whenever $f \in A_2$. It follows from the definition of f that $(z_1, w) \in A(T)$ (notice that $f \in A_1$ implies $w \in A_2$; and $f \in A_2$ implies $w \in A_1$).

If $f \in A_1$ then $w \in A_2$, $(f, z_1) \in A(T)$ is blue (as $(z_1, z_2) \in A(T)$ is blue and because of Claim 2) and $(z_1, w) \in A(T)$ is red $((z_0, z_1) \in A(T)$ is red and because of Claim 3). Hence $C_t = (f, z_1) \cup (z_1, w) \cup (w, \alpha, f) \subseteq T$ $(h < k-2$ so $t < k)$ is a 3-colored cycle, a contradiction.

By analogy, if $f \in A_2$ then $w \in A_1$, $(z_1, w) \in A(T)$ if blue $((z_1, z_2) \in A(T)$ is blue and by Claim 2) and $(f, z_1) \in A(T)$ is red $((z_0, z_1) \in A(T)$ is red and by Claim 3). Hence $C_t = (f, z_1) \cup (z_1, w) \cup (w, \alpha, f) \subseteq T$ $(t < k$ as $r < k-2)$ is a 3-colored cycle, a contradiction again.

In particular we have that $(z_1, p - r = m(k-2)) \in A(T)$, contradicting Claim 4. ■

Corollary 2. *Theorem* 3 *is a generalization of the Theorem A by Shen Minggang.*

5. Remarks and Open Problems

Remark 1. If we only ask in Theorem 2 for every $C_k \subseteq T$ to be an at most bicolor cycle and not to be a $(2, k-2)$-subdivision of C_2-bicolor, then the result does not hold.

Proof. Consider tournaments in Figure 1a (for k=4) and Figure 1b (for k=6). Even though every C_k in such tournaments is an at most bicolor cycle and is not a $(2, k-2)$-subdivision of C_2-bicolor, there exists a 3-colored $C_3 =$

(z_0, z_1, z_2, z_0) and both tournaments do not have a kernel by monochromatic paths. ∎

Remark 2. If we omit the last hypothesis (c) in Theorem 3 then the result will be false. To prove it consider the digraph in Figure 1a (same argument as in the previous remark).

Remark 3. The Sands *et al.* condition, the Shen Minggang condition and the Galeana-Sánchez condition do not imply the conditions of Theorem 2 and Theorem 3.

Proof. To prove it consider digraphs in Figure 2 to 5.

Figure 2. The tournament in Figure 2 proves that the Sands *et al.* Condition does not imply Condition I: Though T is a 2-colored tournament, there exists $C_6 = (z_0, z_1, z_2, z_3, z_4, z_5, z_0)$ a $(2,4)$-subdivision of a bicolor C_2.

Figure 3. The tournament in Figure 3 proves that the Shen Minggang Condition does not imply Condition II: Though every triangle is a 2-colored subdigraph (notice that if there exists a 3-colored triangle then it must contain the arc (z_1, z_2)), there exists $C_6 = (z_0, z_1, z_2, z_3, z_4, z_5, z_0)$ a $(1,1,4)$-subdivision of a 3-colored C_3.

Figure 4. The tournament in Figure 4 proves that the Galeana-Sánchez Condition does not imply Condition I: Though every cycle of length 3 and 4 in T is a quasimonochromatic cycle (notice that every non quasi-monochromatic cycle of length 4 must contain two arcs colored 1), there exists $C_5 = (z_0, z_1, z_2, z_3, z_4, z_0)$ a $(2,3)$-subdivision of a bicolor C_2.

Figure 5. The tournament in Figure 5 proves that the Galeana-Sánchez Condition does not imply Condition II: Though every cycle of length 3 and 4 in T is a quasimonochromatic cycle (notice that there are only 2 arcs not colored 3 and both are adjacent to z_1, so they can not be simultaneously in a cycle), there exists $P_5 = (z_2, z_3, z_4, z_0, z_1) \cup (z_2, z_1)$, a $(1,1,3)$-subdivision of a 3-colored T_3. ∎

Open problem 1. Let T be an m-colored tournament. If there is some fixed integer $k \geq 4$ such that every $C_s \subseteq T$ $(s \leq k)$ is at most bicolor then $C(T)$ is a KP-digraph.

Open problem 2. Let T be an m-colored tournament. If there is some fixed integer $k \geq 3$ such that there is no $(1,1,t-2)$-subdivision of a 3-colored C_3 in T $(3 \leq t \leq k)$ then $C(T)$ is a KP-digraph.

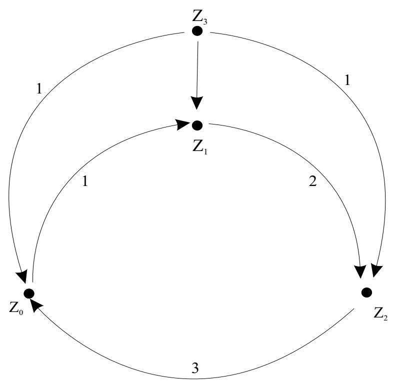

Figure 1a

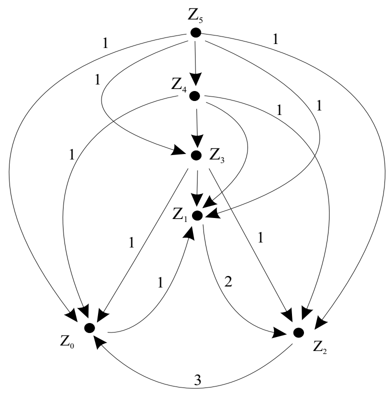

Figure 1b

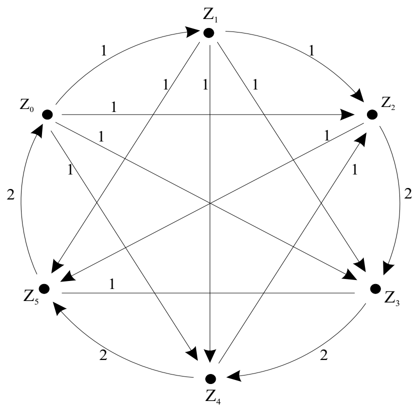

Figure 2

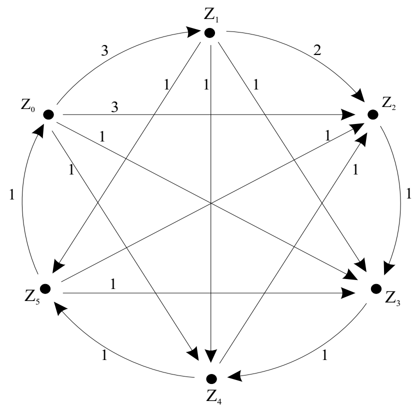

Figure 3

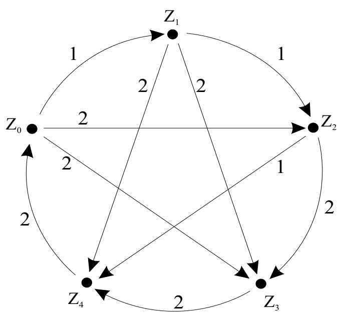

Figure 4

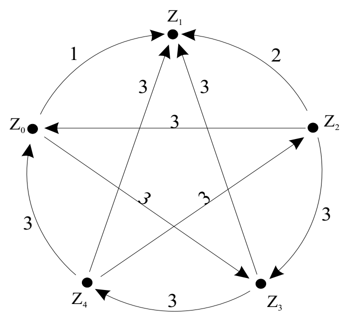

Figure 5

Open problem 3. Let T be an m-colored tournament. If there is some fixed integer $k \geq 3$ such that there is no $(1,1,k-2)$-subdivision of a 3-colored T_3 in T and there is no $(1,1,t-2)$-subdivision of a 3-colored C_3 in T $(3 \leq t < k)$ then $C(T)$ is a KP-digraph.

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References

[1] C. Berge, Graphs (Nort-Holland, Amsterdam, third edition, 1991).

[2] C. Berge and P. Duchet, *Recent problems and results about kernels in directed graphs*, Discrete Math. **86** (1990) 27–31.

[3] E. Boros and V. Gurvich, *A corrected version of the Duchet kernel conjecture*, Discrete Math. **179** (1998) 231–233.

[4] E. Boros and V. Gurvich, Perfect graphs, kernels and cores of cooperative games, 2003, RUTCOR Research Report, 12, Rutgers University, Dedicated to the memory of Claude Berge, New Jersey.

[5] V. Chvátal, On the computational complexity of finding a kernel, Report, CRM-300, Centre de Recherches Mathématiques, Université de Montréal (Montréal, 1973).

[6] P. Duchet, Représentations, Noyaux en Théorie des Graphes et Hypergraphes (Paris, Univ. Paris 6, 1979) 200.

[7] P. Duchet and H. Meyniel, *A note on kernel-critical graphs*, Discrete Math. **33** (1981) 103–105.

[8] A.S. Fraenkel, *Planar kernel and Grundy with $d \leq 3$, $d_{out} \leq 2$, $d_{in} \leq 2$ are NP-complete*, Discrete Appl. Math. **3** (1981) 257–262.

[9] A.S. Fraenkel, *Combinatorial Games: Selected Bibliography with a Succinct Gourmet Introduction*, The Electronic Journal of Combinatorics, **14** Dynamic Survey DS2, 2007.

[10] H. Galeana-Sánchez, *On the existence of kernels and h-kernels in directed graphs*, Discrete Math. **110** (1992) 251–255.

[11] H. Galeana-Sánchez, *On monochromatic paths and monochromatic cycles in edge colored tournaments*, Discrete Math. **156** (1996) 103–112.

[12] H. Galeana-Sánchez and V. Neumann-Lara, *On kernels and semikernels of digraphs*, Discrete Math. **48** (1984) 67–76.

[13] H. Galeana-Sánchez and V. Neumann-Lara, *On kernel-perfect critical digraphs*, Discrete Math. **59** (1986) 257–265.

[14] H. Galeana-Sánchez and H. Rincón-Mejía, *A sufficient condition for the existence of k-kernels in digraphs*, Discuss. Math. Graph Theory **18** (1998) 197–204.

[15] H. Galeana-Sánchez and R. Rojas-Monroy, *On monochromatic paths and monochromatic 4-cycles in edge-colored bipartite tournaments*, Discrete Math. **285** (2004) 313–318.

[16] H. Galeana-Sánchez and R. Rojas-Monroy, *On monochromatic paths and at most 2-colored arc sets in edge colored tournaments*, Graphs and Combinatorics **21** (2005) 307–317.

[17] H. Galeana-Sánchez and R. Rojas-Monroy, *Kernels in edge-coloured orientations of nearly complete graphs*, Discrete Math. **308** (2008) 4599–4607.

[18] M. Kucharska, *On (k,l)-kernels of orientations of special graphs*, Ars Combin. **60** (2001) 137–147.

[19] M. Kucharska and M. Kwaśnik, *On (k,l)-kernels of special superdigraphs of P_m and C_m*, Discuss. Math. Graph Theory **21** (2001) 95–109.

[20] M. Kwaśnik, *The generalization of Richardson theorem*, Discuss. Math. **IV** (1981) 11–14.

[21] M. Kwaśnik, *On (k,l)-kernels of exclusive disjunction, cartesian sum and normal product of two directed graphs*, Discuss. Math. **V** (1982) 29–34.

[22] G. Hahn, P. Ille and R.E. Woodrow, *Absorbing sets in arc-coloured tournaments*, Discrete Math. **283** (2004) 93–99.

[23] J. von Neumann and O. Morgenstern, Theory of games and economic behaviour, (Princeton University Press, Princeton, 2nd edition, 1947).

[24] B. Sands, N. Sauer and R.E. Woodrow, *On monochromatic paths in edge-colored digraphs*, J. Combin. Theory (B) **33** (1982) 271–275.

[25] Shen Minggang, *On monochromatic paths in m-colored tournaments*, J. Combin. Theory (B) **45** (1988) 108–111.

[26] M. Richardson, *Solutions of irreflexive relations*, Annals of Mathematics **58** (1953) 573–590.

[27] J. van Leeuwen, Having a Grundy-numbering is NP-complete, Report, 207, Computer Science Dept., Pennsylvania State University, University Park, PA, 1976.

[28] V. Neumann-Lara, *Seminúcleos y núcleos*, Anales del Instituto de Matemáticas UNAM **11** (1971) 55–62.

[29] A. Włoch and I. Włoch, *On (k, l)-kernels in generalized products*, Discrete Math. **164** (1997) 295–301.

[30] I. Włoch, *On imp-sets and kernels by monochromatic paths of the duplication*, Ars Combin. **83** (2007) 93–100.

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