A NOTE ON k-UNIFORM SELF-COMPLEMENTARY HYPERGRAPHS OF GIVEN ORDER

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Abstract

We prove that a k-uniform self-complementary hypergraph of order \( n \) exists, if and only if \( \binom{n}{k} \) is even.

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Let \( V \) be a set of \( n \) elements. The set of all \( k \)-subsets of \( V \) is denoted by \( \binom{V}{k} \).

A \( k \)-uniform hypergraph \( H \) consists of a vertex-set \( V(H) \) and an edge-set \( E(H) \subseteq \binom{V(H)}{k} \). Two \( k \)-uniform hypergraphs \( G \) and \( H \) are isomorphic, if there is a bijection \( \theta : V(G) \rightarrow V(H) \) such that \( e \in E(G) \) if and only if \( \{ \theta(x) | x \in e \} \in E(H) \). The complement of a \( k \)-uniform hypergraph \( H \) is the hypergraph \( \overline{H} \) such that \( V(\overline{H}) = V(H) \) and the edge set of which consists of all \( k \)-subsets of \( V(H) \) not in \( E(H) \) (in other words \( E(\overline{H}) = \binom{V(H)}{k} - E \)). A \( k \)-uniform hypergraph \( H \) is called self-complementary (s-c for short) if it is isomorphic with its complement \( \overline{H} \). Isomorphism of a \( k \)-uniform self-complementary hypergraph onto its complement is called a self-complementing permutation (or s-c permutation).

The \( k \)-uniform s-c hypergraphs for \( k = 3 \) and \( k = 4 \) are studied in [3] and [6], respectively. The 2-uniform self-complementary hypergraphs are exactly self-complementary graphs. This class of graphs has been independently discovered by Ringel [4] and Sachs [5] who proved that an s-c graph of order \( n \) exists if and only if \( n \equiv 0 \) or \( n \equiv 1 \) (mod 4) or, equivalently, whenever \( \binom{n}{2} \) is even.
We prove a generalisation of this fact for \( k \)-uniform hypergraphs.

**Theorem 1.** Let \( n \) and \( k \) be positive integers, \( k \leq n \). There is a \( k \)-uniform self complementary hypergraph of order \( n \) if and only if \( \binom{n}{k} \) is even.

Let us give first some results which will be needed in the proof of Theorem 1.

For positive integers \( k \) and \( n \) we say that \( n \) contains \( k \) (we write \( k \subset n \)) if when \( k \) has 1 in a certain binary place, then \( n \) also has 1 in the corresponding binary place. That is, the binary representation of \( k \) can be obtained from that of \( n \) by changing some ones to zeros. For example, \( 6 \subset 14 \) since \( 6 = 1 \cdot 2^3 + 1 \cdot 2^1 + 0 \cdot 2^0 \) and \( 14 = 1 \cdot 2^3 + 1 \cdot 2^1 + 0 \cdot 2^0 \) and, clearly, \( 5 \not\subset 14 \). In [2] Hatcher and Riley solved a problem proposed by Kimball by proving the lemma which we give below (Moser has pointed out that this result is contained in [1]).

**Lemma 1.** \( \binom{n}{k} \) is odd if and only if \( k \subset n \).

Any positive integer \( n \) may be, in the unique way, written in the form \( n = 2^l c \), where \( c \) is an odd integer. We denote then \( \lambda(n) = l \). For any finite and nonempty set \( A \) we shall write \( \lambda(A) \) in place of \( \lambda(|A|) \), for short.

The following lemma is proved in [7].

**Lemma 2.** Let \( k, m \) and \( n \) be positive integers, and let \( \sigma : V \to V \) be a permutation of a set \( V \), \( |V| = n \), with orbits \( O_1, \ldots, O_m \). \( \sigma \) is a self-complementing permutation of a self-complementary \( k \)-uniform hypergraph, if and only if, for every \( p \in \{1, \ldots, k\} \) and for every decomposition

\[
\lambda(k) = k_1 + \ldots + k_p
\]

of \( k \) (\( k_j > 0 \) for \( j = 1, \ldots, p \)), and for every subsequence of orbits

\[
O_{i_1}, \ldots, O_{i_p}
\]

such that \( k_j \leq |O_{i_j}| \) for \( j = 1, \ldots, p \), there is a subscript \( j_0 \in \{1, \ldots, p\} \) such that

\[
\lambda(k_{j_0}) < \lambda(O_{i_{j_0}}).
\]

**Proposition 1.** Let \( n \) and \( k \) be two non negative integers, \( k < n \). The following two conditions are equivalent.

1. \( \binom{n}{k} \) is odd.
(2) For every non negative integer \( l \) such that \( k = a2^l + s \), where \( a \) is odd and \( 0 \leq s < 2^l \) we have \( n \in \{2^l + s, \ldots, 2^{l+1} - 1\} \mod 2^{l+1} \).

**Proof.** Put \( k = \sum_{i=0} c_i 2^i \) and \( n = \sum_{i=0} d_i 2^i \), where \( c_i, d_i \in \{0, 1\} \) for every \( i \). Let us suppose first that \( \binom{n}{k} \) is odd. Then, by Lemma 1, for every \( i \), \( c_i = 1 \) implies \( d_i = 1 \). Note that \( k = a2^l + s \), where \( a \) is odd and \( 0 \leq s < 2^l \), means exactly that \( c_l = 1 \) and \( \sum_{i=0}^{l-1} c_i 2^i = s \). Since \( d_i = 1 \) whenever \( c_i = 1 \), we have \( \sum_{i=0}^l d_i 2^i \geq 2^l + s \) for every \( l \) such that \( c_l = 1 \) (and, clearly, \( \sum_{i=0}^l d_i 2^i < 2^{l+1} \)).

If \( \binom{n}{k} \) is even then, again by Lemma 1, there is \( l_0 \) such that \( c_{l_0} = 1 \) and \( d_{l_0} = 0 \). Hence \( k = a2^{l_0} + s \), with \( a \) odd and \( 0 \leq s = \sum_{j=0}^{l_0-1} c_j 2^j < 2^{l_0} \), and \( n = b2^{l_0+1} + \sum_{j=0}^{l_0-1} d_j 2^j \). Since \( \sum_{j=0}^{l_0-1} d_j 2^j < 2^{l_0} \), we have \( n \in \{0, \ldots, 2^{l_0} - 1\} \mod 2^{l_0+1} \subset \{0, \ldots, 2^{l_0} + s - 1\} \mod 2^{l_0+1} \) and the proposition is proved.

Proposition 1 is clearly equivalent to the following.

**Proposition 2.** Let \( n \) and \( k \) be two non negative integers, \( k < n \). The following two statements are equivalent.

1. \( \binom{n}{k} \) is even.
2. There is a non negative integer \( l_0 \) such that \( k = a_0 2^{l_0} + s_0 \), where \( a_0 \) is odd, \( 0 \leq s_0 < 2^{l_0} \), and \( n \in \{0, \ldots, 2^{l_0} + s_0 - 1\} \mod 2^{l_0+1} \).

**Lemma 3.** Let \( l, k, s \) and \( n \) be non negative integers such that \( k < n \), \( k = a2^l + s \), \( a \) is odd, \( s < 2^l \). If \( n \in \{0, \ldots, 2^{l} + s - 1\} \mod 2^{l+1} \) then there is a \( k \)-uniform self-complementary hypergraph of order \( n \).

**Proof.** Let us write \( n \) in the form \( n = b2^{l+1} + r \), where \( 0 \leq r < 2^l + s \), and let \( \sigma \) be a permutation of an \( n \)-set \( V \) such that it has \( b \) orbits \( O_1, \ldots, O_b \), each of which having its cardinality equal to \( 2^{l+1} \), and one orbit \( O_{b+1} \) with \( |O_{b+1}| = r \). Applying Lemma 2 we shall prove that \( \sigma \) is the self-complementing permutation of a self-complementary \( k \)-uniform hypergraph.

Suppose, contrary to our claim, that \( \sigma \) is not \( s \)-c permutation of any \( s \)-c \( k \)-uniform hypergraph. Then, by Lemma 2, there is a decomposition of \( k \), \( k = k_1 + \ldots + k_p \) and a subsequence \( O_{i_1}, \ldots, O_{i_p} \) of \( O_1, \ldots, O_{b+1} \) such that \( 0 < k_j \leq |O_{i_j}| \) and \( \lambda(k_j) \geq \lambda(O_{i_j}) \) for \( j = 1, \ldots, p \). Clearly, we have \( k_j = |O_{i_j}| = 2^{l+1} \), whenever \( i_j \neq b + 1 \). Hence there exists \( j_0 \) such that \( i_{j_0} = b + 1 \) and \( k_{j_0} = k - \sum_{j \neq j_0} k_j = (2^a + s) - (p - 1)2^{l+1} = 2^l(a - 2(p - 1)) + s \). Observe that \( a - 2(p - 1) > 0 \) is positive and odd, so we have \( k_{j_0} \geq 2^l + s > r = |O_{b+1}| \). This contradicts our assumption that \( |O_{b+1}| = k_{j_0} \).
Note that if there is a $k$-uniform $s$-c hypergraph of order $n$ then, clearly, \( \binom{n}{k} \) is even. Now the proof of Theorem 1 follows by Lemma 3 and Proposition 2.

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References


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