

## **$k$ -KERNELS AND SOME OPERATIONS IN DIGRAPHS**

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### **Abstract**

Let  $D$  be a digraph.  $V(D)$  denotes the set of vertices of  $D$ ; a set  $N \subseteq V(D)$  is said to be a  $k$ -kernel of  $D$  if it satisfies the following two conditions: for every pair of different vertices  $u, v \in N$  it holds that every directed path between them has length at least  $k$  and for every vertex  $x \in V(D) - N$  there is a vertex  $y \in N$  such that there is an  $xy$ -directed path of length at most  $k - 1$ .

In this paper, we consider some operations on digraphs and prove the existence of  $k$ -kernels in digraphs formed by these operations from another digraphs.

**Keywords:**  $k$ -kernel,  $k$ -subdivision digraph,  $k$ -middle digraph and  $k$ -total digraph.

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## 1. INTRODUCTION

We refer the reader to [1] for general concepts. In this paper,  $D$  denotes a digraph;  $V(D)$  is the set of vertices and  $A(D)$  denotes the set of arcs.

A directed path is a sequence  $P = (x_0, x_1, \dots, x_n)$  of distinct vertices of  $D$  such that  $(x_i, x_{i+1}) \in A(D)$  for each  $i, 0 \leq i \leq n - 1$ . The length of  $P$  is  $n$  and we denote  $\ell(P) = n$ . For  $x, y \in V(D)$ , the distance from  $x$  to  $y$  in  $D$  is denoted as  $d_D(x, y)$  and defined as:  $d_D(x, y) = \min\{\ell(P) \mid P \text{ is an } xy\text{-directed path}\}$  whenever there exists an  $xy$ -directed path in  $D$ , otherwise, we define  $d_D(x, y) = \infty$ . If  $P$  is a directed path and  $a, b \in V(P)$ , then  $(a, P, b)$  denotes the  $ab$ -directed path contained in  $P$ .

A set  $N \subseteq V(D)$  is said to be  $k$ -independent whenever for any two different vertices  $x, y \in N$  we have  $d_D(x, y) \geq k$  and  $d_D(y, x) \geq k$ .  $N$  is said to be  $(k - 1)$ -absorbent whenever for each  $x \in V(D) - N$  there exists  $y \in N$  such that  $d_D(x, y) \leq k - 1$ . The set  $N$  is said to be a  $k$ -kernel if it is  $k$ -independent and  $(k - 1)$ -absorbent.

We note that a 2-kernel is a kernel of a digraph in the sense of J. von Neumann and O. Morgenstern [20]. The problem of the existence of a kernel in a digraph has been studied in [2, 3, 4, 7, 17, 18].

The existence of kernels of digraphs formed by some operations from another digraphs have been studied by several authors, namely: M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel [16]; M. Harminc and T. Olejníková [11]; J. Topp [19], H. Galeana-Sánchez and V. Neumann-Lara [7, 8].

The concept of  $k$ -kernel was introduced by M. Kwaśnik in [14]. Clearly, this concept generalizes the concept of a kernel of a digraph. It has been studied by several authors: M. Harminc [9], M. Kwaśnik [14, 15], M. Kucharska [12, 13], H. Galeana-Sánchez [5, 6], A. Włoch and I. Włoch [21].

In [10], M. Harminc constructed all kernels of the line digraph of  $D$  from the kernels of  $D$  and in [19] the author considered some special digraphs:  $S(D)$ ;  $Q(D)$ ,  $T(D)$  and  $L(D)$  which were called the subdivision digraph, the middle digraph, the total digraph and the line digraph of  $D$ , respectively and studied some necessary or sufficient conditions for the existence or uniqueness of kernels of these digraphs.

In this paper, for a given digraph  $D$  and any  $k \geq 2$  we define: the  $k$ -subdivision  $S^k(D)$ , a generalization of the subdivision  $S(D)$ , the digraph  $R^k(D)$ , the  $k$ -middle digraph  $Q^k(D)$  and the  $k$ -total digraph  $T^k(D)$ . Also the following results are proved: for any digraph  $D$  and for any  $k \geq 2$  the

digraphs  $S^k(D)$ ,  $R^k(D)$  and  $Q^k(D)$  have a  $k$ -kernel. For any digraph  $D$  and for  $k \geq 3$  the digraph  $T^k(D)$  has a  $k$ -kernel.

2.  $k$ -KERNELS IN:  $S^k(D)$ ,  $R^k(D)$ ,  $Q^k(D)$  AND  $T^k(D)$

Let  $D$  be a digraph. The *line digraph*  $L(D)$  of  $D$  is the digraph defined as follows:  $V(L(D)) = A(D)$  and  $(a = (u, v), b = (z, w)) \in A(L(D))$  if and only if  $v = z$  [1].

[19]: For a given digraph  $D$ , the subdivision digraph  $S(D)$  of  $D$  is defined by:  $V(S(D)) = V(D) \cup A(D)$  and

$$\Gamma^+(x) = \begin{cases} \{x\} \times \Gamma_D^+(x), & \text{whenever } x \in V(D), \\ \{v\}, & \text{whenever } x = (u, v) \in A(D). \end{cases}$$

Notice that for a vertex  $x$  of the *subdivision digraph* of  $D$  we have the following: If  $x$  corresponds to a vertex of  $D$ , then  $x$  is adjacent to the arcs which are incident from  $x$  in  $D$ ; and if  $x$  corresponds to an arc of  $D$ , then  $x$  is adjacent only to the terminal endpoint of  $x$ . Also notice that  $S(D)$  is obtained from  $D$  by changing each arc of  $D$  for a directed path of length two.

Let  $D$  be a digraph. We define the  $k$ -subdivision digraph of  $D$ , denoted  $S^k(D)$ , as follows:

$$S^k(D) = S(D) - \{(u, a) | a \in A(D) \text{ and } u \text{ is the initial endpoint of } a\} \\ \cup \left( \bigcup_{a \in A(D)} \beta_a \right)$$

for each  $a = (u, v) \in A(D)$ ,  $\beta_a = (a_0 = u, a_1, \dots, a_{n(a)k+k-1} = a = (u, v))$  is a  $ua$ -directed path whose length is  $\equiv k - 1 \pmod{k}$  ( $n(a) \in \mathbb{N}$ ) and the following two properties hold:

- (i)  $V(\beta_a) \cap V(S(D)) = \{u, a\}$ ,
- (ii) For any  $a, b \in A(D)$  with  $a \neq b$  we have  $(V(\beta_a) - \{u\}) \cap V(\beta_b) = \emptyset$ .

Notice that  $S^k(D)$  is obtained from  $D$  by substituting each arc of  $D$  for a directed path whose length is  $\equiv 0 \pmod{k}$  (for an example see Figure 1).

We write  $V^0(D) = \{x \in V(D) \mid \delta_D^+(x) = 0\}$ .

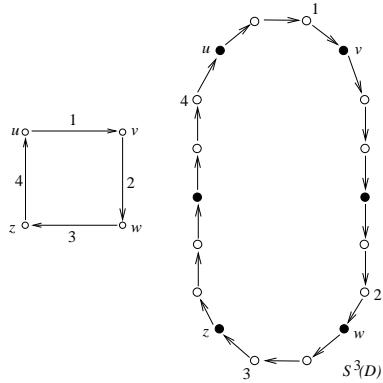


Figure 1

Finally, we define the digraphs  $R^k(D)$ ,  $Q^k(D)$  and  $T^k(D)$  as follows  $R^k(D) = S^k(D) \cup D$ ,  $Q^k(D) = S^k(D) \cup L(D)$  and  $T^k(D) = S^k(D) \cup D \cup L(D)$  (for an example see Figure 2).

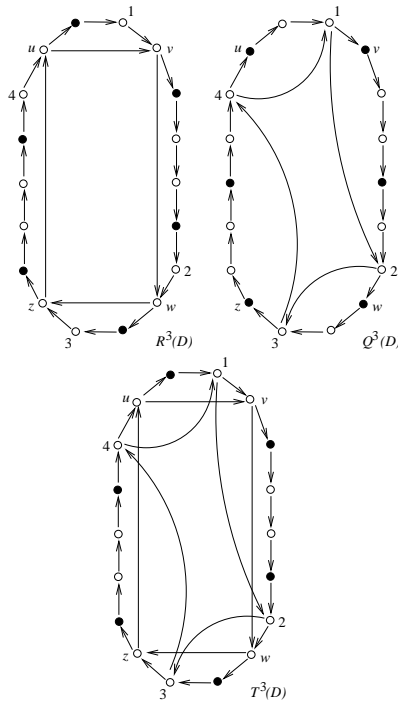


Figure 2

**Theorem 2.1.** *For any digraph  $D$  and for any integer  $k$  ( $k \geq 2$ ), the  $k$ -subdivision digraph  $S^k(D)$  of  $D$  has a  $k$ -kernel.*

**Proof.** Let  $D$  and  $S^k(D)$  be digraphs as in the hypothesis. For each  $a \in A(D)$  we denote  $\mathfrak{N}_a = \{a_i \in V(\beta_a) \mid i \equiv 0 \pmod{k}\}$ . We will prove that  $\mathfrak{N} = V^0(D) \cup \bigcup_{a \in A(D)} \mathfrak{N}_a$  is a  $k$ -kernel of  $S^k(D)$ . Observe that  $V(D) \subseteq \mathfrak{N}$ .

**Claim 1.**  $\mathfrak{N}$  is a  $k$ -independent set of vertices of  $S^k(D)$ .

Let  $x, y \in \mathfrak{N}, x \neq y$ . We will prove  $d_{S^k(D)}(x, y) \geq k$  and  $d_{S^k(D)}(y, x) \geq k$ .

*Case 1.*  $x \in V^0(D)$  and  $y \in V^0(D)$ .

Since  $\delta_{S^k(D)}^+(x) = \delta_D^+(x) = 0$  and  $\delta_{S^k(D)}^+(y) = \delta_D^+(y) = 0$ , it follows that  $d_{S^k(D)}(x, y) = d_{S^k(D)}(y, x) = \infty$ .

*Case 2.*  $x \in V^0(D)$  and  $y \in \bigcup_{a \in A(D)} \mathfrak{N}_a$ .

Since  $\delta_{S^k(D)}^+(x) = \delta_D^+(x) = 0$ , we have  $d_{S^k(D)}(x, y) = \infty$ . Let  $c = (u, v) \in A(D)$  such that  $y \in \mathfrak{N}_c$ . From the definition of  $S^k(D)$  we have  $d_{S^k(D)}(y, x) = d_{S^k(D)}(y, c = (u, v)) + d_{S^k(D)}(c, x)$ . Now since  $y = c_i$  with  $i \equiv 0 \pmod{k}$  and  $\ell(\beta_c) \equiv k - 1 \pmod{k}$  it follows that  $d_{S^k(D)}(y, c = (u, v)) = d_{\beta_c}(y, c) \geq k - 1$ . Clearly,  $d_{S^k(D)}(c, x) \geq 1$  (as  $c \in A(D)$  and  $x \in V^0(D) \subseteq V(D)$ ). Therefore  $d_{S^k(D)}(y, x) \geq (k - 1) + 1 = k$ .

*Case 3.*  $x \in \bigcup_{a \in A(D)} \mathfrak{N}_a$  and  $y \in V^0(D)$ .

Proceed exactly as in Case 2 interchanging  $x$  with  $y$ .

*Case 4.*  $x \in \bigcup_{a \in A(D)} \mathfrak{N}_a$  and  $y \in \bigcup_{a \in A(D)} \mathfrak{N}_a$ .

*Case 4.1.* There exists  $c = (u, v) \in A(D)$  such that  $\{x, y\} \subseteq \mathfrak{N}_c$ .

From the definition of  $\mathfrak{N}_c$  we have  $x = c_{mk}$  and  $y = c_{tk}$  for some  $0 \leq m \leq n(c)$ ,  $0 \leq t \leq n(c)$ . Assume without loss of generality  $t > m$ .

From the definition of  $S^k(D)$  and the fact  $x \neq v$  (as  $\ell(\beta_c) \equiv k - 1 \pmod{k}$ ) we have:  $d_{S^k(D)}(x, y) = d_{\beta_c}(x, y) = (t - m)k \geq k$ . On the other hand, we have  $d_{S^k(D)}(y, x) = d_{S^k(D)}(y, c) + d_{S^k(D)}(c, v) + d_{S^k(D)}(v, x)$ . Since  $d_{S^k(D)}(y, c) = d_{\beta_c}(y, c) \geq k - 1$  and  $d_{S^k(D)}(c, v) = 1$ , we obtain  $d_{S^k(D)}(y, x) \geq k$ .

**Observation 1.** *Observe that in this case we have the same inequalities when we are working in  $Q^k(D)$ , i.e.,  $d_{Q^k(D)}(y, x) \geq k$ , because the definition of  $Q^k(D)$  implies:  $d_{Q^k(D)}(y, x) = d_{Q^k(D)}(y, c) + d_{Q^k(D)}(c, x)$ . And clearly,  $d_{Q^k(D)}(y, c) \geq k - 1$  and  $d_{Q^k(D)}(c, x) \geq 1$ .*

*Case 4.2.*  $x \in \mathfrak{N}_a$  and  $y \in \mathfrak{N}_b$  for some  $a, b \in A(D)$  with  $a \neq b$ . Assume without loss of generality that  $a = (u, v)$  and  $b = (w, z)$ .

$d_{S^k(D)}(x, y) = d_{S^k(D)}(x, a) + d_{S^k(D)}(a, v) + d_{S^k(D)}(v, y)$ . From the definition of  $\mathfrak{N}_a$  we have  $d_{S^k(D)}(x, a) \geq k - 1$  and from the definition of  $S^k(D)$ ,  $d_{S^k(D)}(a, v) = 1$ . Therefore  $d_{S^k(D)}(x, y) \geq k$ .

**Observation 2.** Notice that in this case we have  $d_{Q^k(D)}(x, a) = d_{S^k(D)}(x, a)$  and  $d_{Q^k(D)}(a, y) \geq 1$  (as  $a \neq y$ ). Thus  $d_{Q^k(D)}(x, y) = d_{Q^k(D)}(x, a) + d_{Q^k(D)}(a, y) \geq k$ .

Interchanging  $x$  with  $y$  we obtain  $d_{S^k(D)}(y, x) \geq k$ .

**Claim 2.**  $\mathfrak{N}$  is a  $(k - 1)$ -absorbent set of vertices of  $S^k(D)$ .

Let  $x \in V(S^k(D) - \mathfrak{N})$ . We will prove that there exists  $y \in \mathfrak{N}$  such that  $d_{S^k(D)}(x, y) \leq k - 1$ . Since  $V^0 \subseteq \mathfrak{N}$ , it follows from the definition of  $S^k(D)$  and the fact  $x \in V(S^k(D) - \mathfrak{N})$  that  $x \in \bigcup_{a \in A(D)} \beta_a$ . Let  $c = (u, v) \in A(D)$  be such that  $x \in \beta_c$ .

*Case 1.*  $x \in \beta_c - \{c_i \mid n(c)k + 1 \leq i \leq n(c)k + (k - 1)\}$ .

Since  $x \notin \mathfrak{N}$  (and then  $x \notin \mathfrak{N}_c$ ), it follows that  $x = c_{mk+j}$  for some  $m$  and  $j$  with  $0 \leq m \leq n(c)$  and  $1 \leq j \leq k - 1$ . From the definition of  $S^k(D)$  we have  $d_{S^k(D)}(x, c_{(m+1)k}) = d_{\beta_c}(c_{mk+j}, c_{(m+1)k}) = k - j \leq k - 1$ . Clearly,  $c_{(m+1)k} \in \mathfrak{N}$ .

*Case 2.*  $x \in \{c_i \mid n(c)k + 1 \leq i \leq n(c)k + (k - 1) = (u, v) = c\}$ .

Clearly,  $d_{S^k(D)}(x, v) = d_{S^k(D)}(x, c) + d_{S^k(D)}(c, v)$ ;  $d_{S^k(D)}(x, c) \leq k - 2$  and  $d_{S^k(D)}(c, v) = 1$ . Thus  $d_{S^k(D)}(x, v) \leq k - 1$  with  $v \in \mathfrak{N}$  (recall  $V(D) \subseteq \mathfrak{N}$ ). ■

**Theorem 2.2.** For any digraph  $D$  and for any integer  $k$  ( $k \geq 2$ ), the  $k$ -middle digraph  $Q^k(D)$  of  $D$  has a  $k$ -kernel.

**Proof.** Consider the set  $\mathfrak{N} \subseteq V(S^k(D)) = V(Q^k(D))$  defined in the proof of Theorem 2.1. Since  $S^k(D)$  is a spanning subdigraph of  $Q^k(D)$  and  $\mathfrak{N}$  is a  $(k - 1)$ -absorbent set of vertices of  $S^k(D)$ , it follows that  $\mathfrak{N}$  is a  $(k - 1)$ -absorbent set of vertices of  $Q^k(D)$ .

The proof that  $\mathfrak{N}$  is  $k$ -independent in  $Q^k(D)$  is the same as the proof that  $\mathfrak{N}$  is  $k$ -independent in  $S^k(D)$ , we only need to recall Observations 1 and 2 given along this proof. ■

**Theorem 2.3.** *Let  $D$  be any digraph and for any integer  $k$  ( $k \geq 2$ ), then the digraph  $R^k(D)$  has a  $k$ -kernel.*

**Proof.** Let  $D, k$  and  $R^k(D)$  be as in the hypothesis. For each  $a = (u, v) \in A(D)$  we define  $\mathfrak{N}_a$  as follows:  $\mathfrak{N}_a$  is the unique  $k$ -kernel of  $(\beta_a - \{u\}) \cup \{(a = (u, v), v)\}$  whenever  $\delta_D^+(v) = 0$ . And  $\mathfrak{N}_a = \{a_i \in V(\beta_a) | i \equiv 1 \pmod{k}\}$  whenever  $\delta_D^+(v) > 0$ . We write  $B^0 = \{x \in V(D) | \delta_D^+(x) = \delta_D^-(x) = 0\}$ . We will prove that  $\mathfrak{N} = \bigcup_{a \in A(D)} \mathfrak{N}_a \cup B^0$  is a  $k$ -kernel of  $R^k(D)$ . First, observe that  $V^0(D) \subseteq \mathfrak{N}$ .

**Claim 3.**  $\mathfrak{N}$  is a  $k$ -independent set of  $R^k(D)$ .

Let  $x, y \in \mathfrak{N}$  with  $x \neq y$ . We will prove that  $d_{R^k(D)}(x, y) \geq k$  and  $d_{R^k(D)}(y, x) \geq k$ . Observe that if  $x \in B^0$ , then  $d_{R^k(D)}(x, y) = d_{R^k(D)}(y, x) = \infty \quad \forall y \in V(R^k(D))$ .

*Case 1.* There exists  $c = (u, v) \in A(D)$  such that  $\{x, y\} \subseteq \mathfrak{N}_c$ .

*Case 1.1.*  $\delta_D^+(v) = 0$ . In this case, we have  $\mathfrak{N}_c = \{c_i \in V(\beta_c) | i \equiv 0 \pmod{k}, i > 0\} \cup \{v\}$ .

We assume without loss of generality that  $x = c_{mk}$  with  $1 \leq m \leq n(c)$  and,  $y = c_{tk}$  with  $m < t$  or  $y = v$ .

When  $y = c_{tk}$ , we have  $d_{R^k(D)}(x, y) = (t - m)k \geq k$ . When  $y = v$ , we have  $d_{R^k(D)}(x, v) = d_{R^k(D)}(x, c) + d_{R^k(D)}(c, v)$ . Since  $d_{R^k(D)}(x, c) \geq k - 1$  and  $d_{R^k(D)}(c, v) = 1$ , we conclude  $d_{R^k(D)}(x, v) \geq k$ .

Now, from the definition of  $R^k(D)$  we have:  $d_{R^k(D)}(y, x) = d_{R^k(D)}(y, v) + d_{R^k(D)}(v, x)$ . Since  $\delta_D^+(v) = 0$  we have  $d_{R^k(D)}(v, x) = \infty$ . Thus  $d_{R^k(D)}(y, x) \geq k$ .

*Case 1.2.*  $\delta_D^+(v) > 0$ . In this case we have  $\mathfrak{N}_c = \{c_i \in V(\beta_c) | i \equiv 1 \pmod{k}\}$ . We assume without loss of generality that  $x = c_{mk+1}$ ,  $y = c_{tk+1}$  with  $0 \leq m < t$ . Clearly,  $d_{R^k(D)}(x, y) = (t - m)k \geq k$  and  $d_{R^k(D)}(y, x) = d_{R^k(D)}(y, c) + d_{R^k(D)}(c, v) + d_{R^k(D)}(v, x)$ . From the definition of  $R^k(D)$  we have  $d_{R^k(D)}(y, c) \geq k - 2$ ,  $d_{R^k(D)}(c, v) = 1$  and  $d_{R^k(D)}(v, x) \geq 1$  (because  $v \neq x$ , be as  $m < t$ ). Thus  $d_{R^k(D)}(y, x) \geq k$ .

*Case 2.*  $x \in \mathfrak{N}_b$  and  $y \in \mathfrak{N}_c$  with  $b = (u, v)$ ,  $c = (w, z)$ ,  $b \neq c$ .

From the definition of  $R^k(D)$  we have  $d_{R^k(D)}(x, y) = d_{R^k(D)}(x, b = (u, v)) + d_{R^k(D)}(b, v) + d_{R^k(D)}(v, w) + d_{R^k(D)}(w, y)$ . When  $\delta_D^+(v) = 0$ , we obtain  $d_{R^k(D)}(v, w) = \infty$  and then  $d_{R^k(D)}(x, y) \geq k$ .

When  $\delta_D^+(v) > 0$ , we obtain  $\mathfrak{N}_b = \{b_i \in V(\beta_b) | i \equiv 1 \pmod{k}\}$  and  $d_{R^k(D)}(x, b) \geq k - 2$  also from the definition of  $R^k(D)$ ,  $d_{R^k(D)}(b, v) = 1$ . If  $v \neq w$ , then  $d_{R^k(D)}(v, w) \geq 1$  and we conclude that  $d_{R^k(D)}(x, y) \geq k$ . If  $v = w$ , then  $\delta_D^+(w) > 0$ ,  $w \notin \mathfrak{N}_c$  and  $w \neq y$ ; therefore  $d_{R^k(D)}(w, y) \geq 1$ , and we conclude again that  $d_{R^k(D)}(x, y) \geq k$ .

Analogously, it can be proved  $d_{R^k(D)}(y, x) \geq k$ .

**Claim 4.**  $\mathfrak{N}$  is a  $(k - 1)$ -absorbent set of vertices of  $R^k(D)$ .

We will prove that for any  $z \in V(R^k(D) - \mathfrak{N})$  there exists  $w \in \mathfrak{N}$  such that  $d_{R^k(D)}(z, w) \leq k - 1$ .

Let  $z \in V(R^k(D) - \mathfrak{N})$ . We have observed that  $V^0(D) \subseteq \mathfrak{N}$ . Thus  $z \in \bigcup_{a \in A(D)} V(\beta_a)$ . Take  $c = (u, v) \in A(D)$  such that  $z \in V(\beta_c)$ .

*Case 1.*  $\delta_D^+(v) = 0$ . In this case,  $\mathfrak{N}_c = \{c_i \in V(\beta_c) | i \equiv 0 \pmod{k}, i \geq 1\} \cup \{v\}$ . Since  $z \notin \mathfrak{N}$ , then  $z = c_0$  or  $z = c_{mk+j}$  with  $1 \leq j \leq k - 1$  and  $0 \leq m \leq n(c)$ .

If  $z = c_0 = u$ , then from the definition of  $R^k(D)$  we have  $(z = u, v) \in A(R^k(D))$  and  $d_{R^k(D)}(z, v) = 1 \leq k - 1$  with  $v \in \mathfrak{N}$ . If  $z = c_{mk+j}$ , then  $d_{R^k(D)}(c_{mk+j}, c_{(m+1)k}) = k - j \leq k - 1$  whenever  $m \neq n(c)$ , and  $d_{R^k(D)}(z, v) \leq d_{R^k(D)}(z, c = (u, v)) + d_{R^k(D)}(c = (u, v), v) \leq k - 2 + 1 = k - 1$  whenever  $m = n(c)$  (recall that  $z = c_{mk+j}$ ,  $c = c_{n(k)+(k-1)}$  and  $d_{R^k(D)}(c = (u, v), v) = 1$ ).

*Case 2.*  $\delta_D^+(v) > 0$ . In this case,  $\mathfrak{N}_c = \{c_i \in V(\beta_c) | i \equiv 1 \pmod{k}\}$ . When  $z \in V(\beta_c) - \{c_i | n(c)k + 2 \leq i \leq n(c)k + (k - 1)\}$ , we have two possibilities: If  $z = c_0$ , then  $d_{R^k(D)}(z, c_1) = 1 \leq k - 1$  with  $c_1 \in \mathfrak{N}_c \subseteq \mathfrak{N}$ . If  $z \neq c_0$ , then  $z = c_{mk+j}$  with  $2 \leq j \leq k$ ,  $0 \leq m < n(c)$  and  $d_{R^k(D)}(z, c_{(m+1)k+1}) \leq k - 1$  with  $c_{(m+1)k+1} \in \mathfrak{N}$ .

When  $z \in \{c_i | n(c)k + 2 \leq i \leq n(c)k + (k - 1)\}$ , we recall that  $\delta_D^+(v) > 0$ . Thus there exists  $b = (v, w) \in A(D)$ . We consider  $\beta_b$ . Consider two possibilities: If  $\delta_{R^k(D)}^+(w) > 0$ , then  $\mathfrak{N}_b = \{b_i \in V(\beta_b) | i \equiv 1 \pmod{k}\}$ ; and it follows that  $d_{R^k(D)}(z, b_1) = d_{R^k(D)}(z, c) + d_{R^k(D)}(c, v) + d_{R^k(D)}(v, b_1) \leq k - 3 + 1 + 1 = k - 1$  with  $b_1 \in \mathfrak{N}$ . If  $\delta_{R^k(D)}^+(w) = 0$ , then  $w \in \mathfrak{N}$ , and  $d_{R^k(D)}(z, w) = d_{R^k(D)}(z, c) + d_{R^k(D)}(c, v) + d_{R^k(D)}(v, w) \leq k - 3 + 1 + 1 = k - 1$ . ■

**Theorem 2.4.** *For any digraph  $D$  and for any integer  $k$  ( $k \geq 3$ ), the digraph  $T^k(D)$  has a  $k$ -kernel.*



**Proof.** Let  $k, D$  and  $T^k(D)$  be as in the hypothesis. For each  $a = (u, v) \in A(D)$  we define  $\mathfrak{N}_a$  as follows: If  $\delta_D^+(v) = 0$ , then  $\mathfrak{N}_a$  is the  $k$ -kernel of  $(\beta_a - \{u\}) \cup \{v, a = (u, v)\}$ , i.e.,  $\mathfrak{N}_a = \{a_i | 1 \leq i, i \equiv 0(\text{mod } k)\} \cup \{v\}$ . If  $\delta_D^+(v) > 0$ , then  $\mathfrak{N}_a = \{a_i | i \equiv 1(\text{mod } k)\}$ . We write  $B^0 = \{x \in V(D) | \delta_D^+(x) = \delta_D^-(x) = 0\}$ . We will prove that  $\mathfrak{N} = \bigcup_{a \in A(D)} \mathfrak{N}_a \cup B^0$  is a kernel of  $T^k(D)$ . Observe that  $V^0(D) \subseteq \mathfrak{N}$ .

**Observation 3.** Notice that since  $k \geq 3$ , we have  $a_{n(a)k+1} \neq a = a_{n(a)k+(k-1)}$ , therefore  $a \notin \mathfrak{N}$ , for each  $a \in A(D)$ .

**Claim 5.**  $\mathfrak{N}$  is a  $k$ -independent set of vertices of  $T^k(D)$ .

Let  $x, y \in \mathfrak{N}$  with  $x \neq y$ . We will prove that  $d_{T^k(D)}(x, y) \geq k$  and  $d_{T^k(D)}(y, x) \geq k$ . Observe that if  $x \in B^0$ , then  $d_{T^k(D)}(x, y) = d_{T^k(D)}(y, x) = \infty$  for each  $y \in V(T^k(D))$ .

*Case 1.* There exists  $c = (u, v) \in A(D)$  such that  $\{x, y\} \subseteq \mathfrak{N}_c$ .

*Case 1.1.*  $\delta_D^+(v) = 0$ . In this case,  $\mathfrak{N}_c = \{c_i | 1 \leq i, i \equiv 0(\text{mod } k)\} \cup \{v\}$ . Clearly, we may assume  $x = c_{mk}$  with  $1 \leq m \leq n(c)$  and  $y = c_{tk}$  with  $1 \leq t \leq n(c)$  and  $m < t$  or  $y = v$ .

If  $y = c_{tk}$ , then  $d_{T^k(D)}(x, y) = (t-m)k \geq k$ . If  $y = v$ , then  $d_{T^k(D)}(x, y) = d_{\beta_c}(x, c) + d_{T^k(D)}(c, v) \geq k - 1 + 1 = k$ .

Now from the definition of  $T^k(D)$ , we have  $d_{T^k(D)}(y, x) = d_{T^k(D)}(y, c) + d_{T^k(D)}(c, x)$ .

If  $y \neq v$ , then  $d_{T^k(D)}(y, c) = d_{\beta_c}(y, c) \geq k - 1$ . From Observation 3  $c \neq x$ , so  $d_{T^k(D)}(c, x) \geq 1$  and we conclude that  $d_{T^k(D)}(y, x) \geq k$ .

If  $y = v$ , then  $d_{T^k(D)}(y, x) = \infty$ , as  $\delta_D^+(v) = 0$ .

*Case 1.2.*  $\delta_D^+(v) > 0$ . In this case,  $\mathfrak{N}_c = \{c_i \in \beta_c | i \equiv 1(\text{mod } k)\}$  and clearly, we may assume  $x = c_{mk+1}$ ,  $y = c_{tk+1}$  with  $0 \leq m < t \leq n(c)$ . Therefore  $d_{T^k(D)}(x, y) = (t-m)k \geq k$ . Now from the definition of  $T^k(D)$  we have  $d_{T^k(D)}(y, x) = d_{T^k(D)}(y, c) + d_{T^k(D)}(c, x)$ . Clearly,  $d_{T^k(D)}(y, c) \geq k - 2$ .

Since  $c \in A(D)$ ,  $c = (u, v)$  and  $x \neq v$ , we have  $(c, x) \notin A(T^k(D))$  (recall the definition of  $T^k(D)$ ).

Hence  $d_{T^k(D)}(c, x) \geq 2$ . We conclude that  $d_{T^k(D)}(y, x) \geq k$ .

*Case 2.*  $x \in \mathfrak{N}_b$  and  $y \in \mathfrak{N}_c$  for  $b = (u, v)$ ,  $c = (w, z)$  with  $\{b, c\} \subseteq A(D)$ ,  $b \neq c$ . From the definition of  $T^k(D)$  we have  $d_{T^k(D)}(x, y) = d_{T^k(D)}(x, b) + d_{T^k(D)}(b, y)$ .

*Case 2.1.*  $\delta_D^+(v) = 0$ . In this case,  $x = b_{mk}$  with  $1 \leq m \leq n(b)$  or  $x = v$ . If  $x = b_{mk}$ , then  $d_{T^k(D)}(x, b) \geq k - 1$ ; and from Observation 3  $b \neq y$  which implies  $d_{T^k(D)}(b, y) \geq 1$ . We conclude that  $d_{T^k(D)}(x, y) \geq k$ . If  $x = v$ , then  $d_{T^k(D)}(x, y) = \infty$  (as  $\delta_D^+(v) = \delta_{T^k(D)}^+(v) = 0$ ).

*Case 2.2.*  $\delta_D^+(v) > 0$ . In this case,  $\mathfrak{N}_b = \{b_i \in V(\beta_b) \mid i \equiv 1 \pmod{k}\}$ . From the definition of  $T^k(D)$  we have  $d_{T^k(D)}(x, y) = d_{T^k(D)}(x, b) + d_{T^k(D)}(b, y)$ . Clearly,  $d_{T^k(D)}(x, b) \geq k - 2$ . Since  $b \notin \mathfrak{N}$  (from Observation 3) and  $y \in \mathfrak{N}$ , then  $y \neq b$ . Moreover,  $k \geq 3$  implies  $n(b)k + 1 \neq n(b)k + (k - 1)$  and  $y \neq v$ . Finally,  $d(b, y) = 1$  implies  $y \in A(D)$  and by Observation 3 also  $y \notin \mathfrak{N}$ , a contradiction. Therefore  $d_{T^k(D)}(b, y) \geq 2$ . We conclude that  $d_{T^k(D)}(x, y) \geq k$ . Analogously, it can be proved that  $d_{T^k(D)}(y, x) \geq k$ .

**Claim 6.**  $\mathfrak{N}$  is a  $(k - 1)$ -absorbent set of vertices of  $T^k(D)$ .

Clearly,  $R^k(D)$  is a spanning subdigraph of  $T^k(D)$  and we have proved (Theorem 2.3) that  $\mathfrak{N}$  is a  $k$ -kernel of  $R^k(D)$ , in particular  $\mathfrak{N}$  is a  $(k - 1)$ -absorbent set of vertices of  $R^k(D)$ . Thus  $\mathfrak{N}$  is a  $(k - 1)$ -absorbent set of vertices of  $T^k(D)$ . ■

Observe that the set of black vertices in Figs. 1 and 2 is a 3-kernel.

**Remark 2.1.** It is easy to prove that for  $D = \vec{C}_4$  (the directed cycle of length 4) and  $k = 2$ , the  $k$ -total digraph of  $D$ ,  $T^k(D)$  has no  $k$ -kernel. Thus the assertion given in Theorem 2.4 cannot be improved.

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### REFERENCES

- [1] C. Berge, *Graphs* (North-Holland Mathematical Library, **6** North Holland, Amsterdam, 1985).
- [2] P. Duchet, *Graphes noyau-parfaits*, Ann. Discrete Math. **9** (1980) 93–101.
- [3] P. Duchet, *A sufficient condition for a digraph to be kernel-perfect*, J. Graph Theory **11** (1987) 81–85.
- [4] P. Duchet and H. Meyniel, *A note on kernel-critical graphs*, Discrete Math. **33** (1981) 103–105.
- [5] H. Galeana-Sánchez, *On the existence of  $(k, \ell)$ -kernels in digraphs*, Discrete Math. **85** (1990) 99–102.

- [6] H. Galeana-Sánchez, *On the existence of kernels and  $h$ -kernels in directed graphs*, Discrete Math. **110** (1992) 251–255.
- [7] H. Galeana-Sánchez and V. Neumann-Lara, *On kernel-perfect critical digraphs*, Discrete Math. **59** (1986) 257–265.
- [8] H. Galeana-Sánchez and V. Neumann-Lara, *Extending kernel perfect digraphs to kernel perfect critical digraphs*, Discrete Math. **94** (1991) 181–187.
- [9] M. Harminc, *On the  $(m, n)$ -basis of a digraph*, Math. Slovaca **30** (1980) 401–404.
- [10] M. Harminc, *Solutions and kernels of a directed graph*, Math. Slovaca **32** (1982) 262–267.
- [11] M. Harminc and T. Olejníková, *Binary operations on directed graphs and their solutions* (Slovak with English and Russian summaries), Zb. Ved. Pr. VŠT, Košice (1984) 29–42.
- [12] M. Kucharska, *On  $(k, \ell)$ -kernels of orientations of special graphs*, Ars Combinatoria **60** (2001) 137–147.
- [13] M. Kucharska and M. Kwaśnik, *On  $(k, \ell)$ -kernels of superdigraphs of  $P_m$  and  $C_m$* , Discuss. Math. Graph Theory **21** (2001) 95–109.
- [14] M. Kwaśnik, *The generalization of Richardson theorem*, Discuss. Math. **IV** (1981) 11–13.
- [15] M. Kwaśnik, *On  $(k, \ell)$ -kernels of exclusive disjunction, Cartesian sum and normal product of two directed graphs*, Discuss. Math. **V** (1982) 29–34.
- [16] M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel, *Some operations preserving the existence of kernels*, Discrete Math. **205** (1999) 211–216.
- [17] M. Richardson, *Extensions theorems for solutions of irreflexive relations*, Proc. Mat. Acad. Sci. **39** (1953) 649–655.
- [18] M. Richardson, *Solutions of irreflexive relations*, Ann. Math. **58** (1953) 573–590.
- [19] J. Topp, *Kernels of digraphs formed by some unary operations from other digraphs*, J. Rostock Math. Kolloq. **21** (1982) 73–81.
- [20] J. von Neumann and O. Morgenstern, *Theory of games and economic behavior* (Princeton University Press, Princeton, 1944).
- [21] A. Włoch and I. Włoch, *On  $(k, l)$ -kernels in generalized products*, Discrete Math. **164** (1997) 295–301.

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