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Note

TRIANGLE-FREE PLANAR GRAPHS WITH MINIMUM DEGREE 3 HAVE RADIUS AT LEAST 3

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Abstract

We prove that every triangle-free planar graph with minimum degree 3 has radius at least 3; equivalently, no vertex neighborhood is a dominating set.

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In 1975, Plesník [3] determined all triangle-free planar graphs with diameter 2. They are the stars, the complete bipartite graphs $K_{2,n}$, and a third family that can be described in several ways. One can start with the disjoint union $K_2 + K_1$ and add vertices of degree 2 joined to either nonadjacent pair of the original triple, or start with C_5 and expand two nonadjacent vertices into larger independent sets, or start with $K_{2,n}$ and apply a "vertex split" to one of the high-degree vertices.

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Each graph in Plesník's characterization has a vertex of degree at most 2. Thus his result implies that every triangle-free planar graph with minimum degree 3 has diameter at least 3 (note that no triangle-free planar graph has minimum degree greater than 3). In this note, we strengthen this statement by proving that every triangle-free graph with minimum degree 3 has radius at least 3. That is, it has no vertex whose neighborhood is a dominating set. There are many triangle-free planar graphs with minimun degree 3 and radius equal to 3.

Our result can also be related to other past work about distances in triangle-free or planar graphs. Erdős, Pach, Pollack, and Tuza [1] studied the maximum radius and diameter among graphs with fixed minimum degree. They also solved these problems in the family of triangle-free graphs. In contrast, we are seeking the minimum radius when the family is further restricted to planar graphs.

For planar graphs, Harant [2] proved an upper bound on the radius when the graph is 3-connected and has no long faces (it is $n/6 + q + \frac{3}{2}$ when the graph has n vertices and no face of length more than q). We prove a lower bound on the radius when the graph has no short faces (no triangles), without restriction on connectivity.

We use $\delta(G)$ to denote the minimum degree of G, and we write $[v_1, \ldots, v_k]$ to denote a cycle with vertices $v_1, \ldots, v_k$ in order. Our graphs have no loops or multiple edges. A vertex *dominates* (is adjacent to) any subset of its neighbors.

Theorem 1. *Every triangle-free planar graph with minimum degree* 3 *has radius at least* 3.

Proof. If the radius is 1, then one vertex dominates all others; additional edges would create triangles, so the other vertices cannot reach degree 3. Hence it suffices to forbid radius 2. We assume that our graph G has a vertex v whose neighborhood U dominates the remaining vertices. Let $W = V(G) - U - \{v\}$.

If v lies on no cycle, then each component of $G - v$ is dominated by one vertex of U, which cannot happen since G is triangle-free and $\delta(G) = 3$. If v lies on no cycle of length at most 5, then the shortest path in $G - v$ between any two vertices of U has length at least 4, and the center of such a path is undominated by U.

Fix a planar embedding of G. Define a *trap* to be a cycle of length at most 5 through v. Say that a cycle in G is *empty* if no vertex lies inside the region enclosed by it. Let a *flap* in an embedding of G be the subgraph induced by a nonempty trap and the vertices inside it. If a trap C is empty, then we redraw G so that C is the external face, and now G itself is a flap. Hence a flap exists in some embedding of G.

We obtain a contradiction by proving that every flap P in an embedding of G contains another flap; this contradicts the finiteness of G. The cases appear in Figure 1.

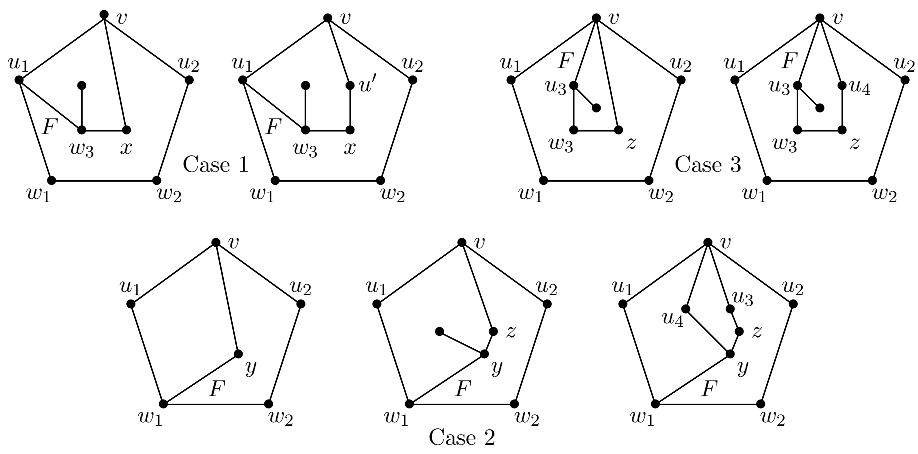

Figure 1. Cases for the proof.

Let C be the external cycle in P. Let u_1 and u_2 be the neighbors of v on C (in U). Let w_1 and w_2 be their neighbors on C other than v, respectively, where $w_1 = w_2$ if C has length 4. Note that $w_1, w_2 \in W$, since U is independent. Let S be the set of vertices of P not on C; call them the *internal* vertices. If $|S| \leq 2$, then $\delta(G) \geq 3$ forces a triangle, since neighbors of adjacent vertices in S cannot alternate on C. Hence we have $|S| \geq 3$.

Case 1. *u_1 or u_2 has an internal neighbor.*
Let u_1 have an internal neighbor. Let w_3 be the internal neighbor of u_1 on the bounded face F of P that contains w_1 and u_1, and let x be the next vertex reached in following F. If $x \in U$, then $[v, u_1, w_3, x]$ is a trap that encloses a smaller flap than P, since w_3 has a third neighbor inside that trap. If $x \notin U$, then x has a neighbor $u' \in U$, and now $[v, u_1, w_3, x, u']$ encloses a smaller flap.

Case 2. *u_1 and u_2 have no internal neighbors, but w_1 or w_2 does.* By symmetry, we may assume that w_1 has an internal neighbor. Let y be the internal neighbor of w_1 following w_1 on the bounded face F of P that contains w_2 and w_1.

If $y \in U$, then there are two cycles formed by v, y, and part of C. Whichever encloses a neighbor of y encloses a smaller flap.

If $y \in W$, then let z be the next vertex after y in following F. If $z \in U$, then the cycle $[v, u_1, w_1, y, z]$ encloses a neighbor of y and yields a smaller flap. If $z \in W$, then let u_3 be a neighbor of z in U. Now $[v, u_1, w_1, y, z, u_3]$ encloses the remaining neighbors of y, which must include a vertex u_4 in U. Since u_4 must have another neighbor in the region enclosed by the 6-cycle, $[v, u_1, w_1, y, u_4]$ or $[v, u_4, y, z, u_3]$ is a trap enclosing a smaller flap.

Case 3. *None of $\{u_1, w_1, w_2, u_2\}$ has an internal neighbor.* Since the interior is nonempty and G is connected, v has an internal neighbor. Let u_3 be the one reached after u_1 and v when following the face F of P whose boundary contains all of C. Let w_3 be the vertex after u_3 on F (since U is independent, $w_3 \in W$), and let z be the vertex after w_3. If $z \in W$, then we can choose $u_4 \in N(z) \cap U - \{u_3\}$. Otherwise, $z \in U$. In the two cases, $[v, u_3, w_3, z, u_4]$ or $[v, u_3, w_3, z]$ encloses another neighbor of u_3 and yields a smaller flap. ■

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