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# EMBEDDING COMPLETE TERNARY TREES INTO HYPERCUBES

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**Abstract**

We inductively describe an embedding of a complete ternary tree $T_h$ of height $h$ into a hypercube $Q$ of dimension at most $\lceil (1.6)h \rceil + 1$ with load 1, dilation 2, node congestion 2 and edge congestion 2. This is an improvement over the known embedding of $T_h$ into $Q$. And it is very close to a conjectured embedding of Havel [3] which states that there exists an embedding of $T_h$ into its optimal hypercube with load 1 and dilation 2. The optimal hypercube has dimension $\lceil (\log_2 3)h \rceil$ $(= \lceil (1.585)h \rceil)$ or $\lceil (\log_2 3)h \rceil + 1$.

**Keywords:** complete ternary trees, hypercube, interconnection network, embedding, dilation, node congestion, edge congestion.

**2000 Mathematics Subject Classification:** 05C05, 05C90, 68E10.

## 1. Introduction

Graph embeddings constitute a central topic in the area of parallel and distributed computing; see [5, 6, 8]. They are natural mathematical models capturing the issues involved in the design of parallel algorithms. We assume that the reader is familiar with the terminology associated with graph embeddings. However, since the terminology is varied in the literature, we recall the most general definition of a graph embedding to avoid any confusion.

Let $G$ and $H$ be any two graphs and let $\wp(H)$ denote the set of all paths in $H$. An *embedding* of a guest graph $G(V, E)$ into a host graph $H(W, F)$

is a pair of functions $(f, \rho)$ where $f : V \to W$ and $\rho : E \to \wp(H)$ such that $\rho$ maps an edge $uv$ of $G$ to a path connecting $f(u)$ and $f(v)$ in $H$. The parameters load, dilation, node congestion, and edge congestion are associated with such an embedding $(f, \rho)$ to measure its qualities.

The *load* of a node $v \in V(H)$ is the number of nodes of $V(G)$ that are mapped onto $v$ by $f$; the *load* of $(f, \rho)$ is the maximum load over all nodes of $H$. Note that, if $f$ is an injective map, then the load is 1.

The *dilation* of an edge $e(uv)$ in $G$ is the length of the path $\rho(e)$. The *dilation* of $(f, \rho)$ is the maximum dilation over all edges of $G$. Note that, if there exists a load 1 and dilation 1 embedding of $G$ into $H$, then $G$ is isomorphic to a subgraph of $H$.

The *congestion of an edge* $e' \in H$ is the number of edges $e \in E(G)$ such that the path $\rho(e)$ contains $e'$. The *edge congestion* of $(f, \rho)$ is the maximum congestion over all edges of $H$.

The *congestion of a node* $v \in H$ is the number of edges $e \in E(G)$ such that $v$ is an internal vertex of the path $\rho(e)$. The *node congestion* of $(f, \rho)$ is the maximum congestion over all nodes of $H$.

All embeddings discussed in this paper have load 1, that is $f$ is an injection. Also, we map every edge $e(uv)$ onto a shortest $(f(u), f(v))$-path. Nevertheless, it is still important which paths we choose, since we are interested in obtaining an embedding with node congestion 2 and edge congestion 2.

There is a vast body of work on embedding of various kinds of trees into hypercubes. In particular, complete $k$-ary trees have received special attention as they represent algorithms that employ divide-and-conquer strategy.

A *complete $k$-ary tree* of height $h$, is a rooted tree in which each internal vertex has exactly $k$ children and the distance from the root to each leaf is exactly $h$.

For $n \geq 1$, the *$n$-dimensional hypercube* (*or $n$-cube*), $Q_n$, is the graph whose vertex set is the set of binary strings $V(Q_n) := \{X := x_1x_2\ldots x_n : x_i \in \{0,1\}, 1 \leq i \leq n\}$ and edge set $E(Q_n) := \{XY : X \text{ and } Y \text{ differ in exactly one position }\}$. Alternatively, hypercubes are recursively defined through the cartesian product $(\times)$ of graphs as $Q_1 = K_2$, and for $n \geq 2$, $Q_n = Q_{n-1} \times K_2$. This definition permits a decomposition of $Q_n$ into two copies of $Q_{n-1}$, say $Q^0_{n-1}$ and $Q^1_{n-1}$ as follows: $V(Q^0_{n-1}) = \{X \in V(Q_n) : X = 0x_2\ldots x_n\}$ and $V(Q^1_{n-1}) = \{X \in V(Q_n) : X = 1x_2\ldots x_n\}$. Any vertex $0x_2\ldots x_n \in V(Q^0_{n-1})$ is adjacent to a unique vertex $1x_2\ldots x_n \in V(Q^1_{n-1})$. Similarly, we can further decompose $Q^0_{n-1}$ and $Q^1_{n-1}$ and obtain four copies $Q^{00}_{n-2}$, $Q^{01}_{n-2}$, $Q^{10}_{n-2}$ and $Q^{11}_{n-2}$ of $Q_{n-2}$.

Given a tree $T$, let $n$ be the smallest integer such that $2^n \geq |V(T)|$. Then $Q_n$ is called the *optimal hypercube* of $T$, and $Q_{n+1}$ is called the *next-to-optimal hypercube*.

Henceforth, $T_h$ will denote a complete 3-ary tree (that is, a ternary tree) of height $h$. The root of $T_h$ will be denoted by $R_h$. The three children of the root $R_h$, namely the left child, the middle child and the right child will be denoted by $c_l$, $c_m$ and $c_r$, respectively. Since $T_h$ has $(3^{h+1} - 1)/2$ vertices, it follows that its optimal hypercube has dimension $\lceil(\log_2 3)h\rceil (\approx \lceil(1.585)h\rceil)$ or $\lceil(\log_2 3)h\rceil + 1$. The following conjecture is open since 1990.

**Conjecture 1.1** (Havel [3])**.** Any complete ternary tree of height $h$ can be embedded with load 1 and dilation 2 into its optimal hypercube.

The following result achieves the smallest dilation, node congestion, and edge congestion known so far for embedding $T_h$ into a $\lceil(1.6)h\rceil + 1$-dimensional hypercube.

**Theorem 1.2** (Gupta *et al.* [2])**.** *Any complete ternary tree of height $h$ can be embedded with load 1, dilation 3 and edge congestion 3 into $Q_{d(h)}$, where $d(h) = \lceil(1.6)h\rceil + 1$.*

The related results on embedding of ternary trees into hypercubes can be found in many papers [1, 4, 7, 8].

In this paper, we prove the following improvement of the above theorem.

**Theorem 1.3.** *Any complete ternary tree of height $h$ can be embedded with load 1, dilation 2, edge congestion 2 and node congestion 2 into $Q_{d(h)}$, where*

$$d(h) = \begin{cases} \lceil(1.6)h\rceil, & \text{if } h \equiv 2 \pmod 5 \text{ or } h \equiv 4 \pmod 5, \\ \lceil(1.6)h\rceil + 1, & \text{if } h \equiv 0 \pmod 5 \text{ or } h \equiv 1 \pmod 5 \text{ or } h \equiv 3 \pmod 5. \end{cases}$$

## 2. EMBEDDING OF COMPLETE TERNARY TREES

Let $\tau_h$ denote the tree obtained from $T_h$ by adding a new vertex $D$ and joining it to the root $R_h$ of $T_h$. Here, we call the vertex $D$ as the *deep root* of the tree $\tau_h$. If a tree $\tau_h$ is embeddable into a hypercube $Q$ with load 1, dilation 2, node congestion 2 and edge congestion 2, such that the root $R_h$ and the deep root $D$ are mapped onto adjacent nodes and the edge $(f(R_h), f(D)) \in Q$ has congestion 1, then we write $\tau_h \hookrightarrow Q$. We first prove the following result which implies our main result, Theorem 1.3, by induction on $h$, since $T_h$ is a subgraph of $\tau_h$. Its proof technique is a refinement of the technique employed in [2].

**Theorem 2.1.** *If* $\tau_h \hookrightarrow Q_d$*, then* $\tau_{h+5} \hookrightarrow Q_{d+8}$.

***Proof.*** We prove the result by describing the following five embeddings.

(i) $\tau_{h+1} \hookrightarrow Q_{d+2}$,

(ii) $\tau_{h+2} \hookrightarrow Q_{d+4}$,

(iii) $\tau_{h+3} \hookrightarrow Q_{d+5}$,

(iv) $\tau_{h+4} \hookrightarrow Q_{d+7}$ and

(v) $\tau_{h+5} \hookrightarrow Q_{d+8}$.

<u>(i) $\tau_{\mathbf{h+1}} \hookrightarrow \mathbf{Q_{d+2}}$</u>: The embedding $(f, \rho)$ of $\tau_{h+1}$ in $Q_{d+2}$ is schematically shown in Figure 1.

To obtain this embedding, we first decompose $Q_{d+2}$ into four copies of $Q_d$. For $i, j \in \{0, 1\}$, we have a copy of $\tau_h$ in $Q_d^{ij}$ denoted by $ij\tau_h$; see Figure 1(b). So, we suitably combine these embeddings, as shown in Figure 1(c), to obtain an embedding of $\tau_{h+1}$, with $10D$ as its deep root and $00D$ as its root. The embedding maps the three children $c_l$, $c_m$, $c_r$ of the root $R_{h+1}$ in $T_{h+1}$ onto the vertices $01R_h$, $00R_h$ and $10R_h$ respectively, with the following properties.

(i) The edges $R_{h+1}c_l$ and $R_{h+1}c_r$ of $T_{h+1}$ are mapped onto the paths $(00D, 01D, 01R_h)$ and $(00D, 00R_h, 10R_h)$ of $Q_{d+2}$, respectively. So, they have dilation 2. The edges $R_{h+1}c_m$ and $R_{h+1}D$ are mapped onto the edges $(00D, 00R_h)$ and $(00D, 10D)$, respectively. So, they have dilation 1. The edge joining the root $R_{h+1}$ and the deep root $D$, (that is, the edge $(00D, 10D)$) has congestion 1. Note that the edge $(00D, 00R_h)$ has congestion 2, since it belongs to $\rho(R_{h+1}c_r)$ and $\rho(R_{h+1}c_m)$.

(ii) The edges of the three trees $T_h$ rooted at $c_l$, $c_m$, $c_r$ in $T_{h+1}$, retain their dilation attained in the embedding $\tau_h \hookrightarrow Q_d$. Every node and every edge of $Q_{d+2}$ retains its congestion attained in the embedding $ij\tau_h \hookrightarrow Q_d^{ij}$, $i, j \in \{0, 1\}$.

Input:

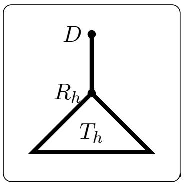


Figure 1(a). $\tau_h \hookrightarrow Q_d$. $D$ and $R_h$ are $d$-bit binary strings.

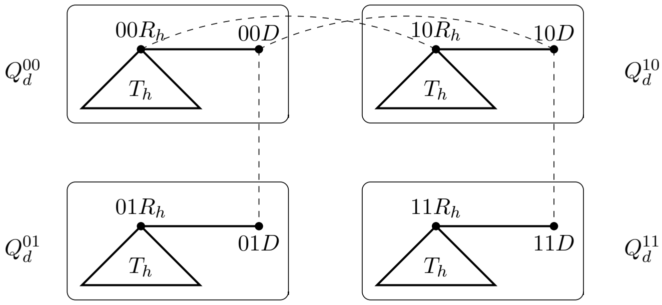


Figure 1(b). $ij\tau_h \hookrightarrow Q_d^{ij}, i, j \in \{0, 1\}$

Embedding:

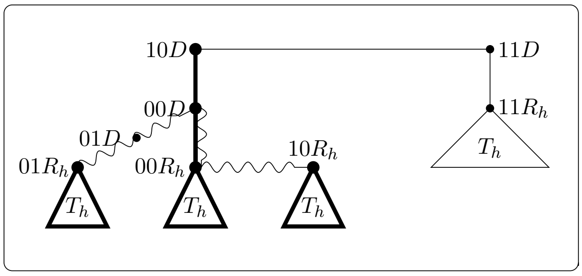


Figure 1(c). The edge $(00D, 00R_h)$ has congestion 2.

Output:

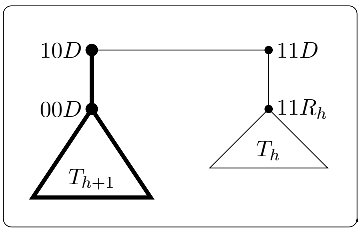


Figure 1(d). $\tau_{h+1} \hookrightarrow Q_{d+2}$. The $\tau_h$ shown on the right with light edges contains unutilized vertices.

Figure 1. The steps involved in an embedding of $\tau_{h+1} \hookrightarrow Q_{d+2}$, with the input $\tau_h \hookrightarrow Q_d$.

(iii) Figure 1(c) also shows a ternary tree $\tau_h$ (with $11D$ as the deep root and $11R_h$ as the root) embedded in $Q_{d+2}$ whose vertices are not yet utilized. These vertices would be used in the subsequent embeddings. Here again, the edge $(11D, 11R_h)$ has congestion 1.

(iv) Every parameter namely, dilation, node congestion and edge congestion of this embedding is bounded by 2.

The output of this step, shown in Figure 1(d), is the input for embedding $\tau_{h+2}$ in $Q_{d+4}$ in the next step.

**(ii) $\tau_{\mathbf{h+2}} \hookrightarrow \mathbf{Q_{d+4}}$:** The embedding $(f, \rho)$ of $\tau_{h+2}$ in $Q_{d+4}$ is described in Figure 2.

Similar to the previous step, we obtain an embedding of $\tau_{h+2}$ with $1010D$ as its deep root and $0010D$ as the root; see Figure 2(b). The embedding maps the three children $c_l$, $c_m$, $c_r$ of the root $R_{h+2}$ in $T_{h+2}$ onto the vertices $0100D$, $0000D$, and $1000D$, respectively. The edges $R_{h+2}c_l$ and $R_{h+2}c_r$ of $T_{h+2}$ have dilation 2. The edges $R_{h+2}c_m$ and $R_{h+2}D$ have dilation 1. The rest of the edges of $T_{h+2}$ retain their dilation attained in the embedding $\tau_{h+1} \hookrightarrow Q_{d+2}$. Note that the node $0110D$ which appears twice (shown inside a circle) and the node $0111D$ which appears twice (shown inside a square) will receive node congestion 2 in the subsequent steps, as the nodes appear in the set of vertices which are yet to be utilized. The edge $\rho(R_{h+2}D) = (0010D, 1010D)$ receives congestion 1. Note also that the edge $(0010D, 0000D)$ has congestion 2, since it belongs to $\rho(R_{h+2}c_r)$ and $\rho(R_{h+2}c_m)$. The remaining nodes and edges of $Q_{d+4}$ retain their congestion attained in the embedding $ij\tau_{h+1} \hookrightarrow Q^{ij}_{d+2}$, for $i, j \in \{0, 1\}$.

Figure 2(b) also shows two copies of $\tau_{h+1}$ (one with $1110D$ as the deep root and $1100D$ as the root $\tau_{h+1}$ and the second with $0111D$ as the deep root and $1111D$ as the root) and a copy of $\tau_h$ (with $0011D$ as the deep root and $0011R_h$ as the root) embedded in $Q_{d+4}$ whose vertices are not yet utilized. In each case, the edge joining the root and the deep root of the tree are mapped onto adjacent nodes and receives congestion 1. Every parameter namely, dilation, node congestion and edge congestion of these embeddings is bounded by 2.

The output of this step, shown in Figure 2(c), is the input for the next step to embed $\tau_{h+3}$ in $Q_{d+5}$.

**(iii) $\tau_{\mathbf{h+3}} \hookrightarrow \mathbf{Q_{d+5}}$:** The embedding $(f, \rho)$ of $\tau_{h+3}$ in $Q_{d+5}$ is described in Figure 3.

Input: The output of the embedding $\tau_{h+1} \hookrightarrow Q_{d+2}$; see Figure 1(d).

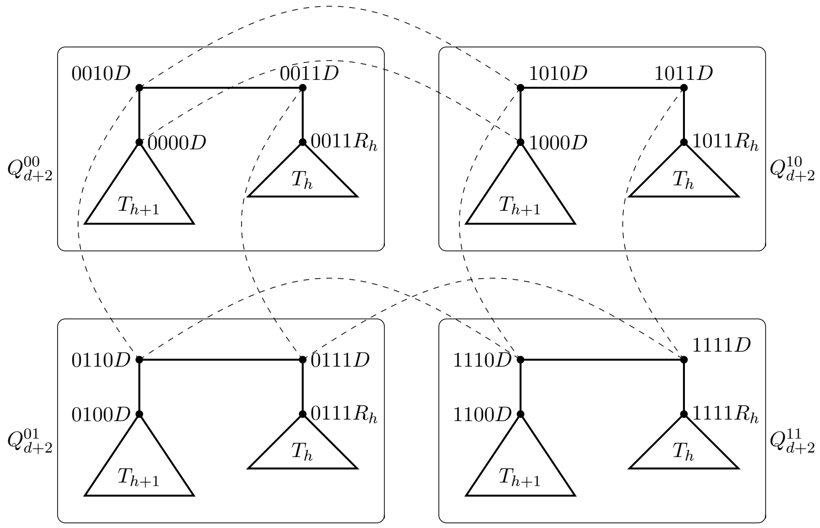


Figure 2(a). $ij\tau_{h+1} \hookrightarrow Q^{ij}_{d+2}, i, j \in \{0, 1\}$

Embedding:

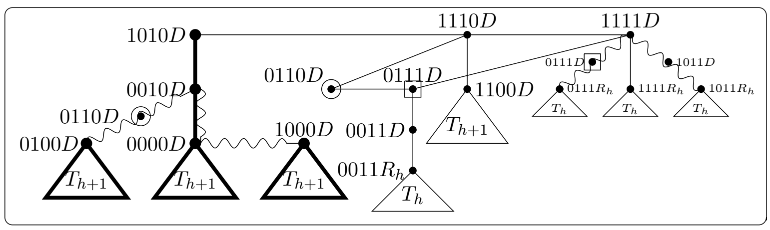


Figure 2(b). The edge $(0010D, 0000D)$ has congestion 2. The nodes $0110D$ and $0111D$ are the candidates to receive node congestion 2 in the subsequent steps.

Output:

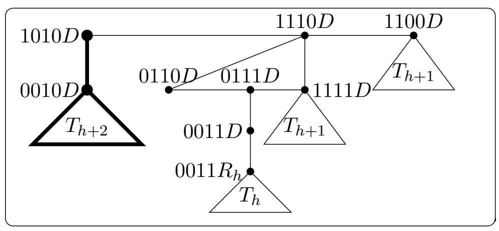


Figure 2(c). $\tau_{h+2} \hookrightarrow Q_{d+4}$. The subgraph shown on the right with light edges contains unutilized vertices.

Figure 2. The steps involved in an embedding of $\tau_{h+2} \hookrightarrow Q_{d+4}$, with the input $\tau_{h+1} \hookrightarrow Q_{d+2}$.

Input: The output of the embedding $\tau_{h+2} \hookrightarrow Q_{d+4}$; see Figure 2(c).

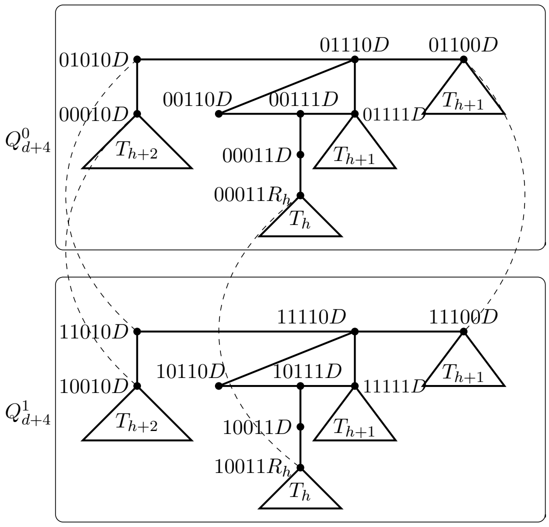


Figure 3(a). $i\tau_{h+2} \hookrightarrow Q^i_{d+4}, i \in \{0,1\}$

Embedding:

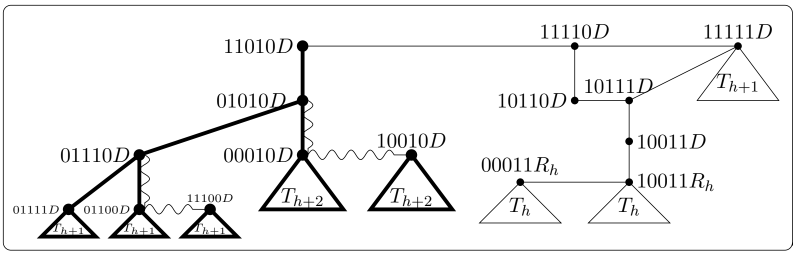


Figure 3(b). The edges $(01110D, 01100D)$ and $(01010D, 00010D)$ have edge congestion 2.

Output:

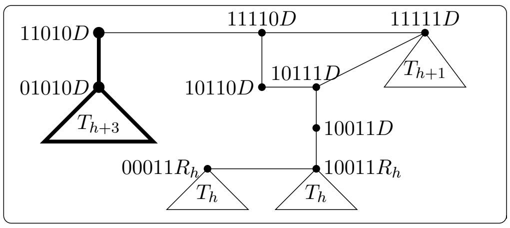


Figure 3(c). $\tau_{h+3} \hookrightarrow Q_{d+5}$. The subgraph shown on the right with light edges contains unutilized vertices.

Figure 3. The steps involved in an embedding of $\tau_{h+3} \hookrightarrow Q_{d+5}$, with the input $\tau_{h+2} \hookrightarrow Q_{d+4}$.

Here, we consider the decomposition of $Q_{d+5}$ into two copies of $Q_{d+4}$ and combine the embeddings such that we obtain an embedding of $\tau_{h+3}$ with $11010D$ as the deep root and $01010D$ as the root; see Figure 3(b). The three children $c_l$, $c_m$ and $c_r$ of the root $R_{h+3}$ are mapped respectively, on to $01110D$, $00010D$ and $10010D$. The edge $R_{h+3}D \in \tau_{h+3}$ (mapped onto the edge $(11010D, 01010D)$) has dilation 1 and the edge $(11010D, 01010D)$ has congestion 1. Each of the parameters namely, dilation, edge congestion and node congestion of the embedding is bounded by 2.

Figure 3(c), also show embeddings of a $T_{h+1}$ (with $11111D$ as the root) and two copies of $T_h$ rooted at $00011R_h$ and $10011R_h$ whose vertices will be utilized in the subsequent steps. In all the embeddings, the edge joining the root and the deep root of the tree are mapped onto the adjacent nodes and they receive edge congestion 1. Every parameter dilation, node congestion and edge congestion of these embeddings is bounded by 2.

Here again, the output of this step, shown in Figure 3(c), is the input for the next step to embed $\tau_{h+4}$ in $Q_{d+7}$.

<u>**(iv)** $\boldsymbol{\tau_{h+4} \hookrightarrow Q_{d+7}}$</u> : The embedding $(f, \rho)$ of $\tau_{h+4}$ in $Q_{d+7}$ is described in Figure 4.

Similar to steps (i) and (ii), we obtain an embedding of $\tau_{h+4}$ with the deep root mapped on to $1011010D$ and the root of the tree $T_{h+4}$ mapped on to $0011010D$; refer to Figure 4(b). The edges $R_{h+4}c_l$ and $R_{h+4}c_r$ have dilation 2. The edges $R_{h+4}c_m$ and $R_{h+4}D$ have dilation 1 and rest of the edges of $T_{h+4}$ retain their dilation attained in the embedding $\tau_{h+3}$ in $Q_{d+5}$. Also, the edge $(1011010D, 0011010D)$ receives congestion 1. Each of the parameters namely, node congestion and edge congestion of the embedding is bounded by 2.

Figure 4(b) also shows two copies of $T_{h+2}$ rooted at $1110111D$ and $0110111D$, and a $T_{h+3}$ rooted at $1101010D$ embedded in $Q_{d+7}$. In all embeddings, the edge joining the root and the deep root of the tree are mapped onto the adjacent nodes and they receive edge congestion 1. The vertices of these trees are not yet utilized. These vertices will be utilized in the next step. The nodes $0010111D$ and $1010111D$ receive congestion 2; see Figure 2(b). Every parameter namely, dilation, node congestion and edge congestion of these embeddings is bounded by 2.

The output of this step, shown in Figure 4(c), is the input to embed $\tau_{h+5}$ in $Q_{d+8}$ in the next step.

Input: The output of the embedding $\tau_{h+3} \hookrightarrow Q_{d+5}$; see Figure 3(c).

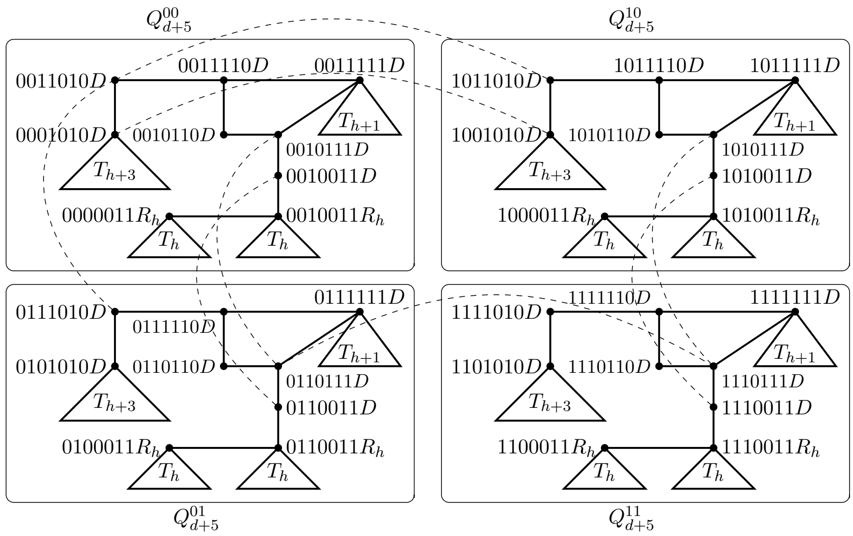


Figure 4(a). $ij\tau_{h+3} \hookrightarrow Q^{ij}_{d+5}, i, j \in \{0, 1\}$

Embedding:

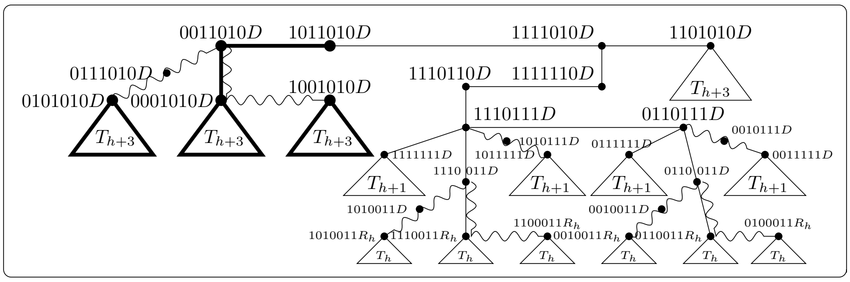


Figure 4(b). The edges $(0011010D, 0001010D)$,$(1110011D, 1110011R_h)$ and $(0110011D, 0110011R_h)$ have edge congestion 2.

Output:

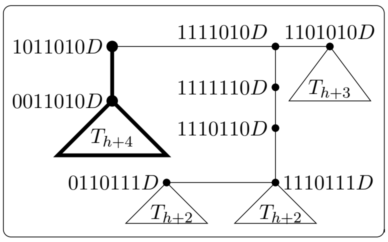


Figure 4(c). $\tau_{h+4} \hookrightarrow Q_{d+7}$. The subgraph shown on the right with light edges contains unutilized vertices.

Figure 4. The steps involved in an embedding of $\tau_{h+4} \hookrightarrow Q_{d+7}$, with the input $\tau_{h+3} \hookrightarrow Q_{d+5}$.

**(v) $\tau_{h+5} \hookrightarrow Q_{d+8}$ :** The embedding $(f, \rho)$ of $\tau_{h+5}$ in $Q_{d+8}$ is described in Figure 5.

We decompose $Q_{d+8}$ into two hypercubes $Q^0_{d+7}$ and $Q^1_{d+7}$. And embed $0\tau_{h+4}$ in $Q^0_{d+7}$ and $1\tau_{h+4}$ in $Q^1_{d+7}$; see Figure 5(a). We combine these two trees, as shown in Figures 5(a) and 5(b), to obtain the required embedding of $\tau_{h+5}$ into $Q_{d+8}$ with $11011010D$ as the deep root and $01011010D$ as the root of the tree $T_{h+5}$. Note that, the edge $R_{h+5}D$ is mapped onto the edge $(01011010D, 11011010D)$. Therefore, $R_{h+5}D$ receives dilation 1 and moreover, $(01011010D, 11011010D)$ has edge congestion 1. The dilation of each of the edges of $T_{h+5}$ is bounded by 2 and that the congestion of each node and edge of $Q_{d+8}$ is also bounded by 2.

Hence, given $\tau_h \hookrightarrow Q_d$, we have obtained an embedding of $\tau_{h+5}$ into $Q_{d+8}$ with load 1, dilation 2, node congestion 2 and edge congestion 2 in five steps (i) to (v). ■

**Theorem 2.2.** *Any complete ternary tree $T_h$ is embeddable with load* 1, *dilation* 2, *node congestion* 2 *and edge congestion* 2 *into $Q_{d(h)}$, where*

$$d(h) = \begin{cases} \lceil (1.6)h \rceil, & \text{if } h \equiv 2 \pmod 5 \text{ or } h \equiv 4 \pmod 5, \\ \lceil (1.6)h \rceil + 1, & \text{if } h \equiv 0 \pmod 5 \text{ or } h \equiv 1 \pmod 5 \text{ or } h \equiv 3 \pmod 5. \end{cases}$$

***Proof.*** We embed $\tau_h$ into $Q_{d(h)}$ by induction on $h \pmod 5$. The theorem follows, since $T_h$ is a subtree of $\tau_h$. For the base case, we have constructed embeddings of $\tau_0$, $\tau_1$, $\tau_2$, $\tau_3$ and $\tau_4$ into $Q_1$, $Q_3$, $Q_4$, $Q_6$ and $Q_7$ respectively, with load 1, dilation 2, node congestion 2 and edge congestion 2. For the inductive step, we assume that $\tau_h \hookrightarrow Q_{d(h)}$ and show that $\tau_{h+5} \hookrightarrow Q_{d(h+5)}$, where $d(h+5) = \lceil (1.6)(h+5) \rceil$ or $\lceil (1.6)(h+5) \rceil + 1$. By Theorem 2.1, we have $\tau_{h+5} \hookrightarrow Q_{d(h)+8}$, where

$$\begin{aligned} d(h) + 8 &= (\lceil (1.6)h \rceil \text{ or } \lceil (1.6)h \rceil + 1) + 8 \\ &= (\lceil (1.6)h + 8 \rceil \text{ or } \lceil (1.6)h + 8 \rceil + 1) \\ &= \lceil (1.6)(h+5) \rceil \text{ or } \lceil (1.6)(h+5) \rceil + 1 \\ &= d(h+5). \end{aligned}$$

Since, in the basic step, we have $\tau_h \hookrightarrow Q_{\lceil (1.6)h \rceil}$, when $h = 2$ or 4, and we have $\tau_h \hookrightarrow Q_{\lceil (1.6)h \rceil + 1}$, when $h = 0, 1$ or 3, the theorem follows. ■

Input: The output of the embedding $\tau_{h+4} \hookrightarrow Q_{d+7}$; see Figure 4(c).

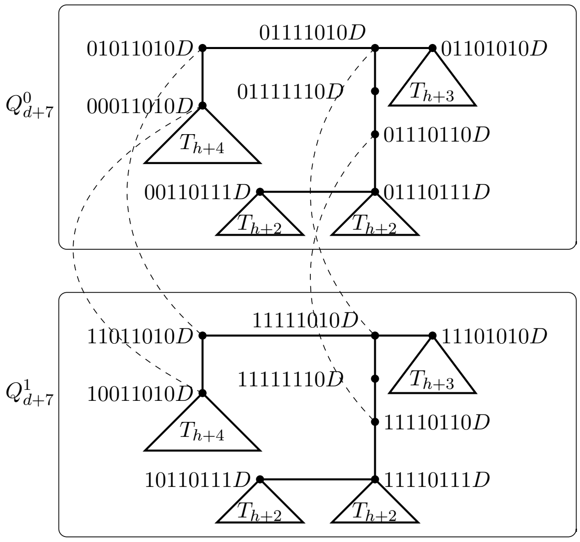


Figure 5(a). $i\tau_{h+4} \hookrightarrow Q^i_{d+7}, i \in \{0,1\}$

Embedding:

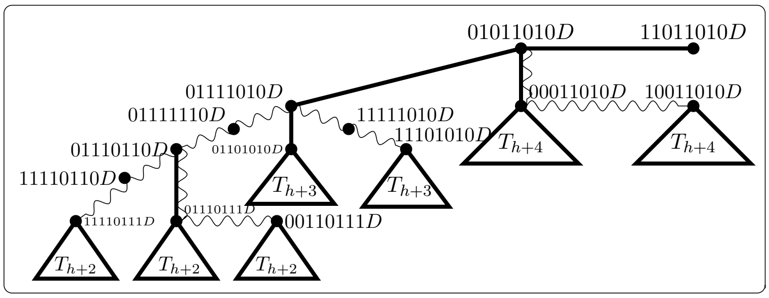


Figure 5(b). The edges $(01110110D, 01110111D)$ and $(01011010D, 00011010D)$ have edge congestion 2.

Output:

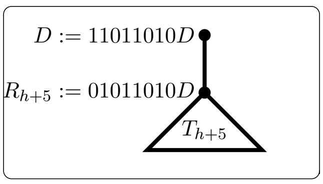


Figure 5(c). $\tau_{h+5} \hookrightarrow Q_{d+8}$

Figure 5. The steps involved in an embedding of $\tau_{h+5} \hookrightarrow Q_{d+8}$, with the input $\tau_{h+4} \hookrightarrow Q_{d+7}$.

### Conclusions and Remarks

1. In this paper, we have obtained a load 1, dilation 2, node congestion 2 and edge congestion 2 embedding of the complete ternary tree of height $h$ into a hypercube of dimension $d(h) = \lceil (1.6)h \rceil$ or $\lceil (1.6)h \rceil + 1$. More precisely,

$$d(h) = \begin{cases} \lceil (1.6)h \rceil, & \text{if } h \equiv 2 \pmod 5 \text{ or } h \equiv 4 \pmod 5, \\ \lceil (1.6)h \rceil + 1, & \text{if } h \equiv 0 \pmod 5 \text{ or } h \equiv 1 \pmod 5 \text{ or } h \equiv 3 \pmod 5. \end{cases}$$

Though the hypercube $Q_{d(h)}$ is not optimal, its dimension is very close to the dimension of the optimal hypercube which is $\lceil (\log_2 3)h \rceil (= \lceil (1.585)h \rceil)$ or $\lceil (\log_2 3)h \rceil + 1$.

2. Let $n(h)$ denote the dimension of the optimal hypercube of $T_h$. Let $d(h)$ be as defined in Theorem 1.3. We have computationally verified that $d(h) = n(h)$ for $2 \leq h \leq 15$, and that $n(h) \leq d(h) \leq n(h)+1$ for $16 \leq h \leq 80$. Therefore, by using our embeddings $\tau_0 \hookrightarrow Q_1$, $\tau_1 \hookrightarrow Q_3$, $\tau_2 \hookrightarrow Q_4$, $\tau_3 \hookrightarrow Q_6$ and $\tau_4 \hookrightarrow Q_7$ and the inductive description of the embedding given in the proof of Theorem 2.1, we conclude that

(i) for $2 \leq h \leq 15$, we have embedded $T_h$ in its optimal hypercube, and

(ii) for $16 \leq h \leq 80$, we have embedded $T_h$ either in its optimal hypercube or next-to-optimal hypercube.

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