Discussiones Mathematicae
Graph Theory 28 (2008) 307–321

CLIQUE IRREDUCIBILITY OF SOME ITERATIVE CLASSES OF GRAPHS

Aparna Lakshmanan S.

and

A. Vijayakumar

Department of Mathematics
Cochin University of Science and Technology
Cochin–682 022, India

e-mail: aparna@cusat.ac.in
e-mail: vijay@cusat.ac.in

Abstract

In this paper, two notions, the clique irreducibility and clique vertex irreducibility are discussed. A graph G is clique irreducible if every clique in G of size at least two, has an edge which does not lie in any other clique of G and it is clique vertex irreducible if every clique in G has a vertex which does not lie in any other clique of G. It is proved that $L(G)$ is clique irreducible if and only if every triangle in G has a vertex of degree two. The conditions for the iterations of line graph, the Gallai graphs, the anti-Gallai graphs and its iterations to be clique irreducible and clique vertex irreducible are also obtained.

Keywords: line graphs, Gallai graphs, anti-Gallai graphs, clique irreducible graphs, clique vertex irreducible graphs.

2000 Mathematics Subject Classification: 05C99.

1. Introduction

We consider only finite, simple graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$.

A clique of a graph G is a maximal complete subgraph of G. A graph G is clique irreducible if every clique in G of size at least two, has an edge which does not lie in any other clique of G and it is clique reducible if it is not clique irreducible [7]. A graph G is clique vertex irreducible if every clique in G has a vertex which does not lie in any other clique of G and it is clique vertex reducible if it is not clique vertex irreducible.

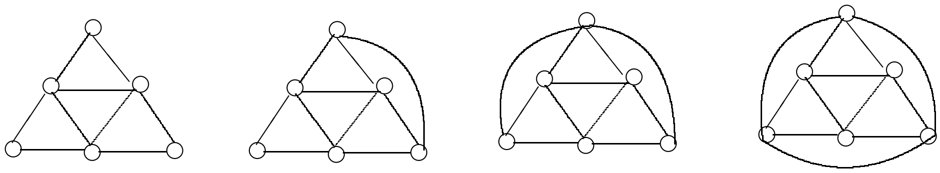
The line graph of a graph G, denoted by $L(G)$, is a graph whose vertex set corresponds to the edge set of G and any two vertices in $L(G)$ are adjacent if the corresponding edges in G are incident. The iterations of $L(G)$ are recursively defined by $L^1(G) = L(G)$ and $L^{n+1}(G) = L(L^n(G))$, for $n \geqslant 1$ [5].

The Gallai graph of a graph G, denoted by $\Gamma(G)$, is a graph whose vertex set corresponds to the edge set of G and any two vertices in $\Gamma(G)$ are adjacent if the corresponding edges in G are incident on a common vertex and they do not lie in a common triangle [4]. The anti-Gallai graph of a graph G, denoted by $\Delta(G)$, is a graph whose vertex set corresponds to the edge set of G and any two vertices in $\Delta(G)$ are adjacent if the corresponding edges lie in a triangle in G [4]. Both the Gallai graph and the anti-Gallai graph are spanning subgraphs of the line graph and their union is the line graph. Though $L(G)$ has a forbidden subgraph characterization, both these do not have the vertex hereditary property and hence cannot be characterized using forbidden subgraphs [4].

In [1], it is proved that there exist infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai and anti-Gallai graphs. The existence of a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be H-free for any finite graph H is proved. The relationship between the chromatic number, the radius and the diameter of a graph and its Gallai and anti-Gallai graphs are also obtained. In [4], it has been proved that $\Gamma(G)$ is isomorphic to G only for cycles of length greater than three. Also, computing the clique number and the chromatic number of $\Gamma(G)$ are NP-complete problems.

A graph G is clique-Helly if any family of mutually intersecting cliques has non-empty intersection [6]. It is hereditary clique-Helly if all the induced subgraphs of G are clique-Helly [6]. It is also proved in [6] that a graph G is hereditary clique-Helly, if it does not contain any Hajós' graph as an induced subgraph.

The complement of a graph G is denoted by G^c and the graph induced by a set of vertices $v_1, v_2, \ldots, v_n$ is denoted by $\langle v_1, v_2, \ldots, v_n \rangle$. A complete graph, a path and a cycle on n vertices are denoted by K_n, P_n and C_n respectively. The complete bipartite graph is denoted by $K_{m,n}$, where m and n are the number of vertices in each of the partition. A vertex of degree one is called a pendant vertex and an edge incident to a pendant vertex is called a pendant edge. A diamond is the graph $K_4 - \{e\}$, where e is any edge of K_4.



Hajós' graphs

In this paper, the graphs G for which $L(G)$ and $L^2(G)$ are clique vertex irreducible are characterized and it is deduced that $L^n(G)$ for $n \geqslant 3$ is clique vertex irreducible if and only if G is $K_3, K_{1,3}$ or P_k where $k \leqslant n+3$. After characterizing the graphs G such that $L(G)$, $L^2(G)$, $L^3(G)$ and $L^4(G)$ are clique irreducible, we prove that $L^n(G)$, $n \geqslant 5$, is clique irreducible if and only if it is non-empty and $L^4(G)$ is clique irreducible. The Gallai graphs which are clique irreducible and clique vertex irreducible are characterized. A forbidden subgraph characterization for clique vertex irreducibility of $\Gamma(G)$ is obtained. Also, the forbidden subgraphs for the anti-Gallai graphs and all its iterations to be clique irreducible and clique vertex irreducible are obtained.

All graph theoretic terminology and notations not mentioned here are from [2].

2. The Iterations of the Line Graph

Theorem 1. *Let G be a graph. The line graph $L(G)$ is clique vertex irreducible if and only if G satisfies the following conditions*

(1) *Every triangle in G has at least two vertices of degree two,*

(2) *Every vertex of degree greater than one in G has a pendant vertex attached to it, except for the vertices of degree two lying in a triangle.*

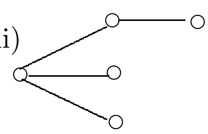
Proof. Let G be a graph which satisfies the conditions (1) and (2). The cliques of $L(G)$ are induced by the vertices corresponding to the edges in G which are incident on a vertex of degree at least three, the edges in G which are incident on a vertex of degree two and which do not lie in a triangle and by the edges in G which lie in a triangle. By (2), the cliques in $L(G)$

induced by the vertices corresponding to the edges in G which are incident on a vertex, have a vertex which does not lie in any other clique of $L(G)$. By (1), the cliques in $L(G)$ induced by the vertices which correspond to the edges in G which lie in a triangle, have a vertex which does not lie in any other clique of $L(G)$. Therefore, G is clique vertex irreducible.

Conversely, assume that $L(G)$ is a clique vertex irreducible graph. Let $\langle u_1, u_2, u_3\rangle$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 in G. $T = \langle e_1, e_2, e_3\rangle$ is a clique in $L(G)$. If $d(u_i) > 2$ for two u_is, u_1 and u_2, then there exist v_1 and v_2 (not necessarily different, but different from u_3) such that u_i is adjacent to v_i for $i = 1, 2$. But then, the vertices e_1 and e_3 will be present in the clique induced by the edges incident on the vertex u_1 and the vertices e_2 and e_3 will be present in the clique induced by the edges incident on the vertex u_2. Therefore, every vertex in T belongs to another clique in $L(G)$ which is a contradiction to the assumption that $L(G)$ is clique vertex irreducible. Hence every triangle in G has at least two vertices of degree two.

Now, let $u \in V(G)$ and $N(u) = \{u_1, u_2, \ldots, u_p\}$, where $p \geqslant 2$ and if $p = 2$ then u_1 is not adjacent to u_2. Let e_i be the vertex in $L(G)$ corresponding to the edge uu_i in G for $i = 1, 2, \ldots, p$. Let C be the clique $\langle e_1, e_2, \ldots, e_p\rangle$ in $L(G)$. If u has no pendant vertex attached to it then every u_i has a neighbor $v_i \neq u$ for $i = 1, 2, \ldots, p$. The v_is are not necessarily pairwise different. Moreover, some v_i can be equal to some u_j with $j \neq i$, except in the case $p = 2$. Therefore, for each i, every e_i in $L(G)$ will be present in another clique, either induced by the edges incident on the vertex u_i in G or by the edges in a triangle containing u and u_i in G. But this is a contradiction to the assumption that $L(G)$ is clique vertex irreducible. Hence, every vertex of degree greater than one in G has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle. ■

Theorem 2. *Let G be a connected graph. The second iterated line graph $L^2(G)$ is clique vertex irreducible if and only if G is one of the following graphs.*

(i) K_2 (ii) K_3 (iii) P_3 (iv) P_4 (v) P_5 (vi) $K_{1,3}$ (vii) 

Proof. By Theorem 1, $L^2(G)$ is clique vertex irreducible if and only if

(1) Every triangle in $L(G)$ has at least two vertices of degree two,

(2) Every vertex of degree greater than one in $L(G)$ has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle.

By (2), every non-pendant edge in G must have a pendant edge attached to it on one end vertex and the degree of that end vertex must be two.

Case 1. $L(G)$ has a triangle.
A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. Let it correspond to a triangle in G. If any of the vertices of this triangle has a neighbor outside the triangle, then two vertices in the corresponding triangle in $L(G)$ have neighbors outside the triangle, which is a contradiction. Therefore, since G is connected, in this case G must be K_3.

If the triangle in $L(G)$ corresponds to a $K_{1,3}$ in G, then two of the edges of this $K_{1,3}$ cannot have any other edge incident on any of its end vertices. Therefore, G cannot have a vertex of degree greater than three. Moreover, two vertices of $K_{1,3}$ in G must be pendant vertices. Again, by (2) and since G is connected, we conclude that G is either $K_{1,3}$ or the graph (vii).

Case 2. $L(G)$ has no triangle.
Since $L(G)$ has no triangle, G cannot have a K_3 or a vertex of degree greater than or equal to 3. Therefore, since G is connected, G must be a path or a cycle of length greater than three. Again, by (2), G cannot be a path of length greater than five or a cycle. Therefore G is K_2, P_3, P_4 or P_5. ■

Corollary 3. *Let G be a connected graph. The n^{th} iterated line graph $L^n(G)$ is clique vertex irreducible if and only if G is $K_3, K_{1,3}$ or P_k where $n+1 \leqslant k \leqslant n+3$, for $n \geqslant 3$.*

Theorem 4. *The line graph $L(G)$ is clique irreducible if and only if every triangle in G has a vertex of degree two.*

Proof. Let G be a graph such that every triangle in G has a vertex of degree two. Let C be a clique in $L(G)$.

Case 1. The clique C is induced by the vertices corresponding to the edges in G which are incident on a vertex of degree at least three.

An edge of C can be present in another clique of $L(G)$ if and only if the corresponding pair of edges in G lies in a triangle. Thus, if every edge of C lies in another clique of $L(G)$, then G has an induced K_p, where p is at least four. But, this contradicts the assumption that every triangle in G has a vertex of degree two.

Case 2. The clique C is induced by the vertices corresponding to the edges in G which are incident on a vertex of degree two and which do not lie in a triangle.

In this case, C is K_2 which always has an edge of its own.

Case 3. The clique C is induced by the vertices corresponding to the edges which lie in a triangle T in G.

Since T has a vertex v of degree two, the vertices corresponding to the edges which are incident on v induce an edge in C which does not lie in any other clique of $L(G)$. Therefore, G is clique irreducible.

Conversely, assume that G is a clique irreducible graph. Let $\langle u_1, u_2, u_3 \rangle$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 of G. $T = \langle e_1, e_2, e_3 \rangle$ is a clique in $L(G)$. If $d(u_i) > 2$ for each i, there exist v_1, v_2, v_3 such that u_i is adjacent to v_i for $i = 1, 2, 3$ (v_1, v_2 and v_3 are not necessarily different, but they are different from u_1, u_2 and u_3). Then the edges e_1e_2, e_2e_3 and e_3e_1 of $L(G)$ will be present in the cliques induced by edges which are incident on the vertices u_1, u_2 and u_3 respectively. Therefore, every edge in T is in another clique of $L(G)$, which is a contradiction. ■

Theorem 5. *The second iterated line graph $L^2(G)$ is clique irreducible if and only if G satisfies the following conditions*

(1) *Every triangle in G has at least two vertices of degree two,*

(2) *Every vertex of degree three has at least one pendant vertex attached to it,*

(3) *G has no vertex of degree greater than or equal to four.*

Proof. Let G be a graph such that $L^2(G)$ is clique irreducible. By Theorem 4, every triangle in $L(G)$ has a vertex of degree two. Then, we have the following cases.

Case 1. The triangle in $L(G)$ corresponds to a triangle in G.

Let $\langle u_1, u_2, u_3 \rangle$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 of G. At least one of the vertices of the triangle $\langle e_1, e_2, e_3 \rangle$ in $L(G)$ must be of degree two. Let e_1 be a vertex of degree two in $L(G)$. Since e_2 and e_3 belong to $N(e_1)$ in $L(G)$, e_1 has no other neighbors in $L(G)$. Therefore, the corresponding end vertices, u_1 and u_2 in G have no other neighbors. Hence (1) holds.

Case 2. The triangle in $L(G)$ corresponds to a $K_{1,3}$ (need not be induced) in G.

Let e_1, e_2, e_3 be the vertices in $L(G)$ corresponding to the edges uu_1, uu_2, uu_3 in G. At least one of the vertices of the triangle $\langle e_1, e_2, e_3 \rangle$ in $L(G)$ must be of degree two. Let e_1 be a vertex of degree two in $L(G)$. Vertices e_2 and e_3 belong to $N(e_1)$ in $L(G)$ and hence e_1 has no other neighbors in $L(G)$. Therefore, the corresponding end vertices, u and u_1 in G have no other neighbors. Since u has no other neighbors (3) holds and since u_1 has no other neighbors (2) holds.

Conversely, assume that G is a graph which satisfies all the three conditions. A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. A triangle in $L(G)$ which corresponds to a triangle in G has at least one vertex of degree two by (1). Again, a triangle in $L(G)$ which corresponds to a $K_{1,3}$ in G has at least one vertex of degree two by (2) and (3). Therefore, every triangle in $L(G)$ has at least one vertex of degree two and by Theorem 4, $L^2(G)$ is clique irreducible. ∎

Theorem 6. *Let G be a connected graph. If $G \neq K_3$ then, $L^3(G)$ is clique irreducible if and only if G satisfies the following conditions*

(1) *G is triangle free,*

(2) *G has no vertex of degree greater than or equal to four,*

(3) *At least two of the vertices of every $K_{1,3}$ in G are pendant vertices,*

(4) *If uv is an edge in G, then either u or v has degree less than or equal to two.*

Proof. Let $L^3(G)$ be clique irreducible. By Theorem 5, $L(G)$ satisfies:

(1′) Every triangle in $L(G)$ has at least two vertices of degree 2,

(2′) Every vertex of degree three in $L(G)$ has at least one pendant vertex attached to it,

(3′) $L(G)$ has no vertex of degree greater than or equal to 4.

A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. Every triangle in $L(G)$ has at least two vertices of degree two implies that every triangle in G has its three vertices of degree two. i.e., G is a triangle, because G is connected. Since $G \neq K_3$, G must be triangle free. Also, every $K_{1,3}$ in G has at least two pendant vertices and the degree of a vertex cannot exceed three. Therefore (1), (2) and (3) hold. Again (3′) implies that no edge in G can have more than three edges incident on its end vertices. Therefore, (4) holds.

Conversely, assume that the given conditions hold. Since G is triangle free, a triangle in $L(G)$ corresponds to a $K_{1,3}$ (need not be induced) in G. Therefore, by (2) and (3) every triangle in $L(G)$ has at least two vertices of degree two.

Let e be a vertex of degree three in $L(G)$ and let uv be the corresponding edge in G. Since e is of degree three in $L(G)$, the number of edges incident on u in G together with the number of edges incident on v in G is three. If u (or v) has three more edges incident on it then u (or v) will be of degree at least four which is a contradiction to the condition (2). Therefore, u has two neighbors and v has one neighbor (or vice versa) in G. Let u_1 and u_2 be the neighbors of u, and let v_1 be the neighbor of v in G. Then $\langle u, v, u_1, u_2 \rangle = K_{1,3}$ in G and hence at least two of v, u_1 and u_2 must be pendant vertices. Since v is not a pendant vertex, u_1 and u_2 must be pendant vertices. Therefore, e has two pendant vertices attached to it in $L(G)$ corresponding to the edges uu_1 and uu_2 in G. Hence (2′) is satisfied.

Again, (2), (3) and (4) together imply (3′). Since the conditions (1′), (2′) and (3′) are satisfied, by Theorem 5, $L^3(G)$ is clique irreducible. ∎

Theorem 7. *Let G be a connected graph. The fourth iterated line graph $L^4(G)$ is clique irreducible if and only if G is $K_3, K_{1,3}, P_n$ with $n \geqslant 5$ or C_n with $n \geqslant 4$.*

Proof. Let $L^4(G)$ be clique irreducible. Then by Theorem 6, if $L(G) \neq K_3$ then $L(G)$ must be triangle free. If $L(G) = K_3$ then G is either K_3 or $K_{1,3}$. If $L(G)$ is triangle free then G is triangle free and cannot have vertices of degree greater than or equal to three. Therefore, G is either a path or a cycle of length greater than three.

Conversely, if G is $K_3, K_{1,3}, P_n$ or C_n then $L^4(G)$ is either a triangle, a path or a cycle and all of them are clique irreducible.

Corollary 8. *For $n \geqslant 5$, $L^n(G)$ is clique irreducible if and only if it is non-empty and $L^4(G)$ is clique irreducible.*

3. The Gallai Graphs

Theorem 9. *The Gallai graph $\Gamma(G)$ is clique vertex irreducible if and only if for every $v \in V(G)$, every maximal independent set I in $N(v)$ with $|I| \geqslant 2$ contains a vertex u such that $N(u) - \{v\} = N(v) - I$.*

Proof. Let G be a graph such that its Gallai graph $\Gamma(G)$ is clique vertex irreducible. A clique C in $\Gamma(G)$ of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex $v \in V(G)$ whose other end vertices form a maximal independent set I of size at least two in $N(v)$. Let $I = \{v_1, v_2, \ldots, v_p\}$, where $p \geqslant 2$, be a maximal independent set in $N(v)$. Let e_i be the vertex in $\Gamma(G)$ corresponding to the edge vv_i in G for $i = 1, 2, \ldots, p$. Let C be the clique $\langle e_1, e_2, \ldots, e_p \rangle$ in $\Gamma(G)$. Let e_i be the vertex in C which does not belong to any other clique in G. Therefore, e_i has no neighbors in $\Gamma(G)$ other than those in C. Hence, $N(v_i) - \{v\} = N(v) - I$.

Conversely, assume that for every $v \in V(G)$, every maximal independent set $I = \{v_1, v_2, \ldots, v_p\}$ in $N(v)$ contains a vertex u such that $N(u) - \{v\} = N(v) - I$. If C is a clique of size one, it contains a vertex of its own. Otherwise, let C be defined as above. By our assumption, there exists a vertex $u = v_i$ such that $N(u) - \{v\} = N(v) - I$. Therefore, e_i has no neighbors outside C. Hence C has a vertex e_i of its own. ■

Theorem 10. *If $\Gamma(G)$ is clique vertex reducible, then G contains one of the graphs in Figure 1 as an induced subgraph.*

Proof. Let G be a graph such that $\Gamma(G)$ is clique vertex reducible and let C be a clique in $\Gamma(G)$ such that each vertex of C belongs to some other clique in $\Gamma(G)$. Consider the order relation $\preceq$ among the vertices of C where $e \preceq e'$ if $N[e] \preceq N[e']$. If $\preceq$ is a total ordering, then every vertex adjacent to the minimum vertex e is also adjacent to all the vertices in C. Therefore, by maximality of C, e cannot have neighbors outside C. This is a contradiction to the assumption that e belongs to some other clique of $\Gamma(G)$. So, there exist two vertices e_1 and e_2 in C which are not comparable. That is, there exist vertices f_1 and f_2 of $\Gamma(G)$ such that e_i is adjacent to f_j

if and only if $i = j$. Let vv_1 and vv_2 be the edges corresponding to e_1 and e_2, respectively. Then v_1 and v_2 are non-adjacent. Let u_1 and u_2 be the end points of f_1 and f_2, respectively, which are both different from v, v_1 and v_2.

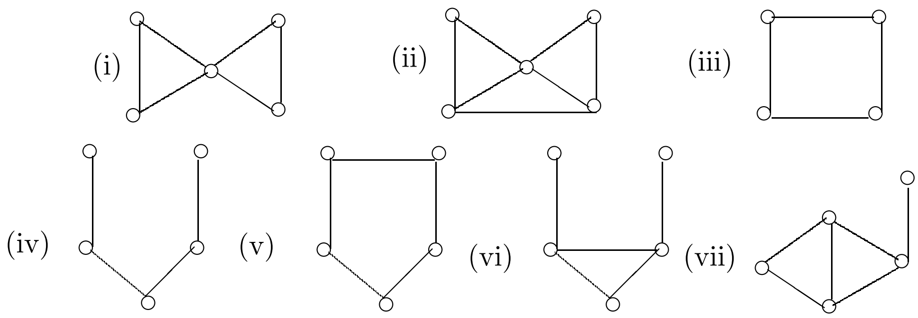


Figure 1

Case 1. Both f_1 and f_2 correspond to the edges incident to v.
In this case, u_1 and u_2 are adjacent to v, u_i is adjacent to v_j if and only if $i \neq j$ and u_1 and u_2 can be either adjacent or not. Therefore $\langle v, v_1, v_2, u_1, u_2 \rangle$ is the graph (i) or (ii) in Figure 1.

Case 2. None of f_1 and f_2 correspond to the edges incident to v.
In this case, u_1 and u_2 are adjacent to v_1 and v_2, respectively, and not to v. If $u_1 = u_2$ then G contains an induced C_4. If $u_1 \neq u_2$ and G does not contain an induced C_4, then $\langle v, v_1, v_2, u_1, u_2 \rangle$ is either P_5 or C_5.

Case 3. Exactly one of f_1 and f_2 correspond to the edges incident to v, say f_1.

In this case, u_1 is adjacent to both v and v_2 and is not adjacent to v_1. The vertex u_2 is adjacent to v_2 and is not adjacent to v. If u_2 is adjacent to v_1 then G contains an induced C_4. Otherwise, $\langle v, v_1, v_2, u_1, u_2 \rangle$ is the graph (vi) or (vii) in Figure 1. ■

Theorem 11. *The Gallai graph $\Gamma(G)$ is clique irreducible if and only if for every $v \in V(G)$, $\langle N(v) \rangle^c$ is clique irreducible.*

Proof. A clique C in $\Gamma(G)$ of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex $v \in V(G)$ whose other end vertices form a maximal independent set I of size at least two in $N(v)$. Therefore, C has an edge which does not belong to any other clique of $\Gamma(G)$ if and only if I has a pair of vertices both of which together does not belong to any other maximal independent set in $N(v)$. But, this happens if and only if every clique of size at least two in $\langle N(v)\rangle^c$ has an edge which does not belong to any other clique in $\langle N(v)\rangle^c$, since a maximal independent set in a graph corresponds to a clique in its complement. ■

Theorem 12. *The second iterated Gallai graph $\Gamma^2(G)$ is clique irreducible if and only if for every $uv \in E(G)$, either $\langle N(u)-N(v)\rangle$ and $\langle N(v)-N(u)\rangle$ are clique vertex irreducible or one among them is a clique and the other is clique irreducible.*

Proof. By Theorem 11, $\Gamma^2(G)$ is clique irreducible if and only if for every $e \in V(\Gamma(G))$, $\langle N(e)\rangle^c$ is clique irreducible.

Let $e = uv \in E(G)$, $N(u)-N(v) = \{u_1, u_2, \ldots, u_p\}$ and $N(v)-N(u) = \{v_1, v_2, \ldots, v_l\}$. Also let $e_i = uu_i$ for $i = 1, 2, \ldots, p$ and $f_j = vv_j$ for $j = 1, 2, \ldots, l$. $N_{\Gamma(G)}(e) = \{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_l\}$. $\langle N(e)\rangle^c$ is clique irreducible if and only if every maximal independent set I in $\langle N(e)\rangle$ has a pair of vertices of its own. e_i is not adjacent to e_j if and only if u_i is adjacent to u_j. Similarly, f_i is not adjacent to f_j if and only if v_i is adjacent to v_j. So, $I = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}, f_{j_1}, f_{j_2}, \ldots, f_{j_l}\}$ if and only if $\{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ is a clique in $\langle N(u) - N(v)\rangle$ and $\{v_{j_1}, v_{j_2}, \ldots, v_{j_l}\}$ is a clique in $N(v) - N(u)$. Therefore, every maximal independent set I in $N_{\Gamma(G)}(e)$ has a pair of vertices of its own if and only if either both $\langle N(u) - N(v)\rangle$ and $\langle N(v) - N(u)\rangle$ are clique vertex irreducible or one among them is a clique and the other is clique irreducible. ■

Theorem [6]**.** *If G is hereditary clique-Helly, then it is clique irreducible.*

Theorem 13. *If $\Gamma(G)$ is clique reducible then G contains one of the graphs in Figure 2 as an induced subgraph.*

Proof. Let $\Gamma(G)$ be a clique reducible graph. By Theorem [6], $\Gamma(G)$ contains at least one of the Hajós' graph as an induced subgraph. The Hajós' graphs is an induced subgraph of $\Gamma(G)$ if and only if G contains one of the graphs in Figure 2 as an induced subgraph. Hence the theorem. ■

Note. The converse is not necessarily true.

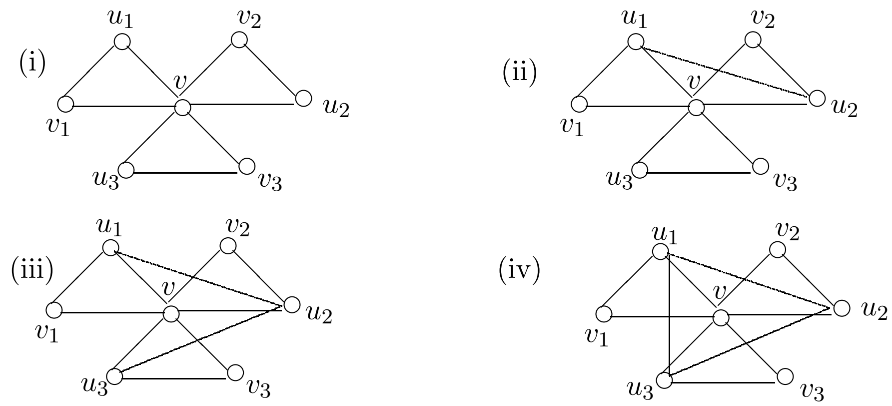


Figure 2

Let G be the graph in Figure 3. $V(G) = \{v, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3, w_4, w_5, w_7, w_7, w_8\}$. Let $\langle v, v_1, v_2, v_3, u_1, u_2, u_3 \rangle$ be the graph (i) in Figure 2 and let w_is for $i = 1, 2, \ldots, 8$ induce a complete graph. Also, let w_1 be adjacent to $\{v_1, v_2, v_3\}$, w_2 be adjacent to $\{v_1, v_2, u_3\}$, w_3 be adjacent to $\{v_1, u_2, v_3\}$, w_4 be adjacent to $\{v_1, u_2, u_3\}$, w_5 be adjacent to $\{u_1, v_2, v_3\}$, w_6 be adjacent to $\{u_1, v_2, u_3\}$, w_7 be adjacent to $\{u_1, u_2, v_3\}$, w_8 be adjacent to $\{u_1, u_2, u_3\}$ and v adjacent to w_i for $i = 1, 2, \ldots, 8$.

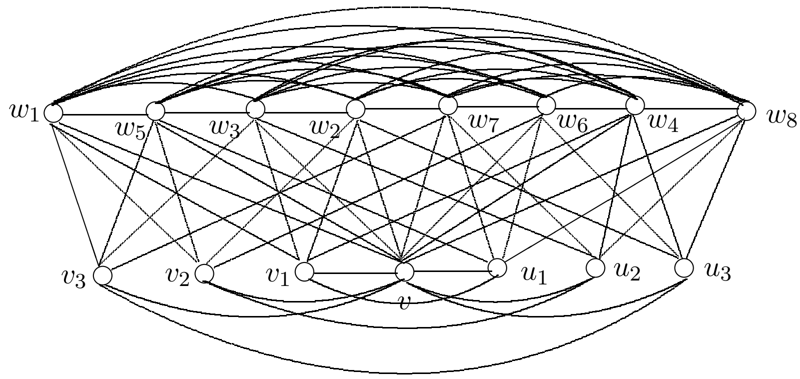


Figure 3

In $\Gamma(G)$ the vertices corresponding to the edges with one end vertex v induces K_6 minus a perfect matching in which the vertices of each of the eight triangles are adjacent to another vertex each. The remaining vertices induce the graph $H = 4K_{1,8}$. Therefore, $\Gamma(G)$ is clique irreducible.

4. The Iterations of the Anti-Gallai Graphs

Theorem 14. *The anti-Gallai graph $\Delta(G)$ is clique vertex irreducible if and only if G neither contains K_4 nor one of the Hajós' graphs as an induced subgraph.*

Proof. Let G be a graph which does neither contain K_4 nor one of the Hajós' graphs as an induced subgraph. The cliques of $\Delta(G)$ are induced by the vertices corresponding to the edges of G incident on a vertex of degree at least 3 whose other end vertices induce a complete graph and by the vertices corresponding to the edges which lie in a triangle. In the first case G contains an induced K_4, which is a contradiction. Therefore, the cliques of $\Delta(G)$ are induced by the edges which lie in a triangle. Let $\langle u_1, u_2, u_3\rangle$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $\Delta(G)$ corresponding to the edges u_1u_2, u_2u_3, u_3u_1 in G. Then $\langle e_1, e_2, e_3\rangle$ is a clique in $\Delta(G)$. If a vertex e_i for $i = 1, 2, 3$ lies in another clique of $\Delta(G)$, then the edge corresponding to e_i lies in another triangle. Therefore, the end vertices of the edge corresponding to e_i in G has a neighbor v_i for $i = 1, 2, 3$. $v_i \neq v_j$ if $i \neq j$ and v_1, v_2, v_3 are not adjacent to u_3, u_1, u_2, respectively, since otherwise G contains a K_4, which is a contradiction. Then, $\langle u_1, u_2, u_3, v_1, v_2, v_3\rangle$ is one of the Hajós' graphs, a contradiction. Hence, G is clique vertex irreducible.

Conversely, assume that G is clique vertex irreducible. If G contains K_4 or one of the Hajós' graphs as an induced subgraph, then there exists a clique in $\Delta(G)$, corresponding to a triangle in G, which shares each of its vertices with some other clique of $\Delta(G)$. ■

Lemma 1. *If G is K_4-free then $\Gamma(G)$ is diamond free.*

Proof. Let G be a graph which does not contain K_4 as an induced subgraph. Therefore, a triangle in $\Delta(G)$ can only be induced by a triangle in G. If two vertices of the triangle in $\Delta(G)$ have a common neighbor, then it forces G to have a K_4, a contradiction. Therefore, $\Delta(G)$ is diamond free. ■

Theorem 15. *The second iterated anti-Gallai graph $\Delta^2(G)$ is clique vertex irreducible if and only if G does not contain K_4 as an induced subgraph.*

Proof. By Theorem 14, $\Delta^2(G)$ is clique vertex irreducible if and only if $\Delta(G)$ does neither contain K_4 nor one of the Hajós' graphs as an induced subgraph.

Let G be a graph which does not contain K_4 as an induced subgraph. Therefore, G does not contain K_5 as an induced subgraph and hence $\Delta(G)$ does not contain K_4 as an induced subgraph. Again, by Lemma 1, $\Delta(G)$ cannot have diamond as an induced subgraph and hence it does not contain any of the Hajós' graph as an induced subgraph. Hence, $\Delta^2(G)$ is clique vertex irreducible.

Conversely, assume that $\Delta^2(G)$ is clique vertex irreducible. If G contains K_4 as an induced subgraph then in $\Delta(G)$ the vertices corresponding to the edges of this K_4 induce K_6 minus a perfect matching which is the fourth Hajós' graph, a contradiction. Therefore, G does not contain K_4 as an induced subgraph. ■

Theorem 16. *The n^{th} iterated anti-Gallai graph $\Delta^n(G)$ is clique vertex irreducible if and only if G does not contain K_{n+2} as an induced subgraph.*

Proof. By Theorem 15, $\Delta^n(G)$ is clique vertex irreducible if and only if $\Delta^{n-2}(G)$ does not contain K_4 as an induced subgraph. $\Delta^{n-2}(G)$ does not contain K_4 as an induced subgraph if and only if $\Delta^{n-3}(G)$ does not contain K_5 as an induced subgraph. Proceeding like this, we get that $\Delta(G)$ does not contain K_{n+1} as an induced subgraph if and only if G does not contain K_{n+2} as an induced subgraph. Therefore, $\Delta^n(G)$ is clique vertex irreducible if and only if G does not contain K_{n+2} as an induced subgraph. ■

Theorem [3]. *If a graph G has no induced diamond, then every edge of G belongs to exactly one clique.*

Theorem 17. *The anti-Gallai graph $\Delta(G)$ is clique irreducible if and only if G does not contain K_4 as an induced subgraph.*

Proof. Let G be a graph which does not contain K_4 as an induced subgraph. By Lemma 1 and Theorem [3], $\Delta(G)$ is clique irreducible.

Conversely, if G contains a $K_4 = \langle u_1, u_2, u_3, u_4 \rangle$, then it follows that the clique in $\Delta(G)$, corresponding to the triangle $\langle u_1, u_2, u_3 \rangle$ in G, shares each of its edges with some other clique. Therefore, if $\Delta(G)$ is clique irreducible, then G cannot have K_4 as an induced subgraph. ■

Theorem 18. *The n^{th} iterated anti-Galli graph $\Delta^n(G)$ is clique irreducible if and only if G does not contain an induced K_{n+3}.*

Proof. By Theorem 17, $\Delta^n(G)$ is clique irreducible if and only if $\Delta^{n-1}(G)$ does not contain an induced K_4. $\Delta^{n-1}(G)$ does not contain an induced K_4 if and only if $\Delta^{n-2}(G)$ does not contain an induced K_5. Proceeding like this, we get, $\Delta(G)$ does not contain an induced K_{n+2} if and only if G does not contain an induced K_{n+3}. Therefore, $\Delta^n(G)$ is clique irreducible if and only if G does not contain an induced K_{n+3}. ■

Acknowledgement

The authors thank the referees for their suggestions for the improvement of this paper.

References

[1] Aparna Lakshmanan S., S.B. Rao and A. Vijayakumar, *Gallai and anti-Gallai graphs of a graph*, Math. Bohem. **132** (2007) 43–54.

[2] R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory (Springer, 1999).

[3] L. Chong-Keang and P. Yee-Hock, *On graphs without multicliqual edges*, J. Graph Theory **5** (1981) 443–451.

[4] V.B. Le, *Gallai graphs and anti-Gallai graphs*, Discrete Math. **159** (1996) 179–189.

[5] E. Prisner, Graph Dynamics (Longman, 1995).

[6] E. Prisner, *Hereditary clique-Helly graphs*, J. Combin. Math. Combin. Comput. **14** (1993) 216–220.

[7] W.D. Wallis and G.H. Zhang, *On maximal clique irreducible graphs*, J. Combin. Math. Combin. Comput. **8** (1990) 187–193.

Received 9 October 2007
Revised 18 March 2008
Accepted 18 March 2008