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Note

SOLUTION TO THE PROBLEM OF KUBESA

Mariusz Meszka

Faculty of Applied Mathematics
AGH University of Science and Technology
Mickiewicza 30, 30–059, Kraków, Poland

e-mail: meszka@agh.edu.pl

Abstract

An infinite family of T-factorizations of complete graphs K_{2n}, where $2n = 56k$ and k is a positive integer, in which the set of vertices of T can be split into two subsets of the same cardinality such that degree sums of vertices in both subsets are not equal, is presented. The existence of such T-factorizations provides a negative answer to the problem posed by Kubesa.

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1. Introduction

Let K_{2n} be the complete graph on $2n$ vertices and T be its spanning tree. A T-*factorization* of K_{2n} is a collection of edge disjoint factors $T_1, T_2, \ldots, T_n$ of K_{2n}, each of which being isomorphic to T.

At the workshop in Krynica in 2004 D. Fronček presented the following problem originally posed by M. Kubesa [2].

Problem. Suppose that there exists a T-factorization of K_{2n}. Is it true that the vertex set of T can be split into two subsets, V_1 and V_2, such that $|V_1| = |V_2| = n$ and $\sum_{v\in V_1} \deg(v) = \sum_{v\in V_2} \deg(v)$?

Notice that there is no requirement on connectness or disconnectness of graphs induced by V_1 or V_2.

Recently, N.D. Tan [3] solved the problem in the affirmative for two narrow classes of trees.

2. Constructions

A tree which becomes a star after removal of its pendant edges is called a *snowflake*. Its central vertex (ie. the central vertex of a star obtained in such a way) is called a *root*, whilst remaining vertices of degrees greater than one are called *inner vertices*.

We define a family of snowlakes $\tilde{T}_{2n}$ of order $2n = 56k$, for every positive integer k. There are 7 vertices of degrees: $28k-18$, $28k-20$, $11, 10, 8, 7, 7$, the remaining $56k-7$ are leaves. The vertex of degree 11 is the root of $\tilde{T}_{2n}$.

Lemma 1. *For every positive integer k, the complete graph K_{56k} has $\tilde{T}_{56k}$-factorization.*

Proof. The snowflake $\tilde{T}_{56k}$ is defined by listing its edges; we use the notation $u \prec u_1, u_2, \ldots, u_m$ if all the vertices u_1, $u_2, \ldots, u_m$ are adjacent to u. Consider two cases.

Case I. $k = 1$. Let $V(K_{56}) = U \cup X \cup Y \cup Z$, where $U = \{u_0, u_1, \ldots, u_{13}\}$, $X = \{x_0, x_1, \ldots, x_{13}\}$, $Y = \{y_0, y_1, \ldots, y_{13}\}$ and $Z = \{z_0, z_1, \ldots, z_{13}\}$. Edges of K_{56} with both endvertices either in U or X or Y or Z are called *pure* edges; the remaining ones are *mixed* edges. To indicate a required $\tilde{T}_{56}$-factorization we prescribe 28 snowflakes split into two classes: $\{T_i : i = 0, 1, \ldots, 13\}$ and $\{T'_i : i = 0, 1, \ldots, 13\}$, each T_i and T'_i being isomorphic to $\tilde{T}_{56}$.

We construct the first class. The vertex u_{12} of degree 11 is the root of T_0 and its inner vertices: $u_0, x_1, x_2, y_0, y_1, z_7$ have degrees $8, 8, 7, 10, 7, 10$, respectively. The remaining pendant edges are: $u_{12} \prec u_1, u_2, u_4, u_7, u_{11}$; $u_0 \prec x_8, x_9, x_{11}, x_{12}, x_{13}, y_4, y_5$; $x_1 \prec u_5, u_9, u_{10}, u_{13}, y_3, y_8, y_{10}$; $x_2 \prec x_3, x_4, x_5, x_6, x_7, x_{10}$; $y_0 \prec u_6, u_8, z_0, z_1, z_2, z_3, z_4, z_6, z_9$; $y_1 \prec y_2, y_6, y_7, y_{11}, y_{12}, y_{13}$; $z_7 \prec u_3, x_0, y_9, z_5, z_8, z_{10}, z_{11}, z_{12}, z_{13}$. Snowflakes $T_1, T_2, \ldots, T_{13}$ can be obtained from T_0 by applying the cyclic permutation $\varphi = (0, 1, \ldots, 13)$ in parallel on the indices of vertices in the sets U, X, Y and Z. One can easily check that the lengths $1, 2, 3, 4, 5, 6$ of all pure edges in K_{56} have been already covered, as well as the following lengths of mixed edges for types: UX: $2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13$; UY: $2, 3, 4, 5, 6, 8$; UZ: $4, 9$; XY: $2, 7, 9$; XZ: 7; YZ: $0, 1, 2, 3, 4, 6, 9, 12$.

To construct the second class we need the snowflake T'_0. Let the vertex u_7 of degree 11 be the root and $x_0, x_8, y_2, y_3, z_0, z_1$ be the inner vertices of degrees $8, 8, 7, 7, 10, 10$, respectively. The remaining pendant edges are: $u_7 \prec u_0, z_3, z_4, z_5, z_6$; $x_0 \prec x_7, y_0, y_8, y_{10}, y_{11}, y_{12}, y_{13}$; $x_8 \prec z_2, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}$; $y_2 \prec x_1, x_{10}, x_{11}, x_{12}, x_{13}, y_9$; $y_3 \prec u_2, u_3, u_4, u_5, u_6, u_{10}$; $z_0 \prec u_8, u_9, u_{11}, u_{12}, u_{13}, y_1, y_6, y_7, z_7$; $z_1 \prec u_1, x_2, x_3, x_4, x_5, x_6, x_9, y_4, y_5$. Six snowflakes T'_i, for $i = 2, 4, \dots, 12$, can be obtained from T'_0 by applying ith power of φ in parallel on the sets U, X, Y and Z. Thus the length 7 of all pure edges is covered completely and still remaining lengths of mixed edges, except the lengths 0 of type UX and 5 of type YZ, are covered in a half. Seven remaining snowflakes T'_j for $j = 1, 3, \dots, 13$ are obtained from T'_0 by replacing the edges $u_0\,u_7$, $x_0\,x_7$, $y_2\,y_9$ and $z_0\,z_7$ with the edges $u_0\,x_0$, $u_7\,x_7$, $y_2\,z_7$ and $y_9\,z_0$, respectively, and then by applying the permutation $(\varphi)^j$ in parallel on the sets U, X, Y and Z. Notice that such a replacement does not result in changing the structure of snowflake, i.e., all T'_j are isomorphic to T'_0. In this way we cover all remaining lengths of mixed edges.

Case II. $k \geq 2$. Let $V(K_{56k}) = \bigcup_{l=1}^{k}(U^l \cup X^l \cup Y^l \cup Z^l)$, where $U^l = \{u_0^l, u_1^l, \dots, u_{13}^l\}$, $X = \{x_0^l, x_1^l, \dots, x_{13}^l\}$, $Y = \{y_0^l, y_1^l, \dots, y_{13}^l\}$ and $Z = \{z_0^l, z_1^l, \dots, z_{13}^l\}$, $l = 1, 2, \dots, k$. In what follows subscripts should be read modulo 14.

In order to construct $28k$ factors, each isomorphic to $\tilde{T}_{56k}$, we proceed in the following way. First, for every snowflake T_i, $i = 0, 1, \dots, 13$, in the $\tilde{T}_{56}$-factorization of K_{56} constructed in Case I we make k copies T_i^l, $l = 1, 2, \dots, k$, by copying every edge $s\,t$ of T_i into k edges $s^l\,t^l$, each being an edge of appropriate T_i^l, where $s, t \in U \cup X \cup Y \cup Z$. Moreover, for every T_i^l among $14k$ trees obtained in this way, where $i = 0, 1, \dots, 13$ and $l = 1, 2, \dots, k$, we add $56(k-1)$ edges: $u_i^l \prec u_j^p, x_j^p, y_j^r, z_j^r$, $y_i^l \prec u_j^r, x_j^r, y_j^p, z_j^p$, where $l < p \leq k$, $1 \leq r < l$, $j = 0, 1, \dots, 13$. Thus every T_i^l is a snowflake with the root u_{12+i}^l of degree 11, and six inner vertices u_i^l, x_{1+i}^l, x_{2+i}^l, y_i^l, y_{1+i}^l, z_{7+i}^l od degrees $28k - 20$, $8, 7$, $28k - 18$, $7, 10$, respectively.

Similarly, for every snowflake T'_i constructed in Case I, $i = 0, 1, \dots, 13$, we built k copies $T_i'^l$, $l = 1, 2, \dots, k$, by copying every edge $s\,t$ of T'_i into k edges $s^l\,t^l$, $s, t \in U \cup X \cup Y \cup Z$. Analogously to the above, for every $T_i'^l$ of $14k$ trees just obtained, $i = 0, 1, \dots, 13$ and $l = 1, 2, \dots, k$, new $56(k-1)$ edges are added: $x_i^l \prec u_j^p, x_j^p, y_j^r, z_j^r$, $z_i^l \prec u_j^r, x_j^r, y_j^p, z_j^p$, where $l < p \leq k$, $1 \leq r < l$, $j = 0, 1, \dots, 13$. Every $T_i'^l$ obtained in this way is a snowflake with the root u^l_{7+i} of degree 11, and six inner vertices x^l_i, x^l_{8+i}, y^l_{2+i}, y^l_{3+i}, z^l_i, z^l_{1+i} od degrees $28k-20$, $8, 7, 7$, $28k-18$, 10, respectively. ■

Lemma 2. *For every set* $\bar{V} \subset V(\tilde{T}_{56k}) = V(K_{56k})$ *such that* $|\bar{V}| = 28k$, $\sum_{v\in\bar{V}} \deg(v) \neq 56k-1$.

Proof. One can check that there are only four sequences of length $28k$ whose terms are degrees of $\tilde{T}_{56k}$ and whose sum of terms is $56k-1$:

(1) $28k-18,\ 10, 10, 1, 1, \ldots, 1$,

(2) $28k-18,\ 7, 7, 7, 7, 1, 1, \ldots, 1$,

(3) $28k-20,\ 11, 11, 1, 1, \ldots, 1$,

(4) $28k-20,\ 8, 8, 7, 1, 1, \ldots, 1$.

None of these sequences is a subsequence of degree sequence of $\tilde{T}_{56k}$. Thus the assertion holds. ■

Notice that every of the sequences (1)–(4) indeed appears as a set of degrees for some vertex in factors of $\tilde{T}_{56k}$-factorization of K_{56k}. It is easily seen that all terms of (1) are degrees of the vertex z^l_i in $\tilde{T}_{56k}$-factorization, similarly (2) is a set of degrees for y^l_i, (3) for u^l_i and (4) for x^l_i, $i = 0, 1, \ldots, 13$, $l = 1, 2, \ldots, k$.

It is still possible that a similar example for the order $2n < 56$ exists. Nevertheless, a computer was used to check that in that case $2n$ cannot be smaller than 38.

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