

ON THE STRUCTURE OF PATH-LIKE TREES

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Abstract

We study the structure of path-like trees. In order to do this, we introduce a set of trees that we call expandable trees. In this paper we also generalize the concept of path-like trees and we call such generalization generalized path-like trees. As in the case of path-like trees, generalized path-like trees, have very nice labeling properties.

Keywords: tree, path-like tree, T_p -tree, expandable tree, α -valuation, super edge-magic labeling, special super edge-magic labeling, harmonious labeling, super edge-antimagic labeling.

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1. INTRODUCTION

For the undefined concepts and notation, the reader is directed to either [5] or [7].

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This paper is mainly devoted to study the structure of a particular family of trees called path-like trees. Path-like trees were first introduced by Barrientos in [4] in his effort to find families of trees with nice labeling properties. They are defined as follows:

We embed the path P_n as a subgraph of the 2-dimensional grid, that is to say the graph $P_k \times P_l$. Given such an embedding, we consider the ordered set of subpaths L_1, L_2, \dots, L_m which are maximal straight segments in the embedding, and such that the end of L_i is the beginning of L_{i+1} . Suppose that $L_i \cong P_2$ for some i and that some vertex u of L_{i-1} is at distance 1 in the grid to a vertex v of L_{i+1} . An elementary transformation of the path consists in replacing the edge of L_i by a new edge uv , namely L'_i . We say that a tree T of order n is a path-like tree, when it can be obtained from some embedding of P_n in the grid, by replacing each edge of $L_i \cong P_2$ by a unique L'_i (note that it is possible to take $L_i = L'_i$).

In [4] Barrientos proved that path-like trees admit α -valuations [11]. Thanks to the relations established by Figueroa *et al.* in [6], we know that path-like trees admit many other labelings. For instance, a particular class of super edge-magic labelings called special super edge-magic labelings, that were defined in [10] by Muntaner, harmonious labelings [8], etc. (See appendix **A** for definitions of different types of labelings). Very recently, Bača *et al.* have studied in [2] the super edge-antimagic properties of path-like trees, and in [3] the same authors have studied structural properties of path-like trees. This is interesting since in [2] Bača *et al.* proposed the question whether it was possible, given a tree T with $\Delta(T) \leq 4$ (where $\Delta(T)$ denotes the maximum degree over the vertices of T), to find an efficient algorithm that allows us to determine whether T is a path-like tree. For what we know this question remains open, and in this paper, although we do not answer it, we provide non trivial conditions for a given tree T with $\Delta(T) \leq 4$ to be a path-like tree. In order to do this, we introduce in this paper the concept of expandable trees. Next we provide the necessary definition and examples.

Let T be a tree with $\Delta(T) \leq 4$. We say that T is an expandable tree if T can be embedded in the two dimensional grid, $P_k \times P_l$, in such a way that there is at most one edge of the tree that joins vertices of two consecutive rows in the grid. A tree drawn in this form is called an expanded tree.

Example 1. Consider the grid $P_6 \times P_7$ with the following expanded tree:

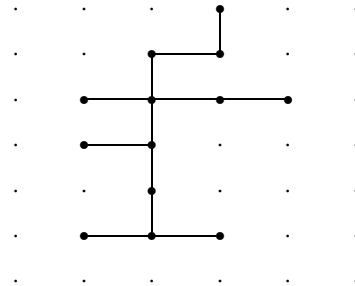


Figure 1. Expanded tree.

Example 2. The following tree, that from now on will be called T^* , will be proved in Lemma 2.1 not to be expandable, and in fact it will play a crucial role in the development of this paper.

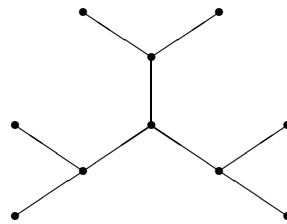


Figure 2. The tree T^* .

Next we define a normalized embedding of the path in the grid $P_k \times P_l$ as follows (see [3]). Let \mathcal{L} be the 2-dimensional grid. If we fix a crossing point as $(0, 0)$ then each crossing point in \mathcal{L} is perfectly determined by an ordered pair (i, j) where i denotes the row (level) and j denotes the column of \mathcal{L} . Let \mathcal{I} be an embedding of a path P in \mathcal{L} such that:

1. One end vertex of the path P is $(0, 0)$.
2. Each row of the embedding contains at least two vertices of the path P , and each vertical subpath is in the embedding isomorphic to P_2 . Furthermore if (i, j) and (i, t) , where $j < t$, are vertices of the path P_n in the embedding, then $(i, j + 1), (i, j + 2), \dots, (i, t - 1)$ are also vertices of the path in the embedding.
3. Assume that i is an even integer and that $(i, j), (i, j + 1), (i, j + 2), \dots, (i, j + t)$ is a maximal straight horizontal subpath in the embedding

of the path P in \mathcal{L} . If $(i+1, m)$ belongs to the embedding of the path P in \mathcal{L} , then $m \leq j+t$.

4. Assume that i is an odd integer and that $(i, j), (i, j-1), (i, j-2), \dots, (i, j-s)$ is a maximal straight horizontal subpath in the embedding of the path P in \mathcal{L} . If $(i+1, m)$ belongs to the embedding of the path P in \mathcal{L} , then $m \geq j-s$.

Then the embedding \mathcal{I} is called a normalized embedding of the path P in the grid \mathcal{L} . The following remark is an immediate consequence of the definition of expandable tree and the fact that every path-like tree can be obtained from a normalized embedding of a path in the 2-dimensional grid (see [3]).

Remark. Every path-like tree is an expandable tree.

In this paper, we take the opportunity to introduce the definition of generalized path-like trees, since they are a natural generalization of path-like trees, and they also have very nice labeling properties.

Let $P = (V, E)$ be a path with $V = \{i\}_{i=1}^n$ and with $E = \{i(i+1)\}_{i=1}^{n-1}$. Choose a subset $S \subseteq E$. For every $x(x+1) \in S$, replace $x(x+1)$ by a new edge ij where $i < j$ and such that $i+j = 2x+1$. Then any graph obtained in this way is called a generalized path-like tree. Notice that the following holds for generalized path-like trees:

1. If $i+j = 2x+1$ ($j > i$) then $d_P(i, x) = d_P(x+1, j)$.
2. A generalized path-like tree is not necessarily a tree.
3. Every path-like tree is a generalized path-like tree.
4. Generalized path-like trees are bipartite graphs.
5. Not all generalized path-like trees which are trees, are in fact path-like trees.
6. Generalized path-like trees admit many different types of labelings as for instance, α -labelings, harmonious labelings, special super edge-magic labelings, etc.

For further information and examples about labelings of generalized path-like trees, see appendices **B** and **C**.

Example 3. The following example is an example of a generalized path-like tree, which is a tree and that is not a path-like tree. It is easy to see it, since it contains a vertex of degree 5.

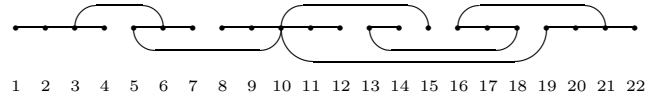


Figure 3. A generalized path-like tree which is not a path-like tree.

Question. Is it possible to find an efficient algorithm that, given a bipartite graph G , allows us to decide whether it is a generalized path-like tree?

We conclude this introduction by defining what we mean by the contraction of a tree T .

Let T be any tree. We define the contraction of T , and we denote it by $\text{contract}(T)$, to be the tree with the following two properties:

1. The tree $\text{contract}(T)$ contains no vertex of degree two.
2. The tree T can be obtained from $\text{contract}(T)$ by subdividing edges.

Example 4.

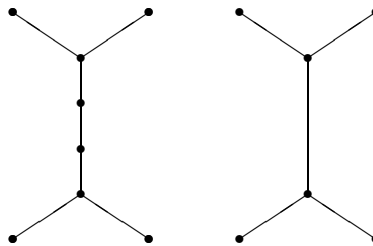


Figure 4. T and $\text{contract}(T)$.

2. PATH-LIKE TREES AND EXPANDABLE TREES

We start this section by providing the following result:

Lemma 2.1.

1. Let T be an expandable tree. Then any subdivision of T is expandable.
2. Let T be an expandable tree. Then $\text{contract}(T)$ is expandable.
3. Let T be any tree for which $\text{contract}(T) \not\cong T^*$. Then $\text{contract}(T)$ is a caterpillar.

4. The tree T^* is not expandable.
5. Let T be any caterpillar with $\Delta(T) \leq 4$. Then T is expandable.

Proof. Since items 1, 2 and 5 are clear, the proofs of these items are left to the reader. Next we prove items 3 and 4.

3. Assume to the contrary, that T is a tree such that $\text{contract}(T) \not\supseteq T^*$ and $\text{contract}(T)$ is not a caterpillar. Let $S = \{\text{leaves of } \text{contract}(T)\}$ and let T' denote the new tree $\text{contract}(T) \setminus S$. Since $\text{contract}(T)$ is not a caterpillar, it follows that there exists $\alpha \in V(T')$ such that $\deg_{T'}(\alpha) \geq 3$. Assume that $\beta_1, \beta_2, \beta_3 \in N(\alpha)$ (where $N(\alpha)$ denotes the neighborhood of α). If $\deg(\beta_i) \geq 3$ for all $i \in \{1, 2, 3\}$, we are done. Therefore there exists $i \in \{1, 2, 3\}$ such that $\deg_{T'}(\beta_i) < 3$. It is clear that β_i is not a leaf in $\text{contract}(T)$, since for if not β_i would not be a vertex of T' . Thus, $\deg_{\text{contract}(T)}(\beta_i) \geq 2$. However, $\text{contract}(T)$ contains no vertex of degree 2, since it is the contraction of some tree. This shows that $\deg_{\text{contract}(T)}(\beta_i) \geq 3$. Therefore $T^* \subseteq \text{contract}(T)$.

4. Assume to the contrary, that T^* is expandable and let $\text{center}(T^*) = \{A\}$. Then the vertices of degree three of T^* connect consecutive rows. Hence a vertex of degree three is adjacent at most to two vertices of degree three. However in T^* , vertex A is adjacent to three vertices of degree three. ■

Next we state and prove the following theorem:

Theorem 2.2. *For any given tree T with $\Delta(T) \leq 4$, the following are equivalent:*

1. T is expandable.
2. $\text{Contract}(T)$ is a caterpillar.
3. T does not contain any subdivision of T^* as a subtree.

Proof. (1) \implies (3). If T is an expandable tree then $\text{contract}(T)$ is also expandable. Thus $T^* \not\subseteq \text{contract}(T)$. Therefore T does not contain any subdivision of T^* as a subgraph.

(3) \implies (2). Let T be a tree such that $\Delta(T) \leq 4$, and such that T does not contain any subdivision of T^* as a subgraph. Then $\text{contract}(T)$ does not contain T^* as a subgraph. Therefore $\text{contract}(T)$ is a caterpillar.

(2) \implies (1). Let T be a tree with $\Delta(T) \leq 4$ such that $\text{contract}(T)$ is a caterpillar. Then $\text{contract}(T)$ is expandable. Therefore since T is a subdivision of $\text{contract}(T)$ it follows that T is expandable. ■

Since any path-like tree is an expandable tree, we obtain the following corollary.

Corollary 2.1.

1. If T is a path-like tree, then $\text{contract}(T)$ is a caterpillar.
2. If T is a path-like tree, then T does not contain any subdivision of T^* as a subgraph.

In [2] Bača *et al.* proved the next result.

Theorem 2.3. *Let T be any path-like tree. Then there are at most two vertices $u, v \in V(T)$ with:*

1. $\deg(u) = \deg(v) = 3$.
2. If $N(u) = \{u_1, u_2, u_3\}$ and $N(v) = \{v_1, v_2, v_3\}$, then $\deg(u_i) = \deg(v_i) = 1$ for $i \in \{1, 2\}$ and $\deg(u_3) = \deg(v_3) = 2$, where $N(u)$ and $N(v)$ denote the neighborhood of u and v respectively.

Next we provide a different proof for this result using the theory developed in this paper.

Proof. We proceed by contradiction. Assume that u, v are two vertices in $V(T)$ with the properties described in the statement of the theorem. Then there is exactly one $u - v$ path in T . Assume to the contrary, that there is a third vertex in T , called w , with the properties of the vertices described in the statement of the theorem. Again, there is a unique $u - w$ path in T and also there is a unique $v - w$ path in T . Furthermore, these $u - w$ and $v - w$ paths are of the form

$$\begin{cases} u = u'_0, u'_1, \dots, u'_l, w_0, w_1, \dots, w_m = w, \\ v = v'_0, v'_1, \dots, v'_s, w_0, w_1, \dots, w_m = w, \end{cases}$$

where $\{u'_i\}_{i=0}^l \cap \{v'_i\}_{i=0}^s = \emptyset$. Then the subtree H of T induced by the vertices

$$\{u'_i\}_{i=1}^l \cup \{v'_i\}_{i=1}^s \cup \{w_i\}_{i=0}^{m-1} \cup \{u, u_1, u_2, v, v_1, v_2, w\} \cup N(w)$$

is a subdivision of the tree $T,^*$ where $N(w)$ denotes the neighborhood of w . ■

3. PARTICULAR FAMILIES OF PATH-LIKE TREES

This section is devoted to obtain necessary and sufficient conditions for different families of trees that guarantee that our trees either are or are not path-like trees. In particular we study the following two families of trees.

- Trees with exactly one vertex of degree three and exactly three vertices of degree one. (The remaining vertices are of degree two).
- Trees with exactly one vertex of degree four and exactly four vertices of degree one. (The remaining vertices are of degree two).

Next we introduce the following lemma.

Lemma 3.1. *Let P be a path embedded in the two dimensional grid. Assume that there is an elementary transformation on a unitary path (without loss of generality we may assume that it is a horizontal path), that (possible together with other elementary transformations) allows us to obtain another path. The following holds:*

1. *All horizontal subpaths are unitary.*
2. *All horizontal subpaths have to be relocated.*
3. *All vertical subpaths have equal length.*

Proof. Let P be an embedding of the path in the grid. Without loss of generality, we may assume that we replace a unitary horizontal edge e . Assume that this movement (possible together with other movements) allows us to obtain the path. Then e has to be replaced between the two end vertices of the vertical subpaths that edge e communicates. For otherwise, the vertex or vertices of degree 3 that will appear after this replacement cannot disappear with other replacements. This shows that the two vertical subpaths that edge e communicates must have the same length. Inductively, we have to replace all horizontal subpaths, since every time that we relocate a unitary path, we create at least one vertex of degree 3, except for the last movements on each side of the embedding. Therefore,

1. All vertical subpaths have the same length.
2. All horizontal subpaths must be relocated.
3. As a consequence of (2), all horizontal subpaths must have unitary length, since for otherwise they could not be relocated. ■

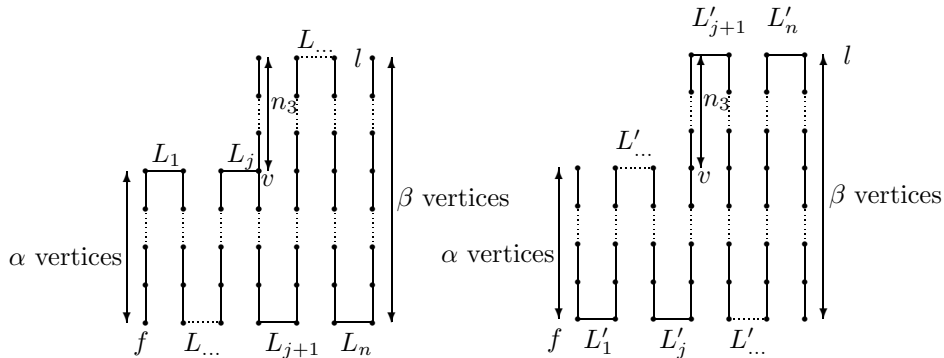


Figure 5. A path-like tree and his associated path.

Theorem 3.2. *Let T be a tree with exactly one vertex of degree three, namely v . Assume that v has attached three paths of lengths n_1, n_2 and n_3 (so that $|V(T)| = n_1 + n_2 + n_3 + 1$). Then T is a path-like tree if and only if for some permutation σ of $\{1, 2, 3\}$ there exist two natural numbers α, β such that $\alpha \mid n_{\sigma(1)}, \beta \mid (n_{\sigma(2)} + n_{\sigma(3)} + 1)$ and $\alpha + n_{\sigma(3)} = \beta$.*

Proof. (\implies) Assume that L_i is a unitary subpath of the path-like tree. We denote by L'_i the unitary subpath from which L_i is the elementary transformation.

Without loss of generality assume that L_i is a unitary horizontal subpath, with a vertex (without loss of generality, the right one) of degree 3. Replacing L_i by L'_i we are left with two cases.

Case 1. The subpath L'_i contains a vertex of degree 2 and a vertex of degree 3. In this case the part of the path-like tree that ends in the "right side" of L_i satisfies the conditions of Lemma 3.1, and therefore, its length is $n_1 = \alpha \cdot h$ where α is the length of the segment of the path-like tree that comes before L_i and $\alpha - 1$ is the length of the one of the subpaths attached to v . (Without loss of generality $\alpha = n_2 + 1$). Therefore, if we take $\beta = n_2 + n_3 + 1$ we get that $\alpha + n_3 = \beta$.

Case 2. The subpath L'_i contains two vertices of degree 3. In this case, like in case 1, the length of the part of the path-like tree that ends at L_i is $n_1 = \alpha \cdot h$ and using a similar argument to the one used in case 1, the back part has length $\beta \cdot h' = n_2 + n_3 + 1$ with $\alpha + n_2 = \beta$.

(\Leftarrow) Let T be a path-like tree that meets the hypothesis of the theorem. Take, without loss of generality, σ to be the identity permutation.

For a proof it suffices to see the figure below, where n_1 is the number of edges of the path $f - v$ and n_2 is the number of edges of the path $v - l$. The horizontal dots, represent paths of lengths $2k\alpha$ and $2h\beta$ ($k, h \in \mathbf{N}$) respectively.

It is easy to check from the left path-like tree in Figure 5, and from the elementary transformation that transforms the L_i 's into L'_i 's, that we obtain the path in the right in Figure 5. \blacksquare

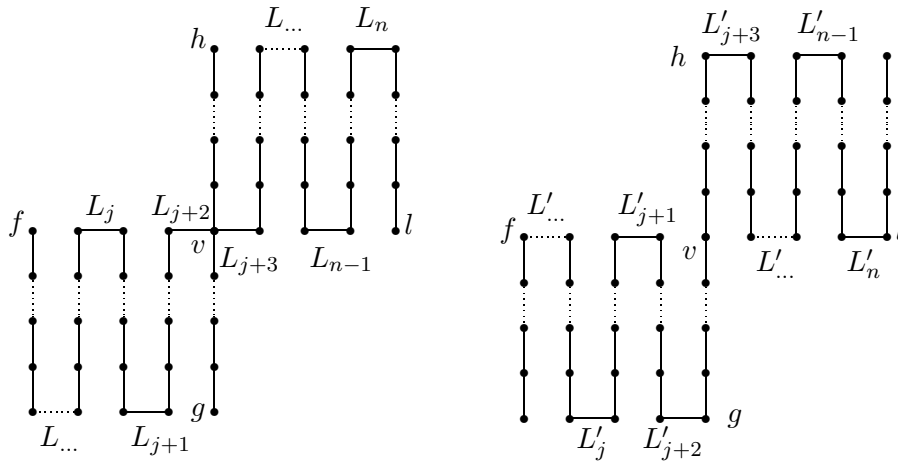


Figure 6. A path-like tree and his associated path.

Theorem 3.3. *Let T be a tree with exactly one vertex, namely v , of degree 4, exactly 4 end vertices and the remaining vertices being of degree 2. Let n_1, n_2, n_3 and n_4 be the lengths of the paths from v to each one of the end vertices. Then T is a path-like tree if and only if there exists a permutation σ of the set $\{1, 2, 3, 4\}$ such that*

- $(n_{\sigma(1)} + 1) | n_{\sigma(3)},$
- $(n_{\sigma(2)} + 1) | n_{\sigma(4)}.$

Proof. (\implies) We can view the tree T as two paths with exactly one vertex v in common. Let P be one of these two paths and let $n_i < n_j$ be the lengths of the segments of P which come before and after v respectively. Assume that L is a unitary segment of P with an end at v that comes from applying

an elementary transformation to a given segment L' . When we substitute again L by L' , the path P meets the conditions of the previous proposition. Therefore it is formed by segments of the same length that are joined by unitary segments, where $n_j = (n_i + 1)$. With the remaining of the tree, that is to say, the other path, we proceed in a similar way.

(\Leftarrow) Let T be a path-like tree that meets the hypothesis of the theorem. Take, without loss of generality, σ to be the identity permutation.

For a proof it suffices to see the Figure 6, where n_3 is the number of edges of the path $f - v$, n_4 is the number of edges of the path $v - l$, n_1 is the number of vertices of the path $g - v$ and n_2 is the number of vertices of the path $v - h$. The horizontal dots, represent paths of lengths $2kn_1$ and $2hn_2$ ($k, h \in \mathbf{N}$) respectively.

It is easy to check from the left path-like tree in Figure 6, and from the elementary transformation that transforms the L_i 's into L'_i 's, that we obtain the path in the right in Figure 6. ■

4. PATH-LIKE TREES AND T_p -TREES

In [9] Hegde and Shetty defined a tree to be a T_p -tree as follows:

Let T be a tree and u_0, v_0 be two adjacent vertices in T . Let there be two pendant vertices u, v in T such that the length of the $u_0 - u$ path and the length of the $v_0 - v$ path are equal. If the edge u_0v_0 is deleted from T , and vertices u and v are joined by an edge uv , such a transformation is called an elementary parallel transformation (or an ept), see [1], and the edge u_0v_0 is called a transformable edge. If by a sequence of ept's T can be reduced to a path, T is called a T_p -tree (transformed tree) and any such sequence regarded as a composition of mappings (ept's) denoted by P , is called a parallel transformation of T . The path, the image of T under P , is denoted by $T(P)$.

T_p -trees were introduced due to their nice labeling properties, and some of such properties were explored in [9].

The reason why we introduce this section is because the definitions of path-like trees and T_p -trees look very similar at a first glance, and hence we feel like it would be interesting to explore the relationship existing among these two concepts. Although, one may be tempted to think that the definitions of path-like trees and T_p -trees are equivalent, this is not true, and the

relationship existing among the two concepts is as shown in the next Venn diagram.

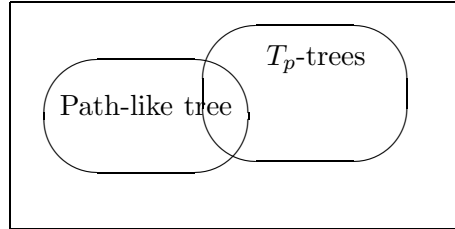


Figure 7. Venn diagram.

It is clear that

$$\{\text{path-like trees}\} \cap \{T_p\text{-trees}\} \neq \emptyset.$$

Therefore, we will show the following two facts:

1. $|\{\text{path-like trees}\} \setminus \{T_p\text{-trees}\}| = \infty$,
2. $|\{T_p\text{-trees}\} \setminus \{\text{path-like trees}\}| = \infty$.

Proof of 1. let T be a tree as described in the statement of Theorem 3.3 with $n_1 = n_2 = 2k$ and $n_3 = n_4 = 2l(2k + 1)$, $k, l \in \mathbf{N}$. By Theorem 3.3 it follows that T is a path-like tree. On the other hand, all paths in T that join two leaves have even length, and hence there is no two adjacent vertices which are at the same distance from any two leaves of T . Therefore T is not a T_p -tree.

Proof of 2. it suffices to find T_p -trees that contain T^* as a subdivision. Such T_p -trees are not path-like trees. It is clear that there are infinitely many of such T_p -trees. ■

Next, we provide an example of a T_p -tree which is not a path-like tree. Consider the tree T in Figure 8:

$$\begin{aligned} ab &\longrightarrow a_0b_0, \\ cd &\longrightarrow c_0d_0, \\ ef &\longrightarrow e_0f_0. \end{aligned}$$

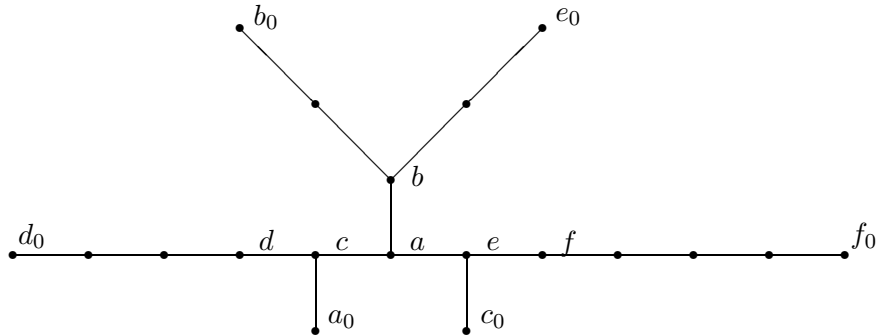


Figure 8. The tree T .

It is clear that T is not a path-like tree, since it contains a subdivision of T^* . Next, if we perform the indicated ept's in the given order, we obtain a path.

Therefore the tree T is a T_p -tree.

A. DEFINITIONS OF LABELINGS

1. **α -valuation.** Let $G = (V, E)$ be a graph of order p and size q . An injective function $f : V \rightarrow \{0, 1, \dots, p\}$ is called an α -valuation of G if

- (a) the set $\{|f(u) - f(v)| : uv \in E\}$ contains q elements,
- (b) there is $\lambda \in \mathbf{Z}$ such that for all $uv \in E$ either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$.

2. **Harmonious labelings.** Let $G = (V, E)$ be a graph of order p and size q . An injective function $f : V \rightarrow \mathbf{Z}_q$ is a harmonious labeling of G if the set $\{f(u) + f(v) \pmod{q} : uv \in E\}$ contains q elements. Furthermore, G is said to be a harmonious graph if such labeling exists. This definition extends to trees (for which $q = p - 1$) if at most one vertex label is allowed to be repeated.

3. **Special super edge-magic labelings.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with bipartite sets V_1 and V_2 with $|V_i| = p_i$ ($i = 1, 2$) and size q . A bijective function $f : V_1 \cup V_2 \cup E \rightarrow \{1, 2, \dots, q + p_1 + p_2\}$ is said to be a special super edge-magic labeling of G if

- (a) $f(V_1) = \{1, 2, \dots, p_1\}$,
- (b) $f(V_2) = \{p_1 + 1, p_1 + 2, \dots, p_1 + p_2\}$,
- (c) $f(u) + f(uv) + f(v)$ is constant for every $uv \in E$.

If a graph admits a special super edge-magic labeling it is called special super edge-magic.

B. SOME LABELINGS

Special super edge-magic labelings, α -valuations and harmonious labelings of generalized path-like trees can be generated from the following labelings of the path. This is true since when interchanging edges, the sums and differences are preserved depending on each one of the three types of labelings.

B.1. Special super edge-magic labeling of P_n

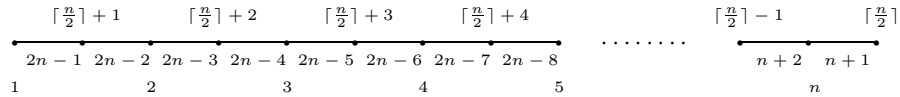


Figure 9. Special super edge-magic labeling of P_n , n odd.

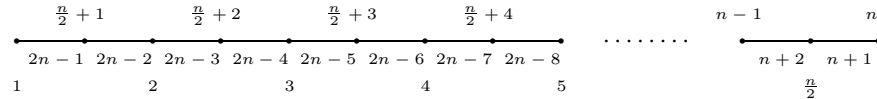


Figure 10. Special super edge-magic labeling of P_n , n even.

B.2. α -valuations of P_n

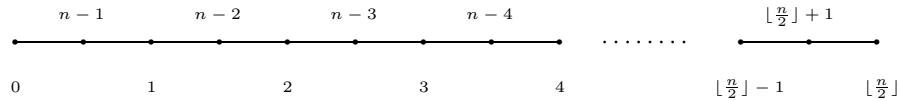


Figure 11. α -valuations of P_n , n odd.

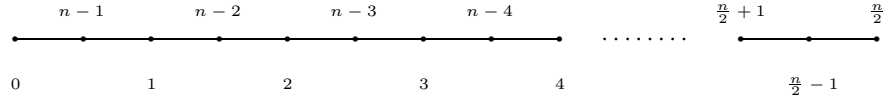


Figure 12. α -valuations of P_n , n even.

B.3. Harmonious labelings

To get harmonious labelings of the path P_n we just have to reduce the vertex labels of the special super edge-magic labelings showed in Figures 9 and 10 by one unit (mod $(n - 1)$).

C. EXAMPLES

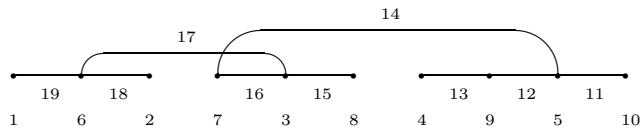


Figure 13. Special super edge magic labeling, even order.

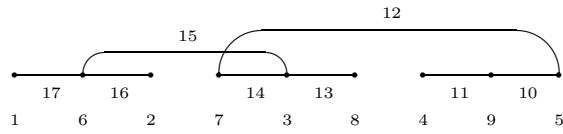


Figure 14. Special super edge magic labeling, odd order.

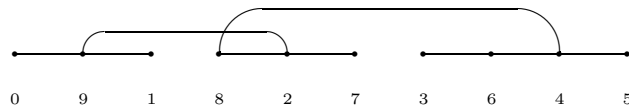


Figure 15. α -valuations, even order.

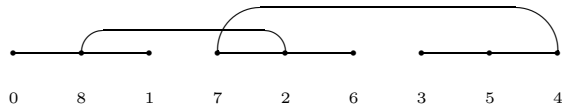


Figure 16. α -valuations, odd order.

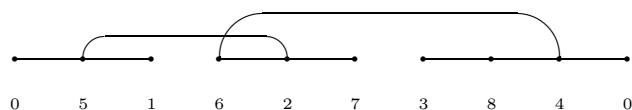


Figure 17. Harmonious labelings, even order.

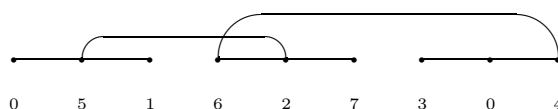


Figure 18. Harmonious labelings, odd order.

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