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# THE CHROMATIC EQUIVALENCE CLASS OF GRAPH $\overline{B_{n-6,1,2}}$ *

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**Abstract**

By $h(G, x)$ and $P(G, \lambda)$ we denote the adjoint polynomial and the chromatic polynomial of graph $G$, respectively. A new invariant of graph $G$, which is the fourth character $R_4(G)$, is given in this paper. Using the properties of the adjoint polynomials, the adjoint equivalence class of graph $B_{n-6,1,2}$ is determined, which can be regarded as the continuance of the paper written by Wang *et al.* [J. Wang, R. Liu, C. Ye and Q. Huang, A complete solution to the chromatic equivalence class of graph $\overline{B_{n-7,1,3}}$, Discrete Math. (2007), doi: 10.1016/j.disc.2007.07.030]. According to the relations between $h(G, x)$ and $P(G, \lambda)$, we also simultaneously determine the chromatic equivalence class of $\overline{B_{n-6,1,2}}$ that is the complement of $B_{n-6,1,2}$.

**Keywords**: chromatic equivalence class, adjoint polynomial, the smallest real root, the second smallest real root, the fourth character.

**2000 Mathematics Subject Classification:** 05C15, 05C60.

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## 1. Introduction

All graphs considered here are finite and simple. Notations and terminology not defined here will conform to those in [1]. For a graph $G$, let $V(G)$, $E(G)$, $p(G)$, $q(G)$ and $\overline{G}$, respectively, be the set of vertices the set of edges, the order, the size and the complement of $G$.

For a graph $G$, we denote by $P(G,\lambda)$ the chromatic polynomial of $G$. A partition $\{A_1, A_2, \ldots, A_r\}$ of $V(G)$, where $r$ is a positive integer, is called an *$r$-independent partition* of a graph $G$ if every $A_i$ is a nonempty independent set of $G$. We denote by $\alpha(G,r)$ the number of $r$-independent partitions of $G$. Thus the chromatic polynomial of $G$ is $P(G,\lambda) = \sum_{r\geq 1} \alpha(G,r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)\cdots(\lambda-r+1)$ for all $r \geq 1$. The readers can turn to [13] for details on chromatic polynomials.

Two graphs $G$ and $H$ are said to be *chromatically equivalent*, denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. By $[G]$ we denote the equivalence class determined by $G$ under " $\sim$ ". It is obvious that " $\sim$ " is an equivalence relation on the family of all graphs. A graph $G$ is called chromatically unique (or simply *$\chi$-unique*) if $H \cong G$ whenever $H \sim G$. See [6, 7] for many results on this field.

**Definition 1.1** ([11])**.** Let $G$ be a graph with $p$ vertices, the polynomial

$$h(G,x) = \sum_{i=1}^{p} \alpha(\overline{G}, i) x^i$$

is called its *adjoint polynomial.*

**Definition 1.2** ([11])**.** Let $G$ be a graph and $h_1(G,x)$ the polynomial with a nonzero constant term such that $h(G,x) = x^{\rho(G)} h_1(G,x)$. If $h_1(G,x)$ is an irreducible polynomial over the rational number field, then $G$ is called *irreducible graph.*

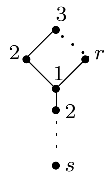
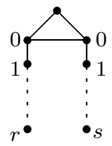
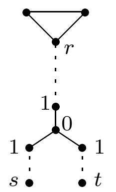
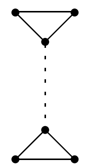
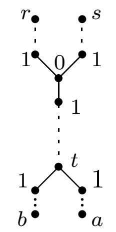
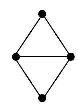
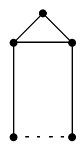
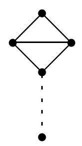
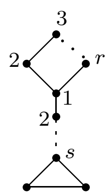
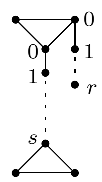
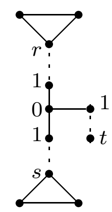
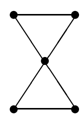
Two graphs $G$ and $H$ are said to be adjointly equivalent, denoted by $G \overset{h}{\sim} H$, if $h(G,x) = h(H,x)$. Evidently, " $\overset{h}{\sim}$ " is an equivalence relation on the family of all graphs. Let $[G]_h = \{H \mid H \overset{h}{\sim} G\}$. A graph $G$ is said to be *adjointly unique* (or simply *$h$-unique*) if $H \cong G$ whenever $H \overset{h}{\sim} G$.

**Theorem 1.1** ([3])**.** (1) *$G \overset{h}{\sim} H$ if and only if $\overline{G} \sim \overline{H}$.*

(2) $[G]_h = \{H \mid \overline{H} \in [\,\overline{G}\,]\}$.

(3) *$G$ is $\chi$-unique if and only if $\overline{G}$ h-unique.*

The graphs with orders $n$ used in the paper are drawn as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| $\xi$ |  |  |  |  |  |  |
| | $C_r(P_s)$ | $Q_{r,s}$ | $B_{r,s,t}$ | $F_p$ | $U_{r,s,t,a,b}$ | $K_4^-$ |
| | $r \geq 4, s \geq 2$ | $r, s \geq 1$ | $r, s, t \geq 1$ | $p \geq 6$ | $r, s, t, a, b \geq 1$ | $p = 4$ |
| $\psi$ |  |  |  |  |  |  |
| | $\psi_p^1$ | $\psi_p^2$ | $\psi_p^3(r,s)$ | $\psi_p^4(r,s)$ | $\psi_p^5(r,s,t)$ | $\psi_5^6$ |
| | $p \geq 5$ | $p \geq 5$ | $r \geq 4, s \geq 2$ | $r, s \geq 1$ | $r, s, t \geq 1$ | $p = 5$ |

Now we define some classes of graphs, which will be used throughout the paper.

(1) $C_n$ (resp. $P_n$) denotes the cycle (*resp.* the path) of order $n$, and write $\mathcal{C} = \{C_n \mid n \geq 3\}$, $\mathcal{P} = \{P_n \mid n \geq 2\}$ and $\mathcal{U} = \{U(1,1,t,1,1) \mid t \geq 1\}$.

(2) $D_n$ ($n \geq 4$) denotes the graph obtained from $C_3$ and $P_{n-2}$ by identifying a vertex $C_3$ with a pendant vertex of $P_{n-2}$.

(3) $T_{l_1,l_2,l_3}$ is a tree with a vertex $v$ of degree 3 such that $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$ and $l_3 \geq l_2 \geq l_1$, write $\mathcal{T}^0 = \{T_{1,1,l_3} \mid (l_3 \geq 1)\}$ and $\mathcal{T} = \{T_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \neq (1,1,1)\}$.

(4) By $B_5$ we denote the graph obtained from $C_3$ by identifying a vertex of $C_3$ with the vertex of degree 2 of the path $P_3$.

(5) $\vartheta = \{C_n, D_n, K_1, T_{l_1,l_2,l_3} \,|\, n \geq 4\}$.

(6) $\xi = \{C_r(P_s), Q_{r,s}, B_{r,s,t}, F_n, U_{r,s,t,a,b}, K_4^-\}$.

(7) $\psi = \{\psi_n^1, \psi_n^2, \psi_n^3(r,s), \psi_n^4(r,s), \psi_n^5(r,s,t), \psi_5^6\}$.

For convenience, we simply denote $h(G,x)$ by $h(G)$ and $h_1(G,x)$ by $h_1(G)$. By $\beta(G)$ and $\gamma(G)$ we denote the smallest and the second smallest real root of $h(G)$, respectively. Let $d_G(v)$, simply denoted by $d(v)$, be the degree of vertex $v$. For two graphs $G$ and $H$, $G \cup H$ denotes the disjoint union of $G$ and $H$, and $mH$ stands for the disjoint union of $m$ copies. By $K_n$ we denote the complete graph with order $n$. Let $n_G(K_3)$ and $n_G(K_4)$ denote the number of subgraphs isomorphic to $K_3$ and $K_4$, respectively. On the real field, let $g(x) \mid f(x)$ (resp. $g(x) \nmid f(x)$) denote $g(x)$ divides $f(x)$ (resp. $g(x)$ does not divide $f(x)$) and $\partial(f(x))$ denote the degree of $f(x)$. By $(f(x), g(x))$ we denote the largest common factor of $f(x)$ and $g(x)$.

It is an interesting problem to determine $[G]$ for a given graph $G$. From Theorem 1.1, it is not difficult to see that the goal of determining $[G]$ can be realized by determining $[\ \overline{G}\ ]_h$. The related topics have been partially discussed in this respect by Dong *et al.* in [3]. In this paper, using the properties of adjoint polynomials, we determine the $[B_{n-6,1,2}]_h$ for graph $B_{n-6,1,2}$, simultaneously, $[\ \overline{B_{n-6,,1,2}}\ ]$ is also determined, where $n \geq 8$.

## 2. Preliminaries

For a polynomial $f(x) = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n$, we define

$$R_1(f(x)) = \begin{cases} -\binom{b_1}{2} + 1 & \text{if } n = 1, \\ b_2 - \binom{b_1 - 1}{2} + 1 & \text{if } n \geq 2. \end{cases}$$

For a graph $G$, we write $R_1(G)$ instead of $R_1(h(G))$.

**Definition 2.1** ([2, 11])**.** Let $G$ be a graph with $q$ edges. The first character of a graph $G$ is defined as

$$R_1(G) = \begin{cases} 0 & \text{if } q = 0, \\ b_2(G) - \binom{b_1(G)-1}{2} + 1 & \text{if } q > 0. \end{cases}$$

The second character of a graph is defined as

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G) - 2)\left(b_2(G) - \binom{b_1(G)}{2}\right) - b_1(G),$$

where $b_i(G)(0 \leq i \leq 3)$ is the first four coefficients of $h(G)$.

**Lemma 2.1** ([2, 11]). *Let $G$ be a graph with $k$ components of $G_1, G_2, \ldots, G_k$. Then*

$$h(G) = \prod_{i=1}^{k} h(G_i) \quad \text{and} \quad R_j(G) = \sum_{i=1}^{k} R_j(G_i) \quad \text{for} \quad j = 1, 2.$$

It is obvious that $R_j(G)$ is an invariant of graphs. So, for any two graphs $G$ and $H$, we have $R_j(G) = R_j(H)$ for $j = 1, 2$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

**Lemma 2.2** ([8, 11]). *Let $G$ be a graph with $p$ vertices and $q$ edges. Denote by $M$ the set of vertices of the triangles in $G$ and by $M(i)$ the number of triangles which cover the vertex $i$ in $G$. If the degree sequence of $G$ is $(d_1, d_2, \ldots, d_p)$, then*

(1) $b_0(G) = 1, b_1(G) = q$.

(2) $b_2(G) = \binom{q+1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2 + n_G(K_3)$.

(3) $b_3(G) = \frac{q}{6}(q^2+3q+4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 + \sum_{ij \in E(G)} d_i d_j - \sum_{i \in M} M(i) d_i$
$+ (q+2)n_G(K_3) + n_G(K_4)$.

For an edge $e = v_1v_2$ of a graph $G$, the graph $G * e$ is defined as follows: the vertex set of $G*e$ is $(V(G)-\{v_1, v_2\}) \cup \{v\}(v \notin G)$, and the edge set of $G*e$ is $\{e' \mid e' \in E(G), e'$ is not incident with $v_1$ or $v_2\} \cup \{uv \mid u \in N_G(v_1) \cap N_G(v_2)\}$, where $N_G(v)$ is the set of vertices of $G$ which are adjacent to $v$.

**Lemma 2.3** ([11]). *Let $G$ be a graph with $e \in E(G)$. Then*

$$h(G, x) = h(G - e, x) + h(G * e, x),$$

*where $G - e$ denotes the graph obtained by deleting the edge $e$ from $G$.*

**Lemma 2.4** ([11]). (1) *For $n \geq 2$, $h(P_n) = \sum_{k \leq n} \binom{k}{n-k} x^k$.*

(2) *For* $n \geq 4$, $h(D_n) = \sum_{k \leq n} \left(\frac{n}{k}\binom{k}{n-k} + \binom{k-2}{n-k-3}\right) x^k$, $h(K_1 \cup D_n) = h(T_{1,2,n-3})$.

(3) *For* $n \geq 4, m \geq 6$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2})), h(D_m) = x(h(D_{m-1}) + h(D_{m-2}))$.

**Lemma 2.5** ([17])**.** *Let* $\{g_i(x)\}$, *simply denoted by* $\{g_i\}$, *be a polynomial sequence with integer coefficients and* $g_n(x) = x(g_n(x) + g_{n-1}(x))$. *Then*

(1) $g_n(x) = h(P_k)g_{n-k}(x) + xh(P_{k-1})g_{n-k-1}(x)$.

(2) $h_1(P_n) \mid g_{k(n+1)+i}(x)$ *if and only if* $h_1(P_n) \mid g_i(x)$, *where* $0 \leq i \leq n$, $n \geq 2$ *and* $k \geq 1$.

**Lemma 2.6** ([4, 10])**.** *Let* $G$ *be a nontrivial connected graph with* $n$ *vertices. Then*

(1) $R_1(G) \leq 1$, *and the equality holds if and only if* $G \cong P_n (n \geq 2)$ *or* $G \cong K_3$.

(2) $R_1(G) = 0$ *if and only if* $G \in \vartheta$.

(3) $R_1(G) = -1$ *if and only if* $G \in \xi$, *especially,* $q(G) = p(G) + 1$ *if and only if* $G \in \{F_n \mid n \geq 6\} \cup \{K_4^-\}$.

(4) $R_1(G) = -2$ *if and only if* $G \cong B_5$ *for* $q(G) = p(G) = 5$, $G \in \psi$ *for* $q(G) = p(G)$ *and* $G \cong K_4^-$ *for* $q(G) = p(G) + 2$.

**Lemma 2.7** ([5])**.** *For* $k \geq 0$, *let* $G^{(-k)}$ *denote the union of the components of* $G$ *whose the first characters are* $-k$ *and* $s_k$ *denote the number of components of* $G^{(-k)}$. *Then*

(1) *If* $k = 0$, *or* $1$, *or* $2$, *then* $q(G^{(-k)}) - p(G^{(-k)}) \leq ks_k$ *and the equality holds if and only if each component* $G_i$ *of* $G^{(-k)}$ *satisfies* $q(G_i) - p(G_i) = k$, *where* $1 \leq i \leq s_k$.

(2) *If* $k = 3$, *then* $q(G^{(-k)}) - p(G^{(-k)}) \leq 2s_3$ *and the equality holds if and only if each component* $G_i$ *of* $G^{(-3)}$ *verifies* $q(G_i) - p(G_i) = 2$, *where* $1 \leq i \leq s_3$.

**Lemma 2.8** ([17])**.** *Let* $G$ *be a connected graph and* $H$ *a proper subgraph of* $G$, *then*

$$\beta(G) < \beta(H).$$

**Lemma 2.9** ([17])**.** *Let* $G$ *be a connected graph. Then*

(1) $\beta(G) = -4$ *if and only if*

$$G \in \{T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, C_4(P_2), Q(2,2), K_4^-, D_8\} \cup \mathcal{U}.$$

(2) $\beta(G) > -4$ *if and only if*

$$G \in \{K_1, T(1,2,i)(2 \le i \le 4), D_i(4 \le i \le 7)\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}^0.$$

**Lemma 2.10** ([17])**.** *Let $G$ be a connected graph. Then $-(2+\sqrt{5}) \le \beta(G) < -4$ if and only if $G$ is one of the following graphs:*

(1) $T_{l_1,l_2,l_3}$ *for* $l_1 = 1, l_2 = 2, l_3 > 5$ *or* $l_1 = 1, l_2 > 2, l_3 > 3$*, or* $l_1 = l_2 = 2, l_3 > 2$*, or* $l_1 = 2, l_1 = l_2 = 3$.

(2) $U_{r,s,t,a,b}$ *for* $r = a = 1$*,* $(r,s,t) \in \{(1,1,2),(2,4,2),(2,5,3),(3,7,3),(3,8,4)\}$*, or* $r = a = 1$*,* $s \ge 1$*,* $t \ge t^*(s,b)$*,* $b \ge 1$*, where* $(s,b) \ne (1,1)$ *and*

$$t^* = \begin{cases} s+b+2, & \text{if } s \ge 3, \\ b+3, & \text{if } s = 2, \\ b, & \text{if } s = 1. \end{cases}$$

(3) $D_n$ *for* $n \ge 9$.

(4) $C_n(P_2)$ *for* $n \ge 5$.

(5) $F_n$ *for* $n \ge 9$.

(6) $B_{r,s,t}$ *for* $r = 5$*,* $s = 1$ *and* $t = 3$*, or* $r \ge 1$*,* $s = 1$ *if* $t = 1$*, or* $r \ge 4$*,* $s = 1$ *if* $t = 2$*, or* $b \ge c+3$*,* $s = 1$ *if* $t \ge 3$.

(7) $G \cong C_4(P_3)$*, or* $G \cong Q_{1,2}$.

**Lemma 2.11** ([14])**.** *Let graph* $G_n \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$*, then*

(1) $b_3(G_n) = b_3(D_n) - n + 5$ *if and only if*

$$G_n \in \{C_r(P_s) \,|\, r \ge 4, s \ge 3\} \cup \{Q_{1,n-4} \,|\, n \ge 6\} \cup \{B_{r,1,t}, B_{1,1,1} \,|\, r,t \ge 2\}.$$

(2) $b_3(G_n) = b_3(D_n) - n + 6$ *if and only if*

$$G_n \in \{Q_{r,s} \,|\, r,s \ge 2\} \cup \{B_{1,1,t}, B_{r,s,t} \,|\, r,s,t \ge 2\}.$$

**Lemma 2.12** ([14])**.** *Let graph* $G_n \in \psi$*, then* $b_3(G_n) = b_3(D_{n+1}) - 2(n+1) + t$*, where* $10 \le t \le 13$.

## 3. The Algebraic Properties of Adjoint Polynomials

### 3.1. The divisibility of adjoint polynomials and the fourth character of graph

**Lemma 3.1.1** ([17]). *For $n, m \geq 2$, $h(P_n) \mid h(P_m)$ if and only if $n+1 \mid m+1$.*

**Theorem 3.1.1.**
(1) *For $n \geq 7$, $\rho(B_{n-6,1,2}) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n-1}{2}, & \text{otherwise.} \end{cases}$*

(2) *For $n \geq 7$, $\partial(h_1(B_{n-6,1,2})) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{otherwise.} \end{cases}$*

(3) *For $n \geq 9$, $h(B_{n-6,1,2}) = x(h(B_{n-7,1,2}) + h(B_{n-8,1,2}))$.*

***Proof.*** (1) Choosing a pendant edge $e = uv \in E(B_{n-6,1,2})$ such that $d(u) = 1, d(v) = 3$, by Lemma 2.3 we have $h(B_{n-6,1,2}) = xh(D_{n-1}) + xh(P_2)h(D_{n-4})$. We have, from Lemma 2.4, that

$$\rho(K_1 \cup D_{n-1}) = 1 + \left\lfloor \frac{n-1}{2} \right\rfloor \quad \text{and} \quad \rho(K_1 \cup P_2 \cup D_{n-4}) = 2 + \left\lfloor \frac{n-4}{2} \right\rfloor.$$

If $n$ is even, then $\rho(K_1 \cup D_{n-1}) = \rho(K_1 \cup P_2 \cup D_{n-4}) = \frac{n}{2}$, which implies that $\rho(B_{n-6,1,2}) = \frac{n}{2}$. If $n$ is odd, then we arrive at $\rho(K_1 \cup D_{n-1}) = \frac{n+1}{2} > \frac{n-1}{2} = \rho(K_1 \cup P_2 \cup D_{n-4})$, which indicates that $\rho(B_{n-6,1,2}) = \frac{n-1}{2}$. Hence the result holds.
(2) obviously follows from (1). ∎

(3) Choosing a pendant edge $e = uv \in E(B_{n-6,1,2})$ such that $d(u) = 1$, $d(v) = 3$, we have, by Lemma 2.4, that

$$\begin{aligned} h(B_{n-7,1,3}) &= xh(D_{n-1}) + xh(P_2)h(D_{n-4}) \\ &= x(xh(D_{n-2}) + xh(D_{n-3})) + xh(P_2)(xh(D_{n-5}) + xh(D_{n-6})) \\ &= x(xh(D_{n-2}) + xh(P_2)h(D_{n-5})) + x(xh(D_{n-3}) + xh(P_2)h(D_{n-6})) \\ &= x(h(B_{n-7,1,2}) + h(B_{n-8,1,2})). \end{aligned}$$
∎

**Theorem 3.1.2.** *For $n \geq 2$ and $m \geq 7$, $h(P_n) \mid h(B_{m-6,1,2})$ if and only if $n = 2$ and $m = 3k + 6$ or $n = 4$ and $m = 5k + 3$, where $k \geq 1$.*

***Proof.*** Let $g_0(x) = -x^4-6x^3-10x^2-6x-1$, $g_1(x) = x^4+5x^3+6x^2+4x+1$ and $g_m(x) = x(g_{m-1}(x) + g_{m-2}(x))$. We can deduce that

$$\begin{aligned} g_0(x) &= -x^4 - 6x^3 - 10x^2 - 6x - 1, \\ g_1(x) &= x^4 + 5x^3 + 6x^2 + 4x + 1, \\ g_2(x) &= -x^4 - 4x^3 - 2x^2, \\ g_3(x) &= x^4 + 4x^3 + 4x^2 + x, \\ g_4(x) &= 2x^3 + x^2, \\ g_5(x) &= x^5 + 6x^4 + 5x^3 + x^2, \\ g_6(x) &= x^6 + 6x^5 + 7x^4 + 2x^3, \\ g_m(x) &= h(B_{m-7,1,3}) \text{ if } m \geq 7. \end{aligned} \tag{3.1}$$

Let $m = (n+1)k+i$, where $0 \leq i \leq n$. It is obvious that $h_1(P_n) \mid h(B_{m-6,1,2})$ if and only if $h_1(P_n) \mid g_m(x)$. From Lemma 2.5, it follows that $h_1(P_n) \mid g_m(x)$ if and only if $h_1(P_n) \mid g_i(x)$, where $0 \leq i \leq n$. We distinguish the following two cases:

*Case* 1. $n \geq 7$.
If $0 \leq i \leq 6$, from (3.1), it is not difficult to verify that $h_1(P_n) \nmid g_i(x)$. If $i \geq 7$, from $i \leq n$, Lemma 2.4 and Theorem 3.1.1, we have that

$$\partial(h_1(P_n)) = \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \partial(h_1(B_{i-6,1,2}) = \left\lfloor \frac{i+1}{2} \right\rfloor. \tag{3.2}$$

The following cases are taken into account:

*Subcase* 1.1. $i = n$.
It follows from (3.2) that $\partial(h_1(B_{i-6,1,2}) = \partial(h_1(P_n)) = \frac{n}{2}$ if $n$ is even and $\partial(h_1(B_{i-6,1,2}) = \partial(h_1(P_n)) + 1 = \frac{n+1}{2}$ if $n$ is odd.

*Subcase* 1.1.1. $\partial(h_1(B_{i-6,1,2}) = \partial(h_1(P_n))$.
Suppose that $h_1(P_n) \mid h_1(B_{i-6,1,2})$, we have $h_1(P_n) = h_1(B_{i-6,1,2})$, which implies $R_1(P_n) = R_1(B_{i-6,1,2})$. By Lemma 2.6 we know it is impossible. Hence $h_1(P_n) \nmid h_1(B_{i-6,1,2})$, together with $(h_1(P_n), x^{\alpha(B_{i-6,1,2})}) = 1$, we have $h_1(P_n) \nmid h(B_{i-6,1,2})$.

*Subcase* 1.1.2. $\partial(h_1(B_{i-6,1,2}) = \partial(h_1(P_n)) + 1$.
Assume that $h_1(P_n) \mid h_1(B_{i-6,1,2})$, it follows that $h_1(B_{i-6,1,2}) = (x+a)h_1(P_n)$. Note that $R_1(B_{i-6,1,2}) = -1$ and $R_1(P_n) = 1$, so $R_1(x+a) = -2$, which brings about $a = 4$. This implies that $\beta(B_{i-6,1,2}) = -4$, which contradicts to (6) of Lemma 2.10. Hence $h_1(P_n) \nmid h_1(B_{i-6,1,2})$, together with $(h_1(P_n), x^{\alpha(B_{i-6,1,2})}) = 1$, we have $h_1(P_n) \nmid h(B_{i-6,1,2})$.

*Case* 1.2. $i \leq n-1$.
It follows by (3.2) that $\partial(B_{i-6,1,2}) \leq \partial(h_1(P_n))$. Assume that $h_1(P_n) \mid h_1(B_{i-6,1,2})$, we have that $\partial(B_{i-6,1,2}) = \partial(h_1(P_n))$ and $h_1(P_n) = h_1(B_{i-6,1,2})$. So we can turn to Subcase 1.1.1 for the same contradiction.

*Case* 2. $2 \leq n \leq 6$.
From (1) of Lemma 2.4 and (3.1), we can verify $h_1(P_n) \mid g_i(x)$ if and only if $n = 2$ and $i = 6$ or $n = 4$ and $i = 3$ for $0 \leq i \leq n \leq 7$. From Lemma 2.5, we have $h_1(P_n) \mid h(B_{m-6,1,2})$ if and only if $n = 2$ and $m = 3k+6$ or $n = 4$ and $m = 5k+3$. From $\alpha(P_3) = 2$, $\alpha(P_6) = 3$ and $\alpha(B_{m-6,1,2}) = \lfloor \frac{m}{2} \rfloor \geq 3$ for $m \geq 7$, we obtain that the result holds. ∎

**Theorem 3.1.3.** *For $m \geq 7$, $h^2(P_2) \nmid h(B_{m-6,1,2})$ and $h^2(P_4) \nmid h(B_{m-6,1,2})$.*

***Proof.*** Suppose that $h^2(P_2) \mid h(B_{m-6,1,2})$, from Theorem 3.1.2 we have $m = 3k+6$, where $k \geq 1$. Let $g_m(x) = h(B_{m-6,1,2})$ for $m \geq 7$. By (3) of Theorem 3.1.1, (1) of Lemma 2.5, it follows that

$$
\begin{aligned}
g_m(x) &= h(P_2)g_{m-2}(x) + x^2 g_{m-3}(x) \\
&= h^2(P_2)g_{m-4}(x) + 2x^2 h(P_2)g_{m-5}(x) + x^4 g_{m-6}(x) \\
&= h^2(P_2)(g_{m-4}(x) + 2x^2 g_{m-7}(x)) + 3x^4 h(P_2)g_{m-8}(x) + x^6 g_{m-9}(x) \\
&= h^2(P_2)(g_{m-4}(x) + 2x^2 g_{m-7}(x) + 3x^4 g_{m-10}(x)) \\
&\quad + 4x^6 h(P_2)g_{m-11}(x) + x^8 g_{m-12}(x) \\
&= \cdots \\
&= h^2(P_2)\sum_{s=1}^{k-2} g_{m-3s-1}(x) + (k-1)x^{2k-4}h(P_2)g_{m+1-3(k-1)}(x) \\
&\quad + x^{2k-2}g_{m-3(k-1)}(x).
\end{aligned}
$$

According to the assumption and $m = 3k+6$, we arrive at, by (3.1), that

$$h^2(P_2)\Big|((k-1)x^{2k-4}h(P_2)g_{10}(x) + x^{2k-1}g_9(x)),$$

that is,

$$h(P_2)\Big|(kx^{2k+8} + (11k-2)x^{2k+7} + (44k-18)x^{2k+6} + (80k-53)x^{2k+5} + (70k-60)x^{2k+4} + (28k-27)x^{2k+3} + (4k+4)x^{2k+2}).$$

By direct calculation, we obtain that $k = -6$, which contradicts $k \geq 1$. Using the similar methods, we can also prove $h^2(P_4) \not| \, h(B_{m-6,1,3})$. ■

**Lemma 3.1.2** ([12]). *For* $t \geq 13$ *and* $1 \leq t_1 \leq 11$, *we have that* $\gamma(U(1,2,t,2,1)) < \gamma(U(1,2,12,2,1)) = -4 < \gamma(U(1,2,t_1,2,1))$.

**Lemma 3.1.3.** (1) *For* $r, t \geq 1$, $h(U(1,2,r,1,t)) = h(K_1 \cup B_{r,1,t})$.
(2) *For* $n \geq 13$ *and* $7 \leq n_1 \leq 11$, $\gamma(B_{n-6,1,2}) < \gamma(B_{12,1,2}) = -4 < \gamma(B_{n_1-6,1,2})$.

***Proof.*** (1) From Lemma 2.3 and by calculation, we can get that the equality holds. Here the details are omitted.
(2) From Lemma 3.1.2 and (1) of the lemma, the result obviously holds. ■

**Theorem 3.1.4.** *For* $n \geq 7$, $h(K_4^-) \,|\, h(B_{n-6,1,2})$ *if and only if* $n = 18$.

***Proof.*** According to Theorem 3.1.2, we arrive at $h_1(P_2) \,|\, h(B_{12,1,2})$, that is, $(x+1) \,|\, h(B_{12,1,2})$. From Lemma 2.10, we obtain $\gamma(B_{n-6,1,2}) < -4$. In terms of (2) of Lemma 3.1.3, we get $(x+4)|h(B_{n-6,1,2})$ if and only if $n = 18$. Noting that $(x+1, x+4) = 1$ and $h_1(K_4^-) = (x+1)(x+4)$, we obtain $h_1(K_4^-) \,|\, h(B_{n-6,1,2})$ if and only if $n = 18$, together with $\alpha(K_4^-) = 2$ and $\alpha(B_{12,1,2}) = 9$, we know that the theorem holds. ■

## 3.2. The smallest real roots and the fourth characters of graphs

An *internal* $x_1x_k$*-path* of a graph $G$ is a path $x_1x_2x_3 \cdots x_k$ (possibly $x_1 = x_k$) of $G$ such that $d(x_1)$ and $d(x_k)$ are at least 3 and $d(x_2) = d(x_3) = \cdots = d(x_{k-1})$ (unless $k = 2$).

**Lemma 3.2.1** ([17]). *Let $T$ be a tree. If $uv$ is an edge on an internal path of $T$ and $T \ncong U(1,1,t,1,1)$ for $t \geq 1$, then $\beta(T) < \beta(T_{xy})$, where $T_{xy}$ is the graph obtained from $T$ by inserting a new vertex on the edge $xy$ of $T$.*

**Lemma 3.2.2** ([17]). (1) *For $m \geq 9$, $\beta(C_{m-1}(P_2)) \leq \beta(F_m)$, with the equality holds if and only if $m = 9$.*

(2) *For $n, m \geq 5$, $\beta(C_n(P_2)) < \beta(C_{n-1}(P_2)) \leq \beta(D_m) < \beta(D_{m-1})$.*

(3) *For $m \geq 6$ and $n \geq 4$, $\beta(F_m) < \beta(F_{m+1}) < \beta(D_n)$.*

(4) *For $m \geq 6$ and $n \geq 4$, $\beta(B_{m-5,1,1}) < \beta(B_{m-4,1,1}) < \beta(D_n)$.*

From Lemma 2.3, by calculation we have the following lemma.

**Lemma 3.2.3.** (1) $B_{4,1,2} \cup K_{1,4} \overset{h}{\sim} 2K_1 \cup D_4 \cup C_8(P_2)$, $B_{12,1,2} \overset{h}{\sim} D_4 \cup D_8 \cup C_5(P_2)$, $B_{6,1,2} \overset{h}{\sim} D_5 \cup C_6(P_2)$, $B_{4,1,2} \overset{h}{\sim} C_4 \cup Q_{1,2} \overset{h}{\sim} D_4 \cup C_4(P_3)$.
(2) $B_{2n-6,1,2} \overset{h}{\sim} D_{n-1} \cup B_{n-4,1,1}$, $B_{n-6,1,2} \overset{h}{\sim} K_1 \cup F_{n-1}$.

***Proof.*** We only give the proof of $B_{2n-6,1,2} \overset{h}{\sim} D_{n-1} \cup B_{n-4,1,1}$, the others can be proved similarly. We choose the edge $e \in E(B_{2n-6,1,2})$ such that $B_{2n-6,1,2} = D_{n-1} \cup T_{1,2,n-3}$. In the light of Lemmas 2.3 and 2.4, we obtain that

$$\begin{aligned} h(B_{2n-6,1,2}) &= h(D_{n-1})h(T_{1,2,n-3}) + xh(D_{n-2})h(T_{1,2,n-4}) \\ &= xh(D_{n-1})h(D_n) + x^2 h(D_{n-2})h(D_{n-1}) \\ &= h(D_{n-1})(xh(D_{n-1}) + x^2 h(D_{n-2})). \end{aligned} \tag{3.3}$$

Choosing one of the pendant edges in $B_{n-4,1,1}$, we conclude, from Lemma 2.3, that

$$h(B_{n-4,1,1}) = xh(D_n) + x^2 h(D_{n-2}). \tag{3.4}$$

Combining (3.3) with (3.4), we have that the result holds.

**Theorem 3.2.1.** (1) $\beta(B_{4,1,2}) = \beta(C_8(P_2)) = \beta(C_4(P_3)) = \beta(Q_{1,2})$, $\beta(B_{6,1,2}) = \beta(C_6(P_2))$, $\beta(B_{12,1,2}) = \beta(C_5(P_2))$.

(2) $\beta(B_{2n-5,1,2}) = \beta(B_{n-4,1,1})$, $\beta(B_{n-6,1,2}) = \beta(F_{n-1})$.

(3) *For $n \geq 7$, $\beta(B_{n-6,1,2}) < \beta(B_{n-5,1,2})$.*

(4) *For* $n_1 \geq 9, 7 \leq n_2 \leq 11$ *and* $n_3 \geq 13$, $\beta(B_{1,1,2}) < \beta(B_{2,1,2}) < \beta(B_{3,1,2}) < \beta(C_{n_1}(P_2)) < \beta(C_8(P_2)) = \beta(B_{4,1,2}) < \beta(C_7(P_2)) < \beta(C_6(P_2)) = \beta(B_{6,1,2}) < \beta(B_{n_2,1,2}) < \beta(B_{12,1,2}) = \beta(C_5(P_2)) < \beta(B_{n_3,1,2}) < \beta(C_4(P_2))$.

(5) *For* $n \geq 7$ *and* $m \geq 6$, $\beta(B_{n-6,1,2}) = \beta(F_m)$ *if and only if* $m = n - 1$.

(6) *For* $n \geq 7$, $\beta(Q_{1,2}) = \beta(C_4(P_3)) = \beta(B_{n-6,1,2})$ *if and only if* $n = 10$.

(7) *For* $n \geq 7$ *and* $m \geq 4$, $\beta(B_{n-6,1,2}) < \beta(D_m)$.

(8) *For* $n \geq 7$ *and* $m \geq 6$, $\beta(B_{n-6,1,2}) = \beta(B_{m-5,1,1})$ *if and only if* $n = 2k$ *and* $m = k + 1$.

(9) *For* $t \geq 3$ *and* $n \geq m$, $\beta(B_{m-t-4,1,t}) < \beta(B_{n-6,1,2})$.

(10) *If a graph* $G$ *satisfies* $R_1(G) \leq -2$, *then* $\beta(G) < -2 - \sqrt{5}$.

***Proof.*** The first two results follow from Lemma 3.2.2.

(3) From (2) of Lemma 3.2.3 and (3) of Lemma 3.2.2, we obtain the result.

(4) The result follows from Lemmas 2.9 and 2.10, (2) of Lemma 3.2.2 and (1),(3) of the theorem.

(5) The result follows from (2) and (3) of the theorem and (3) of Lemma 3.2.2.

(6) The result follows from (1) and (3) of the theorem.

(7) In terms of (2) of the theorem and (3) of Lemma 3.2.2, we arrive at the result.

(8) In view of (2) of the theorem and (4) of Lemma 3.2.2, we arrive at the result.

(9) From (1) of Lemma 3.1.3 and Lemma 3.2.1, we have that

$$\beta(B_{m-t-4,1,t}) < \beta(B_{n-t-4,1,t}) \leq \beta(B_{n-7,1,t}) < \beta(B_{n-6,1,t}) < \beta(B_{n-6,1,2}).$$

(10) In terms of Lemmas 2.6 and 2.10, we arrive at the result. ■

**Definition 3.2.1.** Let $G$ be a graph with $p$ vertices and $q$ edges. The fourth character of a graph is defined as follows:

$$R_4(G) = R_2(G) + p(G) - q(G).$$

From Lemmas 2.1 and 2.2, we obtain the following two theorems:

**Theorem 3.2.2.** *Let $G$ be a graph with $k$ components $G_1, G_2, \ldots, G_k$. Then*

$$R_4(G) = \sum_{i=1}^{k} R_4(G_k).$$

■

**Theorem 3.2.3.** *If graphs $G$ and $H$ such that $h(G) = h(H)$ and $h_1(G) = h_1(H)$, then*

$$R_4(G) = R_4(H).$$

■

From Definitions 3.1.2 and 2.1, we have the following theorem.

**Theorem 3.2.4.** (1) $R_4(C_n) = 0$ *for* $n \geq 4$ *and* $R_4(C_3) = -2$; $R_4(K_1) = 1$.

(2) $R_4(B_{r,1,1}) = 3$ *for* $r \geq 1$ *and* $R_4(B_{r,1,t}) = 4$ *for* $r, t > 1$.

(3) $R_4(F_6) = 4, R_4(F_n) = 3$ *for* $n \geq 7$ *and* $R_4(K_4^-) = 2$.

(4) $R_4(D_4) = 0$ *and* $R_4(D_n) = 1$ *for* $n \geq 5$, $R_4(T_{1,1,1}) = 0$.

(5) $R_4(T_{1,1,l_3}) = 1$, $R_4(T_{1,l_2,l_3}) = 2$ *and* $R_4(T_{l_1,l_2,l_3}) = 3$ *for* $l_3 \geq l_2 \geq l_1 \geq 2$.

(6) $R_4(C_r(P_2)) = 3$ *for* $m \geq 4$ *and* $R_4(C_4(P_3)) = R_4(Q_{1,2}) = 4$.

(7) $R_4(P_2) = 0$ *and* $R_4(P_n) = -1$ *for* $n \geq 3$. ■

## 4. The Chromaticity of Graph $\overline{B_{n-6,1,2}}$

**Lemma 4.1** ([16]). *For $n \geq 4$, $D_n$ is adjointly unique if and only if $n \neq 4, 8$.*

**Lemma 4.2** ([9]). *Let $f(x)$ be a monadic polynomial in $x$ having integral coefficients. If all the roots of $f(x)$ are non-negative and there exists a positive integer $k$ such that $f(k)$ is a prime number, then $f(x)$ is a irreducible polynomial over the rational number field.*

**Lemma 4.3.** $[Q_{1,1}]_h = [C_4(P_3)]_h = \{Q_{1,1}, C_4(P_3), K_1 \cup K_4^-\}$.

***Proof.*** The most understandable proof is that we list all the graphs with orders 5 and sizes 5, then we obtain the lemma. We can also prove the lemma by the method used in Theorem 4.3. Here the details are omitted.

**Theorem 4.1.** *Let $G$ be a graph such that $G \overset{h}{\sim} B_{n-6,1,2}$, where $n \geq 7$. Then $G$ contains at most two components whose first characters are 1, furthermore, one of both is $P_2$ and the other is $P_4$, or one of both is $P_2$ and the other is $C_3$.*

***Proof.*** Let $G_1$ be one of the components of $G$ such that $R_1(G) = 1$. From Lemma 2.6, it follows, from Theorem 3.1.2, that $h(G_1) \mid h(B_{n-6,1,2})$ if and only if $G_1 \cong P_2$ and $n = 3k + 6$, or $G_1 \cong P_4$ and $n = 5k + 3$. According to (1) of Lemma 2.5, we obtain the following equality:

$$h(B_{15k+12,1,2}) = h(P_{15})h(B_{15(k-1)+12,1,2}) + xh(P_{14})h(B_{15(k-1)+11,1,2}). \tag{4.1}$$

Noting that $\{n \mid n = 3k + 6, k \geq 1\} \cap \{n \mid n = 5k + 3, k \geq 1\} = \{n \mid n = 15k + 18, k \geq 0\}$, we have that

$$h(P_2)h(P_4) \mid h(B_{15(k-1)+12,1,2}). \tag{4.2}$$

By Lemma 3.1.1, we get $h(P_2) \mid h(P_{14})$ and $h(P_4) \mid h(P_{14})$, together with $(h_1(P_2), h_1(P_4)) = 1$, which leads to

$$h(P_2)h(P_4) \mid h(P_{20}). \tag{4.3}$$

From (4.1) to (4.3), we obtain $h(P_2)h(P_4) \mid h(B_{15k+12,1,2})$. Noting $h(P_4) = h(K_1 \cup C_3)$, we also have $h(P_2)h(C_3) \mid h(B_{15k+12,1,2})$, together with Theorem 3.1.3, so the theorem holds. ■

**Theorem 4.2.** *If graph $G$ with order $n$ satisfies $h(G) = h(B_{n-6,1,2})$, then it contains $K_4^-$ as its component if and only if $n = 18$.*

***Proof.*** From Theorem 3.1.4, we know that the theorem holds.

**Theorem 4.3.** *Let $G$ be a graph such that $G \overset{h}{\sim} B_{n-6,1,2}$, where $n \geq 7$. Then*

(1) *If $n = 7$, then $[G]_h = \{K_1 \cup F_6,\ B_{2,1,2},\ Q_{2,2}\}$.*

(2) *If $n = 8$, then $[G]_h = \{K_1 \cup F_7,\ B_{2,1,2},\ C_3 \cup B_5\}$.*

(3) *If $n = 10$, then $[G]_h = \{K_1 \cup F_9,\ B_{4,1,2},\ C_4 \cup B_{1,1,1},\ D_4 \cup B_{1,1,1},\ C_4 \cup Q_{1,2},\ D_4 \cup Q_{1,2},\ C_4 \cup C_4(P_3),\ D_4 \cup C_4(P_3)\}$.*

(4) *If $n = 12$, then $[G]_h = \{K_1 \cup F_{11},\ B_{6,1,2},\ D_5 \cup B_{2,1,1},\ D_5 \cup C_6(P_2)\}$.*

(5) *If* $n = 18$, *then* $[G]_h = \{K_1 \cup F_{17},\ B_{12,1,2},\ K_1 \cup C_3 \cup C_4 \cup K_4^- \cup C_5(P_2),\ K_1 \cup C_3 \cup D_4 \cup K_4^- \cup C_5(P_2), K_1 \cup C_3 \cup K_4^- \cup B_{5,1,1},\ C_3 \cup C_4 \cup Q_{1,1} \cup C_5(P_2),\ C_3 \cup D_4 \cup Q_{1,1} \cup C_5(P_2),\ C_3 \cup C_4 \cup C_4(P_2) \cup C_5(P_2),\ C_3 \cup D_4 \cup C_4(P_2) \cup C_5(P_2),\ C_3 \cup C_4(P_2) \cup B_{5,1,1},\ C_3 \cup Q_{1,1} \cup B_{5,1,1},\ P_4 \cup K_4^- \cup C_4 \cup C_5(P_2),\ P_4 \cup K_4^- \cup B_{5,1,1},\ P_4 \cup K_4^- \cup D_4 \cup C_5(P_2),\ C_4 \cup D_8 \cup C_5(P_2),\ D_4 \cup D_8 \cup C_5(P_2),\ D_8 \cup B_{5,1,1}\}$.

(6) *If* $n$ *is even such that* $n \geq 14$ *and* $n \neq 18$, *then*

$$[G]_h = \{B_{n-6,1,2},\ K_1 \cup F_{n-1},\ D_{\frac{n-2}{2}} \cup B_{\frac{n-8}{2},1,1}\}.$$

(7) *If* $n$ *is odd such that* $n \geq 9$, *then* $[G]_h = \{B_{n-6,1,2}, K_1 \cup F_{n-1}\}$.

***Proof.*** (1) When $n = 7$, let graph $G$ satisfy $h(G) = h(B_{1,1,2})$. From Lemmas 2.1, 2.2 and 2.6, we obtain that $p(G) = q(G) = 7$ and $R_1(G) = -1$. By direct calculation, we arrive at $h(G) = h(B_{1,1,2}) = x^3(x^4 + 7x^3 + 13x^2 + 7x + 1)$. We distinguish the following cases:

*Case* 1. $G$ is a connected graph.
From $b_3(G) = b_3(B_{1,1,2}) = 7$ and (2) of Lemma 2.11, it follows that $G \in \{Q_{2,2}, B_{1,1,2}\}$. By calculation, we have that $Q_{2,2}, B_{1,1,2} \in [G]_h$.

*Case* 2. $G$ is not a connected graph.
Noting that $h_1(B_{1,1,2}, 1) = 29$ and from Lemma 4.2, we have that $h_1(B_{1,1,2})$ is a irreducible polynomial over the rational number field, which leads to $G = aK_1 \cup G_1$, where $a \geq 1$ and $G_1$ is a connected graph. It is not difficult to see that $q(G_1) - p(G_1) \geq 1$. By $R_1(G_1) = -1$ and Lemma 2.7, we arrive at $q(G_1) - p(G_1) \leq 1$. So $q(G_1) = p(G_1) + 1$. By Lemma 2.6, it follows that $G_1 \cong F_6$, which leads to $G_1 = K_1 \cup F_6$. From (2) of Lemma 3.2.3, we arrive at $K_1 \cup F_6 \in [G]_h$.

(2) When $n = 8$, let $G$ be a graph such that $h(G) = h(B_{2,1,2})$, which leads to $p(G) = q(G) = 8$ and $R_1(G) = -1$. We distinguish the following cases:

*Case* 1. $G$ is a connected graph.
By $b_3(G) = b_3(B_{2,1,3})$ and (1) of Lemma 2.11, we obtain that $G \in \{C_4(P_5), C_5(P_4), C_6(P_3), Q_{1,4}, B_{2,1,2}\}$. By calculation, we have that $B_{2,1,2} \in [G]_h$.

*Case* 2. $G$ is not a connected graph.
By calculation, we have $h(G) = h(B_{2,1,2}) = x^4 f_1(x) f_2(x)$,where $f_1(x) = x^2 + 3x + 1$ and $f_2(x) = x^2 + 5x + 3$. By calculation, we have that $R_1(f_1(x)) = 1$.

Noting that $b_1(f_1(x)) = 3$, we obtain that $f_1(x) = h_1(P_4)$ or $f_1(x) = h_1(C_3)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph.

*Case* 2.1. Neither $P_4$ nor $C_3$ is not a component of $G$.
Since $G$ is not connected, then the expression of $G$ is $G = aK_1 \cup G_1$, where $a \geq 1$ and $G_1$ is connected. It is not difficult to obtain that $q(G_1)-p(G_1) \geq 1$. We conclude, from Lemma 2.7, that $q(G_1) - p(G_1) \leq 1$. Thus $q(G_1) = p(G_1) + 1$. From Lemma 2.6, it follows that $G_1 \cong F_7$ and $G = K_1 \cup F_7$. In terms of (2) of Lemma 3.2.3, we arrive at $G = K_1 \cup F_7 \in [G]_h$.

*Case* 2.2. Either $P_4$ or $C_3$ is a component of $G$.

*Subcase* 2.2.1. $P_4$ is a component of $G$.
Let $G = P_4 \cup G_1$, where $h_1(G_1) = x^2 + 5x + 3$. The following cases are taken into account:

*Subcase* 2.2.1.1. $G_1$ is a connected graph.
Noting that $R_1(G_1) = -2$ and $q(G_1) = p(G_1)+1 = 5$, we have, from Lemma 2.6, that $G_1 \in \psi$. Nevertheless it contradicts that the order of any graph belonging to $\psi$ is not less than 5.

*Subcase* 2.2.1.2. $G_1$ is not connected.
It follows that $G = P_4 \cup aK_1 \cup G_1$, where $a \geq 1$ and $h_1(G_1) = x^2 + 5x + 3$. It is not difficult to get that $q(G_1) - p(G_1) \geq 2$. Remarking that $R_1(G_1) = -2$, we obtain, from Lemma 2.7, that $q(G_1) - p(G_1) \leq 2$, which results in $q(G_1) = p(G_1) + 2$. Thus we conclude, from Lemma 2.6, that $G_1 \cong K_4^-$, which contradicts to $q(G_1) = 5$.

*Subcase* 2.2.2. $C_3$ is a component of $G$.
Let $G = C_3 \cup G_1$, where $h_1(G_1) = x^2 + 5x + 3$. The following cases are taken into account:

*Subcase* 2.2.2.1. $G_1$ is a connected graph.
Noting that $R_1(G_1) = -2$ and $q(G_1) = p(G_1) = 5$, we have, from Lemma 2.6, that $G_1 \cong B_5$. By calculation, we arrive at $C_3 \cup B_5 \in [G]_h$.

*Subcase* 2.2.2.2. $G_1$ is not connected.
It follows that $G = C_3 \cup aK_1 \cup G_1$, where $a \geq 1$ and $h_1(G_1) = x^2+5x+3$. It is not difficult to get that $q(G_1) - p(G_1) \geq 1$. Remarking that $R_1(G_1) = -2$,

we conclude, from Lemma 2.6, that $1 \leq q(G_1) - p(G_1) \leq 2$. If $q(G_1) = p(G_1) + 1$ or $q(G_1) = p(G_1) + 2$, then we can turn to Subcase 2.2.1 for the same contradiction.

(3) When $n = 9$, let $G$ be a graph such that $h(G) = h(B_{3,1,2})$, which brings $p(G) = q(G) = 9$ and $R_1(G) = -1$. We distinguish the following cases:

*Case* 1. $G$ is a connected graph.
By $b_3(G) = b_3(B_{3,1,2})$, we have, from (2) of Lemma 2.11, that $G \in \{C_4(P_6), C_5(P_5), C_6(P_4), C_7(P_3), Q_{1,5}, B_{3,1,2}\}$. By calculation, we have that $B_{3,1,2} \in [G]_h$.

*Case* 2. $G$ is not a connected graph.
By calculation, we obtain that $h(G) = h(B_{3,1,2}) = x^4 f_1(x) f_2(x)$, where $f_1(x) = x + 1$ and $f_2(x) = x^4 + 8x^3 + 18x^2 + 9x + 1$. Remarking that $R_1(f_1(x)) = 1$ and $b_1(f_1(x)) = 1$, from (1) of Lemma 2.6, we have $f_1(x) = h_1(P_2)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph.

*Case* 2.1. $P_2$ is not a component of $G$.
Since $G$ is not connected, then the expression of $G$ is $G = aK_1 \cup G_1$, where $a \geq 1$ and $G_1$ is a connected graph. It is not difficult to obtain that $q(G_1) - p(G_1) \geq 1$. Noting that $R_1(G_1) = -1$, we have, from Lemma 2.7, that $q(G_1) - p(G_1) \leq 1$. Thus $q(G_1) = 9 = p(G_1) + 1$, which leads to $G_1 \cong F_8$ by Lemma 2.6. By calculation, we arrive at $K_1 \cup F_8 \in [G]_h$.

*Case* 2.2. $P_2$ is a component of $G$.
Let $G = P_2 \cup G_1$, where $h_1(G_1) = x^4 + 8x^3 + 18x^2 + 9x + 1$. The following subcases must be considered:

*Subcase* 2.2.1. $G_1$ is a connected graph.
From $R_1(G_1) = -2$ and $q(G_1) = p(G_1) + 1$, we have that $G_1 \in \psi$ by (4) of Lemma 2.6. By Lemma 2.12, we obtain that $b_3(G_1) \geq 11$, which contradicts $b_1(G_1) = 9$.

*Subcase* 2.2.2. $G_1$ is not a connected graph.
From $h_1(G_1, 1) = 37$ and Lemma 2.13, we have that $h_1(G_1)$ is a irreducible polynomial over the rational number field, which leads to $G_1 = aK_1 \cup G_2$ and $G = P_6 \cup aK_1 \cup G_2$, where $a \geq 1$ and $G_2$ is a connected graph. It is not difficult to get that $q(G_2) - p(G_2) \geq 2$. From (1) of Lemma 2.7, we have

that $q(G_2) - p(G_2) \leq 2$. Hence $q(G_2) - p(G_2) = 2$, which leads to $G_2 \cong K_4$ by (4) of Lemma 2.6. This contradicts $q(G_2) = 8$.

(4) When $n \geq 10$, let $G = \bigcup_{i=1}^{t} G_i$. From Lemma 2.1, we have that

$$h(G) = \prod_{i=1}^{t} h(G_i) = h(B_{n-6,1,2}), \tag{4.4}$$

which results in $\beta(G) = \beta(B_{n-6,1,2}) \in [-2-\sqrt{5}, -4)$ by Lemma 2.10. Let $s_i$ denote the number of components $G_i$ such that $R_1(G_i) = -i$, where $i \geq -1$. From Theorem 4.1, Lemmas 2.1 and 2.2, it follows that $0 \leq s_{-1} \leq 2$, $R_1(G) = \sum_{i=1}^{t} R_1(G_i) = -1$ and $q(G) = p(G)$, which results in

$$\begin{aligned} & -3 \leq R_1(G_i) \leq 1, \\ & s_{-1} = s_1 + 2s_2 + 3s_3 - 1, \\ & \sum_{-3 \leq R_1(G_i) \leq 0} (q(G_i) - p(G_i)) = s_{-1}. \end{aligned} \tag{4.5}$$

According to (4.5) and Lemma 2.7, we have that

$$-1 + s_3 + s_1 \leq \sum_{R_1(G_i)=-1} (q(G_i) - p(G_i)) \leq s_1, \tag{4.6}$$

We distinguish the following cases by $0 \leq s_{-1} \leq 2$:

*Case* 1. $s_{-1} = 0$.
It follows, from (4.5) and (4.6), that

$$s_3 = 0, s_2 = 0, s_1 = 1, \quad \text{and} \quad 0 \leq q(G_1) - p(G_1) \leq 1 \tag{4.7}$$

with $R_1(G_1) = -1$.

From (4.7), we set

$$G = G_1 \cup \Big(\bigcup_{i \in A} C_i\Big) \cup \Big(\bigcup_{j \in B} D_i\Big) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup \Big(\bigcup_{T \in \mathcal{T}_0} T_{l_1,l_2,l_3}\Big), \tag{4.8}$$

where $\bigcup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\bigcup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\bigcup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\bigcup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} \,|\, l_3 \geq 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} \,|\, l_3 \geq l_2 \geq 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} \,|\, l_3 \geq l_2 \geq l_1 \geq 2\}$, $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree $T_{l_1,l_2,l_3}$ is denoted by $T$ for short,

$A = \{i \,|\, i \geq 4\}$ and $B = \{j \,|\, j \geq 5\}$. From Theorems 3.2.2, 3.2.3 and 3.2.4, we arrive at

$$R_4(G) = R_4(B_{n-6,1,2}) = 4 = R_4(G_1) + |\,B\,| + a + |\,\mathcal{T}_1\,| + 2\,|\,\mathcal{T}_2\,| + 3\,|\,\mathcal{T}_3\,|. \tag{4.9}$$

We distinguish the following cases by $0 \leq q(G_1) - p(G_1) \leq 1$:

*Case* 1.1. $q(G_1) = p(G_1) + 1$.
From Lemmas 2.6 and 2.10, we have $G_1 \in \{F_m, K_4^- \,|\, m \geq 9\}$. Recalling that $q(G) = p(G)$, we obtain the following equality:

$$a + b + |\,\mathcal{T}_1\,| + |\,\mathcal{T}_2\,| + |\,\mathcal{T}_3\,| = 1. \tag{4.10}$$

*Subcase* 1.1.1. $G_1 \cong F_m$.
If $m \geq 9$, from (3) of Theorem 3.1.6, (4.9) and (4.10), we arrive at $|\,B\,| + a + |\,\mathcal{T}_1\,| + 2\,|\,\mathcal{T}_2\,| + 3\,|\,\mathcal{T}_3\,| = 1$, which leads to $|\,B\,| + a + |\,\mathcal{T}_1\,| = 1$, $|\,\mathcal{T}_2\,| = |\,\mathcal{T}_3\,| = 0$ and $a + b + |\,\mathcal{T}_1\,| = 1$. Thus, we have the following three cases to be considered:
If $|\,B\,| = 1$, then $a = |\,\mathcal{T}_1\,| = 0$ and $b = 1$, which results in

$$G = F_m \cup (\bigcup_{i \in A} C_i) \cup D_j \cup fD_4 \cup T_{1,1,1}.$$

If $a = 1$, then $|\,B\,| = |\,\mathcal{T}_1\,| = b = 0$, which leads to

$$G = F_m \cup (\bigcup_{i \in A} C_i) \cup fD_4 \cup K_1.$$

If $|\,\mathcal{T}_1\,| = 1$, then $|\,B\,| = a = b = 0$, which brings about

$$G = F_m \cup (\bigcup_{i \in A} C_i) \cup fD_4 \cup T_{1,1,l_3}.$$

As stated above, we always have, from Lemmas 2.9 and 2.10, that $\beta(G) = \beta(F_m)$. From (5) of Theorem 3.2.1 and $\beta(G) = \beta(B_{n-6,1,2})$, it follows that $\beta(F_m) = \beta(B_{n-6,1,2})$ if and only if $m = n - 1$. Note that $p(G) = p(B_{n-6,1,2}) = n$, so we arrive at $a = 1, A = B = \mathcal{T} = \emptyset, f = b = 0$, which leads to $G = K_1 \cup F_{n-1}$. From (2) of Lemma 3.2.3, we know that $K_1 \cup F_{n-1} \in [G]_h$.

*Subcase* 1.1.2. $G_1 \cong K_4^-$.
From Theorem 4.2, we know that $p(G) = 18$, that is to say,

$$
\begin{aligned}
& h(G) = h(B_{12,1,2}) \\
(4.11)\quad & = x^9(x+1)(x+4)(x^2+3x+1)(x^2+4x+2)(x^3+6x^2+8x+1) \\
& = x^9 h_1(C_3 \cup D_4 \cup C_5(P_2)).
\end{aligned}
$$

Eliminating the common factors $h(K_4^-)$ and $x$ of $h(G)$ and $h(B_{12,1,2})$, by (4.8) and (4.11) we get $h_1(H_1) = h_1(H_2)$, where $H_1 = (\bigcup_{i\in A} C_i) \cup (\bigcup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\bigcup_{\mathcal{T}_0} T_{l_1,l_2,l_3})$ and $H_2 = C_3 \cup D_4 \cup C_5(P_2)$. It is clear that $R_1(H_1) = 0$ and $R_1(H_2) = -1$, which contradicts to $R_1(H_1) = R_1(H_2)$.

*Case* 1.2. $q(G_1) = p(G_1)$.
Recalling that $q(G) = p(G)$, we arrive at, from (4.8), $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, which leads to

$$G = G_1 \cup \Big(\bigcup_{i\in A} C_i\Big) \cup \Big(\bigcup_{j\in B} D_i\Big) \cup f D_4. \tag{4.12}$$

From (3) of Lemma 2.6 and Lemma 2.10, it follows that

$$G_1 \in \{B_{m-t-4,1,t}, C_m(P_2), Q_{1,2}, C_4(P_3)\}, \tag{4.13}$$

where $m-t-4, t$ and $m$ satisfy the conditions of Lemma 2.10.

We distinguish the following cases by (4.13):

*Subcase* 1.2.1. $G_1 \cong C_m(P_2)$.
From lemma 2.9 and (2) of Lemma 3.2.2, it follows that $\beta(G) = \beta(C_m(P_2))$. Since $\beta(G) = \beta(B_{n-7,1,3})$, we have, from (4) of Theorem 3.2.1, that $\beta(B_{n-6,1,2}) = \beta(C_m(P_2))$ if and only if $n = 10, m = 8$, or $n = 12, m = 6$, or $n = 18, m = 5$. The four subcases will be discussed:

*Subcase* 1.2.1.1. $n = 10, m = 8$.
In this subcase, it contradicts to $p(G) = p(B_{n-6,1,2})$.

*Subcase* 1.2.1.2. $n = 12, m = 6$.
From (4.12) and $p(G) = 12$, we only have that $G = C_6(P_2) \cup C_5$ or $G = C_6(P_2) \cup D_5$, By calculation, we arrive at $C_6(P_2) \cup D_5 \in [G]_h$.

*Subcase* 1.2.1.3. $n = 18, m = 5$.
By (4.12) and $p(G) = 18$, we get that $G \in \{3D_4 \cup C_5(P_2), 3C_4 \cup C_5(P_2), 2D_4 \cup C_4 \cup C_5(P_2), 2C_4 \cup D_4 \cup C_5(P_2), C_4 \cup C_8 \cup C_5(P_2), C_4 \cup D_8 \cup C_5(P_2), D_4 \cup C_8 \cup C_5(P_2), D_4 \cup D_8 \cup C_5(P_2), C_{12} \cup C_5(P_2), D_{12} \cup C_5(P_2), C_5 \cup C_7 \cup C_5(P_2), C_5 \cup D_7 \cup C_5(P_2), D_5 \cup C_7 \cup C_5(P_2), D_5 \cup D_7 \cup C_5(P_2), 2C_6 \cup C_5(P_2), 2D_6 \cup C_5(P_2), C_6 \cup D_6 \cup C_5(P_2), \}$. By Lemma 2.3 and by calculation, it follows that $C_4 \cup D_8 \cup C_5(P_2), D_4 \cup D_8 \cup C_5(P_2) \in [G]_h$.

*Subcase* 1.2.2. $G_1 \cong Q_{1,2}$ or $G_1 \cong C_4(P_3)$.
From (6) and (7) of Theorem 3.2.1 and Lemma 2.9, we have that $\beta(G) = \beta(G_1) = \beta(B_{n-7,1,3})$ if and only if $n = 10$, which brings about $G \in \mathcal{G}_1 = \{Q_{1,2} \cup C_4, Q_{1,2} \cup D_4, C_4(P_3) \cup C_4, C_4(P_3) \cup D_4\}$ by (4.12). By calculation, we have $\mathcal{G}_1 \subseteq [G]_h$.

*Subcase* 1.2.3. $G_1 \cong B_{m-t-4,1,t}$.
We distinguish the following subcases:

*Subcase* 1.2.3.1. $t = 1$.
From Lemma 2.9 and (4) of Lemma 3.2.2, it follows that $\beta(G) = \beta(B_{m-5,1,1})$. According to (8) of Theorem 3.2.1, we obtain that $\beta(B_{m-5,1,1}) = \beta(B_{n-7,1,3})$ if and only if $n = 2k, m = k + 1$, where $k \geq 1$. By (2) of Lemma 3.2.3 and eliminating the common factor $h(B_{k-4,1,1})$ of $h(G)$ and $h(B_{2k-6,1,2})$, we obtain that $h((\bigcup_{i\in A} C_i) \cup (\bigcup_{j\in B} D_j) \cup fD_4) = h(D_{k-1})$.

If $k = 4$, then $4 \in A$ and $B = \emptyset$ or $f = 1$ and $B = \emptyset$, which results in $G \in \mathcal{G}_2 = \{C_4 \cup B_{1,1,1}, D_4 \cup B_{1,1,1}\}$. By direct calculation, we have that $\mathcal{G}_2 \subseteq [G]_h$.

If $k = 9$, from Lemma 2.6, we have $\beta(D_8) = -4$, which leads to $A = \emptyset$, $f = 0$ and $8 \in B$. Thus $G = D_8 \cup B_{5,1,1}$. By calculation, we have $D_8 \cup B_{5,1,1} \in [G]_h$.

If $k \neq 5$ and $k \neq 9$, from Lemma 4.1, we obtain $A = \emptyset, f = 0$ and $k - 1 \in B$. So $G = D_{k-1} \cup B_{k-4,1,1} = D_{\frac{n-2}{2}} \cup B_{\frac{n-8}{2},1,1}$, where $n \geq 10$ is even. In terms of Lemma 3.2.3, we arrive at $D_{\frac{n-2}{2}} \cup B_{\frac{n-8}{2},1,1} \in [G]_h$.

*Subcase* 1.2.3.2. $t = 2$.
From (3) and (7) of Theorem 3.2.1 and Lemma 2.9, it follows that $\beta(G) = \beta(B_{m-6,1,2}) = \beta(B_{n-6,1,2})$ if and only if $m = n$, which leads to $G \cong B_{n-6,1,2}$.

*Subcase* 1.2.3.3. $t \geq 3$.
From (7) and (9) of Theorem 3.2.1, we arrive at $\beta(G) = \beta(B_{m-t-4,1,t}) < \beta(B_{n-6,1,2})$, which contradicts to $\beta(G) = \beta(B_{n-6,1,2})$.

*Case* 2. $s_{_{-1}} = 1$.
It follows, from (4.5), that $s_1 + 2s_2 + 3s_3 = 2$, which leads to $s_3 = 0$ and $s_1 + 2s_2 = 2$. Hence

$$s_2 = 1, s_1 = 0, \text{ or } s_2 = 0, s_1 = 2. \tag{4.14}$$

We distinguish the following cases by (4.14):

*Case* 2.1. $s_2 = 1, s_1 = 0$.
Without loss of generality, let $G_1$ be the component such that $R_1(G_1) = -2$. From Corollary 2.1, we know that $\beta(G_1) < -2 - \sqrt{5}$, which contradicts $\beta(B_{n-6,1,2}) \in [-2 - \sqrt{5}, -4)$.

*Case* 2.2. $s_2 = 0, s_1 = 2$.
Without loss of generality, let

$$\begin{aligned} G = {} & G_1 \cup G_2 \cup G_3 \cup \Big(\bigcup_{i\in A} C_i\Big) \cup \Big(\bigcup_{j\in B} D_i\Big) \cup f D_4 \cup aK_1 \\ & \cup b\, T_{1,1,1} \cup \Big(\bigcup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3}\Big), \end{aligned} \tag{4.15}$$

where $\bigcup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\bigcup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\bigcup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\bigcup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} \,|\, l_3 \geq 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} \,|\, l_3 \geq l_2 \geq 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} \,|\, l_3 \geq l_2 \geq l_1 \geq 2\}$, $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree $T_{l_1,l_2,l_3}$ is denoted by $T$ for short, $G_1 \in \{P_2, P_6\}$, $R_1(G_2) = R_2(G_3) = -1$, $A = \{i \,|\, i \geq 4\}$ and $B = \{j \,|\, j \geq 5\}$.

*Subcase* 2.2.1. $G_1 \cong C_3$.
From (4.6), we obtain that

$$1 \leq \sum_{i=2}^{3} (q(G_i) - p(G_i)) \leq 2. \tag{4.16}$$

We distinguish the following cases by (4.16):

*Subcase* 2.2.1.1. $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 1$.
From Lemmas 2.6 and 2.10, it follows that $G_2 \in \{C_m(P_2), B_{r,1,t}, C_4(P_3), Q_{1,2}\}$ and $G_3 \in \{F_m, K_4^- \mid m \geq 9\}$, where $m, r$ and $t$ satisfy the conditions of Lemma 2.10. Recalling that $q(G) = p(G)$, we have, from (4.15), that

$$a + b + |\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = 1. \tag{4.17}$$

In terms of Theorem 3.2.4, we have that $R_4(G) = R_4(C_3) + R_4(G_2) + R_4(G_3) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 4$ if and only if $G_2 \in \{C_s(P_2), B_{r,1,1}\}$, $G_3 \cong F_m$ and $|B| = a = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, or $G_2 \in \{B_{r,1,t}, C_4(P_3), Q_{1,2}\}$, $G_3 \in \{K_4^-\}$ and $|B| = a = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, or $G_2 \in \{C_s(P_2), B_{r,1,1}\}$, $G_3 \in \{K_4^-\}$, $|\mathcal{T}_2| = |\mathcal{T}_3| = 0$ and $a + |B| + |\mathcal{T}_1| = 1$, where $s, r$ and $m$ satisfy the conditions of Lemma 2.10.

*Subcase* 2.2.1.1.1. $G_2 \in \{C_s(P_2), B_{r,1,1}\}$, $G_3 \cong F_m$ and $|B| = a = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$.

From (4.17), we arrive at $b = 1$. If $G_2 \cong C_s(P_2)$ and $m = 9$, we have, from (1) of lemma 3.2.2 and Lemma 2.9, that $\beta(G) = \beta(G_2) = \beta(C_8(P_2)) = \beta(F_9)$, which contradicts to $p(G) = 10$. If $m \geq 10$, then $\beta(G) = \beta(C_s(P_2)) = \beta(B_{n-6,1,2})$ if and only if $s = 6, n = 12$, or $s = 5, n = 18$, which contradicts to $p(G) = p(B_{n-6,1,2})$.

If $G_2 \cong B_{r,1,1}$, it is clear that $\beta(G) = \beta(B_{r,1,1})$. Or else, if $\beta(G) = \beta(F_m) = \beta(B_{n-6,1,2})$, we obtain, from (5) of Theorem 3.2.1, that $m = n - 1$, which contradicts to $p(B_{r,1,1}) \geq 6$. So $\beta(G) = \beta(B_{r,1,1}) = \beta(B_{n-6,1,2})$ iff $n = 2k$ and $r = k - 4$. By (2) of Lemma 3.2.3 and eliminating the common factor $h(B_{k-4,1,1})$ of $h(G)$ and $h(B_{2k-6,1,2})$, we conclude, from (4.4), (4.15) and Lemma 2.9, that $\beta(F_m) = \beta(D_{k-1})$, which is impossible by (3) of Lemma 3.2.2.

*Subcase* 2.2.1.1.2. $G_2 \in \{B_{r,1,t}, C_4(P_3), Q_{1,2}\}$, $G_3 \in \{K_4^-\}$ and $|B| = a = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$.

From (4.18), it follows that $b = 1$. Noting that $\beta(Q_{1,2}) = \beta(C_4(P_3)) = \beta(B_{4,1,2})$ and by Lemma 2.9, we have that $\beta(G) = \beta(B_{r,1,t})$. It is clear that $p(B_{r,1,t}) \leq n - 7$, which implies, from (9) of Theorem 3.2.1, that $\beta(B_{r,1,t}) \neq \beta(B_{n-6,1,2})$ for $r, t \geq 2$.

*Subcase* 2.2.1.1.3. $G_2 \in \{C_s(P_2), B_{r,1,1}\}$, $G_3 \in \{K_4^-\}$, $|\mathcal{T}_2| = |\mathcal{T}_3| = 0$ and $a + |B| + |\mathcal{T}_1| = 1$.

If $a = 1$, then $|B| = |\mathcal{T}_1| = 0$. By Theorem 4.2, we have that $K_4^-$ is a component of $G$ if and only if $n = 18$. By (4) and (8) of Theorem 3.2.1 and Lemma 2.9, we arrive at $\beta(G) = \beta(G_2) = \beta(B_{12,1,2})$ if and only if $s = 5$ or $r = 5$. We conclude, from (4.11), that $G \in \mathcal{G}_3 = \{K_1 \cup C_3 \cup C_4 \cup K_4^- \cup C_5(P_2),\ K_1 \cup C_3 \cup D_4 \cup K_4^- \cup C_5(P_2),\ K_1 \cup C_3 \cup K_4^- \cup B_{5,1,1}\}$. From Lemma 4.3, we arrive a $G \in \mathcal{G}_4 = \{Q_{1,1} \cup C_3 \cup C_4 \cup C_5(P_2),\ Q_{1,1} \cup C_3 \cup D_4 \cup C_5(P_2),\ C_3 \cup C_4 \cup C_4(P_2) \cup C_5(P_2),\ C_3 \cup D_4 \cup C_4(P_2) \cup C_5(P_2),\ Q_{1,1} \cup C_3 \cup B_{5,1,1},\ C_4(P_2) \cup C_3 \cup B_{5,1,1}\}$. By calculation, we arrive at $\mathcal{G}_3 \cup \mathcal{G}_4 \subseteq [G]_h$.

*Subcase* 2.2.1.2. $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 2$.
From Lemmas 2.6 and 2.10, we have $G_i \in \{F_m, K_4^- \mid m \geq 9\}$ for $i = 2, 3$. In terms of (4.15), we arrive at

$$a + b + |\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = 2. \tag{4.18}$$

In terms of Theorem 3.2.4, we have that

$$\begin{aligned} R_4(G) &= R_4(C_3) + R_4(G_2) + R_4(G_3) + |B| + a + |\mathcal{T}_1| \\ &\quad + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 4. \end{aligned} \tag{4.19}$$

By (3) of Theorem 3.2.4, it follows that $R_4(G_2) + R_4(G_3) \geq 4$, which leads to $|\mathcal{T}_3| = 0$ from (4.19). If $|\mathcal{T}_2| = 1$, then we conclude, from (4.19), that $G_2 \cong G_3 \cong K_4^-$. Since $h(K_4^-) = x^3(x+1)(x+4)$, we know that $(x+1)^2 \mid h(G)$, that is, $h_1(P_2)^2 \mid h(B_{n-6,1,2})$, which contradicts to Theorem 3.1.3. Hence $|\mathcal{T}_2| = 0$. In the light of (4.19) again, we obtain that $R_4(G) = 4$ if and only if $G_2 \cong F_{m_1}, G_3 \cong F_{m_2}$ and $|B| = a = |\mathcal{T}_1| = 0$, or $G_2 \cong F_m, G_3 \cong K_4^-$ and $|B| + a + |\mathcal{T}_1| = 1$.

*Subcase* 2.2.1.2.1. $G_2 \cong F_{m_1}, G_3 \cong F_{m_2}$ and $|B| = a = |\mathcal{T}_1| = 0$.
From (4.18) we arrive at $b = 2$. In terms of Lemmas 2.9 and 2.10, we have that $\beta(G) = min\{\beta(F_{m_1}), \beta(F_{m_2})\} = \beta(F_{m_1})$ if $m_1 \leq m_2$. By (5) of Theorem 3.2.1, it follows that $\beta(G) = \beta(F_{m_1}) = \beta(B_{n-6,1,2})$ if and only if $m_1 = n - 1$, which contradicts to $m_2 \geq 9$.

*Subcase* 2.2.1.2.2. $G_2 \cong F_m, G_3 \cong K_4^-$ and $|B| + a + |\mathcal{T}_1| = 1$.
If any one in $\{a, |B|, |\mathcal{T}_1|\}$ is equal to 1, then we obtain, from Lemmas 2.9 and (3) of Lemma 3.2.2, that $\beta(G) = \beta(F_m)$. Thus we can turn to Subcase 2.2.1.2.1 for the same contradiction.

*Case* 2.2.2. $G_1 \cong P_4$.
With the same analytic method as that of Case 2.2.1, we conclude, from Theorem 3.2.4, that $R_4(G) = R_4(P_4)+R_4(G_2)+R_4(G_3)+|B|+a+|\mathcal{T}_1|+2|\mathcal{T}_2|+3|\mathcal{T}_3| = 4$ if and only if $G_2 \in \{C_s(P_2), B_{r,1,1}\}, G_3 \cong K_4^-$ and $|B| = a = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, or $G_2 \cong G_3 \cong K_4^-$, $|B|+a+|\mathcal{T}_1| = 1$ and $|\mathcal{T}_2| = |\mathcal{T}_3| = 0$, where $r$ and $s$ satisfy the conditions of Lemma 2.10.

*Subcase* 2.2.2.1. $G_2 \in \{C_s(P_2), B_{r,1,1}\}, G_3 \cong K_4^-$ and $|B| = a = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$.

From Theorem 4.2, we know that $K_4^-$ is a component of $G$ if and only if $n = 18$. According to (1) and (4) of Lemma 3.2.2, (4) and (8) of Theorem 3.2.1 and Lemma 2.9, we obtain that $\beta(G) = \beta(G_2) = \beta(B_{12,1,2})$ if and only if $s = 5$, or $r = 5$. Thus $G \in \mathcal{G}_5 = \{P_4 \cup K_4^- \cup B_{5,1,1},\ P_4 \cup K_4^- \cup C_4 \cup\ C_5(P_2),\ P_4 \cup K_4^- \cup D_4 \cup C_5(P_2)\}$. By calculation, we arrive at $\mathcal{G}_5 \subseteq [G]_h$.

*Subcase* 2.2.2.2. $G_2 \cong G_3 \cong K_4^-$, $|B|+a+|\mathcal{T}_1| = 1$ and $|\mathcal{T}_2| = |\mathcal{T}_3| = 0$.

If $G_2 \cong G_3 \cong K_4^-$, then we can turn to Subcase 2.2.1.2 for the same contradiction.

*Case* 2.2.3. $G_1 \cong P_2$.
From Lemmas 2.6 and 2.10, it follows that $G_2 \in \{C_m(P_2), B_{r,1,t}, C_4(P_3), Q_{1,2}\}$ and $G_3 \in \{F_m, K_4^- \mid m \geq 9\}$, where $m, r$ and $t$ satisfy the conditions of Lemma 2.10. we conclude, from Theorem 3.2.2, that

$$R_4(G) = R_4(P_2) + R_4(G_2) + R_4(G_3) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|.$$

Noting that $R_4(P_2) = 0$ and $R_4(G_2) + R_4(G_3) \geq 5$, we conclude, from the above equality, that $R_4(G) \geq 5$, which contradicts $R_4(G) = 4$.

*Case* 3. $s_{-1} = 2$.
From (4.5), we arrive at $s_1 + 2s_2 + 3s_3 = 3$, which brings about the following cases:

*Case* 3.1. $s_3 = 1$ and $s_1 = s_2 = 0$.
Let the component $G_1$ such that $R_1(G_1) = -3$, which contradicts $\beta(G) \in [-2-\sqrt{5}, -4)$ by Corollary 2.1.

*Case* 3.2. $s_2 = 1$ and $s_1 = s_3 = 0$.
According to the same reason as that of case 3.1, we have a contradiction.

*Case* 3.3. $s_1 = 3$ and $s_2 = s_3 = 0$.

*Case* 3.3.1. The components of $G$, with the first characters 1, are $P_2$ and $P_4$.

Without loss of generality, from Theorem 4.1, we set

$$G = P_2 \cup P_6 \cup \Big(\bigcup_{k=1}^{3} G_k\Big) \cup \Big(\bigcup_{i\in A} C_i\Big) \cup \Big(\bigcup_{j\in B} D_i\Big) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup \Big(\bigcup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3}\Big), \tag{4.20}$$

where $\bigcup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\bigcup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\bigcup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\bigcup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} \,|\, l_3 \geq 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} \,|\, l_3 \geq l_2 \geq 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} \,|\, l_3 \geq l_2 \geq l_1 \geq 2\}$, $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree $T_{l_1,l_2,l_3}$ is denoted by $T$ for short, $R_1(G_k) = -1$ for $1 \leq k \leq 3$, $A = \{i \,|\, i \geq 4\}$ and $B = \{j \,|\, j \geq 5\}$.

From (4.6), it follows that

$$2 \leq \sum_{k=1}^{3}(q(G_k) - p(G_k)) \leq 3. \tag{4.21}$$

We distinguish the following cases by (4.21):

*Subcase* 3.3.1.1. $\sum_{k=1}^{3}(q(G_k) - p(G_k)) = 2$.
From $\beta(G) \in [-2-\sqrt{5}, -4)$ and Lemmas 2.6 and 2.10, we have that $G_1 \in \{C_r(P_2),\, B_{r,1,t},\, C_4(P_3),\, Q_{1,2}\}$ and $G_2, G_3 \in \{F_r, K_4^- \,|\, r \geq 9\}$, where $r$ and $t$ satisfy the conditions of Lemma 2.10. Recalling that $q(G) = p(G)$, we have, from (4.20), that

$$a = b = |\,\mathcal{T}_1\,| = |\,\mathcal{T}_2\,| = |\,\mathcal{T}_3\,| = 0,$$

which leads to

$$R_4(G) = 4 = R_4(P_2) + R_4(P_6) + \sum_{k=1}^{3} R_4(G_k) + |\,B\,|. \tag{4.22}$$

From Theorem 3.2.4 and (4.22), we obtain that $R_4(G) = 4 \geq |\,B\,| + 6$, which results in $|\,B\,| \leq -2$. This is obviously a contradiction.

*Subcase* 3.3.1.2. $\sum_{k=1}^{3}(q(G_k) - p(G_k)) = 3$.
From lemmas 2.6 and 2.10, we have $G_k \in \{F_r, K_4^- \,|\, r \geq 9\}$. By (4.20) and Theorem 3.1.4, we obtain

$$R_4(G) = 4 = R_4(P_2)+R_4(P_6)+\sum_{k=1}^{3} R_4(G_k)+a+|B|+|\mathcal{T}_1|+2|\mathcal{T}_2|+3|\mathcal{T}_3|.$$

In terms of Theorem 3.2.4, we arrive at $R_4(G) = 4 \geq 5+a+|B|+|\mathcal{T}_1|+2|\mathcal{T}_2|+3|\mathcal{T}_3|$, which leads to $a+|B|+|\mathcal{T}_1|+2|\mathcal{T}_2|+3|\mathcal{T}_3| \leq -1$. This is also a contradiction.

*Case* 3.3.2. The components of $G$, with the first characters 1, are $P_2$ and $C_3$.

Without loss of generality, from Theorem 4.1, we set

$$G = P_2 \cup C_3 \cup \Big(\bigcup_{k=1}^{3} G_k\Big) \cup \Big(\bigcup_{i\in A} C_i\Big) \cup \Big(\bigcup_{j\in B} D_i\Big) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup \Big(\bigcup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3}\Big), \tag{4.23}$$

where $\bigcup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\bigcup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\bigcup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\bigcup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} \,|\, l_3 \geq 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} \,|\, l_3 \geq l_2 \geq 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} \,|\, l_3 \geq l_2 \geq l_1 \geq 2\}$, $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree $T_{l_1,l_2,l_3}$ is denoted by $T$ for short, $R_1(G_k) = -1$ for $1 \leq k \leq 3$, $A = \{i \,|\, i \geq 4\}$ and $B = \{j \,|\, j \geq 5\}$.

From (4.6), it follows that

$$2 \leq \sum_{k=1}^{3}(q(G_k) - p(G_k)) \leq 3. \tag{4.24}$$

We distinguish the following cases by (4.24):

*Subcase* 3.3.2.1. $\sum_{k=1}^{3}(q(G_k) - p(G_k)) = 2$.
From $\beta(G) \in [-2-\sqrt{5}, -4)$ and Lemmas 2.6 and 2.10, we get that $G_1 \in \{C_r(P_2),\, B_{r,1,t},\, C_4(P_3),\, Q_{1,2}\}$ and $G_2, G_3 \in \{F_r, K_4^- \,|\, r \geq 9\}$, where $r$ and $t$ satisfy the conditions of Lemma 2.10. Recalling that $q(G) = p(G)$, we have, from (4.23), that

$$a+b+|\mathcal{T}_1|+|\mathcal{T}_2|+|\mathcal{T}_3| = 1.$$

Recalling that the number of component $K_4^-$ is 1, we conclude, from Theorem 3.2.4, that $\sum_{k=1}^{3} R_4(G_k) \geq 8$, which lead to $R_4(G) = 4 \geq R_4(P_2) + R_4(C_3) + \sum_{k=1}^{3} R_4(G_k) + |B| = 6+|B|$, that is, $|B| \leq -2$ which is impossible.

*Subcase* 3.3.2.2. $\sum_{k=1}^{3}(q(G_k) - p(G_k)) = 3$.
From $\beta(G) \in [-2-\sqrt{5}, -4)$ and Lemmas 2.6 and 2.10, we have that $G_1, G_2, G_3 \in \{F_m, K_4^- \,|\, r \geq 9\}$, where $m$ satisfies the conditions of Lemma 2.10. Recalling that $q(G) = p(G)$, we have, from (4.23), that

$$a + b + |\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = 2.$$

Recalling that the number of component $K_4^-$ is 1, we conclude, from (3) of Theorem 3.2.4, that $\sum_{k=1}^{3} R_4(G_k) \geq 8$, which lead to $R_4(G) = 4 \geq R_4(P_2) + R_4(C_3) + \sum_{k=1}^{3} R_4(G_k) + |B| = 6 + |B|$, that is, $|B| \leq -2$ which is impossible.

This completes the proof of the theorem. ■

**Corollary 4.1.** *For $n \geq 7$, the chromatic equivalence class of $\overline{B_{n-6,1,2}}$ only contains the complements of graphs described in Theorem* 4.3.

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