Discussiones Mathematicae
Graph Theory 28 (2008) 59–66

TREES WITH EQUAL TOTAL DOMINATION AND TOTAL RESTRAINED DOMINATION NUMBERS

Xue-Gang Chen*

Department of Mathematics
North China Electric Power University
Beijing 102206, China

e-mail: gxc_xdm@163.com

Wai Chee Shiu

Department of Mathematics
Hong Kong Baptist University
224 Waterloo Road, Kowloon Tong, Hong Kong, China

and

Hong-Yu Chen

The College of Information Science and Engineering
Shandong University of Science and Technology
Qingdao, Shandong Province 266510, China

Abstract

For a graph $G = (V, E)$, a set $S \subseteq V(G)$ is a *total dominating set* if it is dominating and both $\langle S \rangle$ has no isolated vertices. The cardinality of a minimum total dominating set in G is the *total domination number*. A set $S \subseteq V(G)$ is a *total restrained dominating set* if it is total dominating and $\langle V(G) - S \rangle$ has no isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number*. We characterize all trees for which total domination and total restrained domination numbers are the same.

Keywords: total domination number, total restrained domination number, tree.

2000 Mathematics Subject Classification: 05C69.

*Partially Supported by CERG Research Grant Council of Hong Kong and Faculty Research Grant of Hong Kong Baptist University. Supported by Doctoral Research Grant of North China Electric Power University (200722026).

1. Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Arumugam [1].

Let $G = (V, E)$ be a simple graph of order n. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. For a subset S of V, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *diameter diam(G)* of a connected graph G is the maximum distance between two vertices of G, that is $diam(G) = \max_{u,v \in V(G)} d_G(u, v)$. Let P_n denote a path with n vertices. Let $K_{1,r}$ denote the star with $r+1$ vertices. Define $K_{1,r,4}$ as follows: for each edge of $K_{1,r}$, we subdivide by two vertices. The vertex of degree r is called the central vertex of $K_{1,r,4}$. Let η be a family of graphs and $\eta = \{K_{1,r,4} | r \geq 1$ and r is an integer $\}$.

A subset S of V is called a *dominating set* if every vertex in $V - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. A set $S \subseteq V(G)$ is a *total dominating set* if it is dominating and $\langle S \rangle$ has no isolated vertices. The cardinality of a minimum total dominating set in G is the *total domination number* and is denoted by $\gamma_t(G)$. Cockayne *et al.* [6] studied total dominating functions in trees: minimality and convexity.

The total restrained domination number of a graph was defined by D. Ma *et al.* in [4]. A set $S \subseteq V(G)$ is a *total restrained dominating set* if it is total dominating and $\langle V(G) - S \rangle$ has no isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number* and is denoted by $\gamma_r^t(G)$.

A total dominating set S with cardinality $\gamma_t(G)$ is called a γ_t-set. A total restrained dominating set S with cardinality γ_r^t is called a γ_r^t-set. Let $S \subset V(G)$ and $x \in S$, we say that x has a private neighbour (with respect to S) if there is a vertex in $V(G) - S$ whose only neighbour in S is x. Let $PN(x, S)$ denote the private neighbours set of x with respect to S.

A vertex of degree one is called a *leaf.* A vertex v of G is called a *support* if it is adjacent to a leaf. If T is a tree, $L(T)$ and $S(T)$ denote the set of leaves and supports, respectively. Any vertex of degree greater than one is called an *internal vertex.*

For any graph theoretical parameters λ and μ, we define G to be (λ, μ)-graph if $\lambda(G) = \mu(G)$. In this paper we provide a constructive characterization of (γ_t, γ_r^t)-trees.

2. A Characterization of (γ_t, γ_r^t)-trees

As a consequence of the definition of total restrained domination number, we have the following observations.

Observation 1. *Let G be a graph without isolated vertices. Then*

(i) *every leaf belongs to every γ_r^t-set;*

(ii) *every support belongs to every γ_r^t-set;*

(iii) $\gamma_t(G) \leq \gamma_r^t(G)$.

Observation 2. *Let T be a (γ_t, γ_r^t)-tree. Then each $\gamma_r^t(T)$-set is a $\gamma_t(T)$-set.*

Let τ_1 and τ_2 be the following two operations defined on a tree T.

• **Operation τ_1.** Assume $x \in V(T)$ is a leaf or support. Then add one or more trees of η and the edges between x and each central vertex.

• **Operation τ_2.** Assume $x \in N(S(T)) - L(T)$. Then add one or more paths P_3 and the edges between x and one leaf of each P_3.

Let τ be the family of trees such that $\tau = \{T : T$ is obtained from P_6 by a finite sequence of operations τ_1 or $\tau_2\} \cup \{P_2, P_6\}$. We show first that each tree in the family τ has equal total domination number and total restrained domination number.

Lemma 1. *If T belongs to the family τ, then T is a (γ_t, γ_r^t)-tree.*

Proof. We proceed by induction on the number of operations $s(T)$ required to construct the tree T. If $s(T) = 0$, then $T \in \{P_2, P_6\}$ and clearly T is a (γ_t, γ_r^t)-tree. Assume now that T is a tree with $s(T) = k$ for some positive integer k and each tree $T' \in \tau$ with $s(T') < k$ is a (γ_t, γ_r^t)-tree. Then T can be obtained from a tree T' belonging to τ by operation τ_1 or τ_2. We now consider two possibilities depending on whether T is obtained from T' by operation τ_1 or τ_2.

Case 1. T is obtained from T' by operation τ_1. Without loss of generality, we can assume that T is obtained from T' by adding k trees $K_{1,r_1,4}, K_{1,r_2,4}, \ldots, K_{1,r_k,4}$ of η and the edges between x and each central vertex, where $r_1 \leq r_2 \leq \cdots \leq r_k$. It is obvious that $\gamma_t(T) \leq \gamma_t(T') + 2\sum_{1\leq i\leq k} r_i$. Let D be a γ_t-set of T such that $D \cap L(T) = \emptyset$. Then $|D \cap K_{1,r_i,4}| \geq 2r_i$ for each $K_{1,r_i,4}$. Let $D' = D \cap V(T')$.

Case 1.1. x is a support of T'. Then $x \in D'$. If $N_{T'}(x) \cap D' \neq \emptyset$, then D' is a total dominating set of T'. So $\gamma_t(T') \leq |D'| \leq \gamma_t(T) - 2\sum_{1\leq i\leq k} r_i$. If $N_{T'}(x) \cap D' = \emptyset$, then there exists a tree $K_{1,r_i,4}$ such that $|D \cap K_{1,r_i,4}| \geq 2r_i + 1$ and its central vertex belongs to D. Let $y \in N_{T'}(x)$ and $D'' = D' \cup \{y\}$. Then D'' is a total dominating set of T'. So $\gamma_t(T') \leq |D''| = |D'| + 1 \leq \gamma_t(T) - 2\sum_{1\leq i\leq k} r_i$.

Case 1.2. x is a leaf of T'. Let $y \in N_{T'}(x)$. If $y \in D$, then D' is a total dominating set of T'. Suppose $y \notin D$. Then there exists a tree $K_{1,r_i,4}$ such that $|D \cap K_{1,r_i,4}| \geq 2r_i + 1$ and its central vertex belongs to D. Let $D'' = D' \cup \{y\}$. Then D'' is a total dominating set of T'. So $\gamma_t(T') \leq |D''| = |D'| + 1 \leq \gamma_t(T) - 2\sum_{1\leq i\leq k} r_i$.

By Case 1.1 and 1.2, $\gamma_t(T') \leq \gamma_t(T) - 2\sum_{1\leq i\leq k} r_i$. Hence, $\gamma_t(T) = \gamma_t(T') + 2\sum_{1\leq i\leq k} r_i$. It is obvious that $\gamma_r^t(T) \leq \gamma_r^t(T') + 2\sum_{1\leq i\leq k} r_i$. Since $\gamma_r^t(T') + 2\sum_{1\leq i\leq k} r_i = \gamma_t(T') + 2\sum_{1\leq i\leq k} r_i = \gamma_t(T) \leq \gamma_r^t(T)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2\sum_{1\leq i\leq k} r_i$. So $\gamma_t(T) = \gamma_r^t(T)$.

Case 2. T is obtained from T' by operation τ_2. Without loss of generality, we can assume that T is obtained from T' by adding paths v_{1j}, v_{2j}, v_{3j} and the edges between x and v_{1j} for $j = 1, 2, \cdots, k$. It is obvious that $\gamma_t(T) \leq \gamma_t(T') + 2k$. Let D be a γ_t-set of T such that $D \cap L(T) = \emptyset$. Then $v_{1j}, v_{2j} \in D$. Let $D' = D \cap V(T')$. Then D' is a total dominating set of T'. So $\gamma_t(T') \leq \gamma_t(T) - 2k$. Hence $\gamma_t(T) = \gamma_t(T') + 2k$. Let D'' be a γ_r^t-set of T'. Since T' is a (γ_t, γ_r^t)-tree, it follows that $x \notin D''$. Otherwise, assume $N_{T'}(x) \cap S(T') = \{y\}$ and $N_{T'}(y) \cap L(T') = \{z\}$. Then $D'' - \{z\}$ is a total dominating set of T' with cardinality less than $|D''|$, which is a contradiction. So, $\gamma_r^t(T) \leq \gamma_r^t(T') + 2k$. Since $\gamma_r^t(T') + 2k = \gamma_t(T') + 2k = \gamma_t(T) \leq \gamma_r^t(T)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2k$. So $\gamma_t(T) = \gamma_r^t(T)$. ■

We show next that every (γ_t, γ_r^t)-tree belongs to the family τ.

Lemma 2. *Let T be a (γ_t, γ_r^t)-tree. Then*

(i) *for each support $v \in S(T)$, $|N(v) \cap L(T)| = 1$;*

(ii) *for any two supports $u, v \in S(T)$, $d(u, v) \geq 3$.*

Proof. (i) Suppose that there exists a support v such that $|N(v) \cap L(T)| \geq 2$. Let $N(v) \cap L(T) = \{v_1, \ldots, v_k\}$ where $k \geq 2$. Let D be a γ_r^t-set of T. Then, by Observation 1, it follows that $D - \{v_2, \ldots, v_k\}$ is a total dominating set of T with cardinality less than $\gamma_t(T)$, which is a contradiction. Hence, $|N(v) \cap L(T)| = 1$ for each support $v \in S(T)$.

(ii) Suppose that there exist two supports u and v such that $d(u, v) \leq 2$. Let $u_1 \in N(u) \cap L(T)$ and $v_1 \in N(v) \cap L(T)$. Let D be a γ_r^t-set of T. If u is adjacent to v, then, by Observation 1, it follows that $D - \{u_1\}$ is a total dominating set of T with cardinality less than $\gamma_t(T)$, which is a contradiction. Suppose $d(u, v) = 2$. Assume $w \in N(u) \cap N(v)$. Then by Observation 1, it follows that $(D - \{u_1, v_1\}) \cup \{w\}$ is a total dominating set of T with cardinality less than $\gamma_t(T)$, which is a contradiction. Hence, $d(u, v) \geq 3$ for any two supports $u, v \in S(T)$. ■

Lemma 3. *If T is a (γ_t, γ_r^t)-tree, then T belongs to the family τ.*

Proof. Let T be a (γ_t, γ_r^t)-tree. If $diam(T) \leq 5$, then T is P_2 or P_6. It is clear that the statement is true. For this reason, we only consider only trees T with $diam(T) \geq 6$.

Let T be a (γ_t, γ_r^t)-tree and assume that the result holds for all trees on $n(T) - 1$ and fewer vertices. We proceed by induction on the number of vertices of a (γ_t, γ_r^t)-tree. Let $P = (v_0, v_1, \ldots, v_l)$, $l \geq 6$, be a longest path in T and let D be a $\gamma_r^t(T)$-set. Then $v_0, v_1 \in D$. By Lemma 2, it follows that $d(v_1) = d(v_2) = 2$. It is obvious that $v_2, v_3 \notin D$. Otherwise $D - \{v_0\}$ is a total dominating set with cardinality less than $|D|$, which is a contradiction.

Now we have the following claim.

Claim 1. $|N_T(v_3) \cap D| = 1$.

Proof. Without loss of generality, we can assume $|N_T(v_3) \cap D| = t$ and $t > 1$. Then $N_T(v_3) \cap D \subseteq S(T) \cup \{v_4\}$. By Lemma 2, $|N_T(v_3) \cap D \cap S(T)| = 1$. So, $t = 2$. We can assume $N_T(v_3) \cap D = \{v_{31}, v_4\}$, where $v_{31} \in S(T)$. By Lemma 2, it is easy to prove that $v_5 \in D$. Let $A_1 = N_T(v_5) - \{v_4\}$.

Then for any $v \in A_1$, $v \notin D$. Otherwise, let T_1 denote the component of $T - \{v_5\}$ containing v_4. Then $(D - (L(T_1) \cup \{v_4\})) \cup (N_{T_1}[S(T_1)] - L(T_1))$ is a total dominating set of T with cardinality less than $|D|$, which is a contradiction. Let $B_1 = N_T(A_1) \cap (V(T) - D)$, $A_2 = N_T(B_1) \cap D$ and $B_2 = N_T(A_2) \cap D$. For $i \geq 1$, let $A_{2i+1} = N_T(B_{2i}) \cap (V(T) - D)$, $B_{2i+1} = N_T(A_{2i+1}) \cap (V(T) - D)$, $A_{2i+2} = N_T(B_{2i+1}) \cap D$ and $B_{2i+2} = N_T(A_{2i+2}) \cap D$. It is obvious that $|B_{2i+1}| \leq |A_{2i+2}| \leq |B_{2i+2}|$ for $i \geq 0$.

Now we prove that if $N_T(B_{2i+2}) \cap D - A_{2i+2} \neq \emptyset$, then $|N_T(v) \cap D| \geq 2$ for any $v \in N_T(B_{2i+2}) \cap D - A_{2i+2}$. Otherwise, we can assume t is the maximum i satisfying $N_T(B_{2i+2}) \cap D - A_{2i+2} \neq \emptyset$ and there exists a vertex $v \in N_T(B_{2i+2}) \cap D - A_{2i+2}$ such that $|N_T(v) \cap D| = 1$. Without loss of generality, we can assume that $u \in B_{2t+2}$ and $uv \in E(T)$.

Define $C_1 = N_T(v) \setminus \{u\}$. Then for any $w \in C_1$, $w \notin D$. Let $D_1 = N_T(C_1) \cap (V(T) - D)$. Let $C_2 = N_T(D_1) \cap D$ and $D_2 = N_T(C_2) \cap D$. For $i \geq 1$, let $C_{2i+1} = N_T(D_{2i}) \cap (V(T) - D)$, $D_{2i+1} = N_T(C_{2i+1}) \cap (V(T) - D)$, $C_{2i+2} = N_T(D_{2i+1}) \cap D$ and $D_{2i+2} = N_T(C_{2i+2}) \cap D$. It is obvious that $|D_{2i+1}| \leq |C_{2i+2}| \leq |D_{2i+2}|$ for $i \geq 0$. Let $D' = (D - \{v\} - \bigcup_{0 \leq i \leq t} D_{2i+2}) \cup \bigcup_{0 \leq i \leq t} D_{2i+1}$. It is obvious that D' is a total dominating set of T with cardinality less than $|D|$, which is a contradiction.

Let $w \in A_1$. Let $\overline{D} = (D - (L(T_1) \cup \{v_4, v_5\}) - \bigcup_{0 \leq i \leq t} B_{2i+2}) \cup \bigcup_{0 \leq i \leq t} B_{2i+1} \cup \{w\} \cup (N_{T_1}[S(T_1))] - L(T_1))$. It is obvious that $\overline{D}$ is a total dominating set of T with cardinality less than $|D|$, which is a contradiction. Hence, $|N_T(v_3) \cap D| = 1$. ■

By the above claim, we consider the following three cases. Assume $d_T(v_4) = j$.

Case 1. $v_4 \in D$ and $v_4 \in S(T)$. Let T_1 denote the component of $T - \{v_4\}$ containing v_5. Let $N_T(v_4) \cap L(T) = \{l\}$ and $N_T(v_4) - \{v_5, l\} = \{v_{41}, \cdots, v_{4(j-2)}\}$. Denote $T' = \langle V(T_1) \cup \{v_4, l\} \rangle$. Then it is easy to prove that $\gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq (j-2)}(d_T(v_{4i}) - 1)$. It is obvious that $\gamma_r^t(T') \leq \gamma_r^t(T) - 2\sum_{1 \leq i \leq (j-2)}(d_T(v_{4i}) - 1)$. Since T is a (γ_t, γ_r^t)-tree, it follows that $\gamma_r^t(T) = \gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq (j-2)}(d_T(v_{4i}) - 1) \leq \gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-2)}(d_T(v_{4i}) - 1)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-2)}(d_T(v_{4i}) - 1)$. So $\gamma_t(T') = \gamma_r^t(T')$. Consequently, T' is a (γ_t, γ_r^t)-tree and by induction hypothesis, $T' \in \tau$. As v_4 is a support in T', we deduce that T may be obtained from T' by operation τ_1.

Case 2. $v_4 \in D$ and $v_4 \notin S(T)$. Let T_1 denote the component of $T - \{v_4\}$ containing v_5. Then $v_5 \in D$. Let $N_T(v_4) - \{v_5\} = \{v_{41}, \cdots, v_{4(j-1))}\}$. Denote $T' = \langle V(T_1) \cup \{v_4\}\rangle$. Then it is obvious that $\gamma_t(T) = \gamma_t(T') + 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1)$. It is obvious that $\gamma_r^t(T') \le \gamma_r^t(T) - 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1)$. Since T is a (γ_t, γ_r^t)-tree, it follows that $\gamma_r^t(T) = \gamma_t(T) = \gamma_t(T') + 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1) \le \gamma_r^t(T') + 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1)$. So $\gamma_t(T') = \gamma_r^t(T')$. Consequently, T' is a (γ_t, γ_r^t)-tree and by induction hypothesis, $T' \in \tau$. As v_4 is a leaf in T', we deduce that T may be obtained from T' by operation τ_1.

Case 3. $v_4 \notin D$. Then there exists exactly one vertex $x \in N_T(v_3) \cap D$ and x is a support. Assume $N_T(x) \cap L(T) = \{l\}$. Let T_1 denote the component of $T - \{v_3\}$ containing v_4. Denote $T' = \langle V(T_1) \cup \{v_3, x, l\}\rangle$. It is obvious that $\gamma_t(T) = \gamma_t(T') + 2(d_T(v_3) - 2)$. It is obvious that $x, l \in D$. Hence $\gamma_r^t(T') \le \gamma_r^t(T) - 2(d_T(v_3) - 2)$. Since T is a (γ_t, γ_r^t)-tree, it follows that $\gamma_r^t(T) = \gamma_t(T) = \gamma_t(T') + 2(d_T(v_3) - 2) \le \gamma_r^t(T') + 2(d_T(v_3) - 2)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2(d_T(v_3) - 2)$. So $\gamma_t(T') = \gamma_r^t(T')$. Consequently, T' is a (γ_t, γ_r^t)-tree and by induction hypothesis, $T' \in \tau$. As v_3 is a vertex adjacent to a support in T', we deduce that T may be obtained from T' by operation τ_2. ■

As an immediate consequence of Lemmas 2 and 3 we have the following characterization of (γ_t, γ_r^t)-trees.

Theorem 3. *A tree T is a (γ_t, γ_r^t)-tree if and only if T belongs to the family τ.*

References

[1] S. Arumugam and J. Paulraj Joseph, *On graphs with equal domination and connected domination numbers*, Discrete Math. **206** (1999) 45–49.

[2] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar and L.R. Marcus, *Restrained domination in graphs*, Discrete Math. **203** (1999) 61–69.

[3] F. Harary and M. Livingston, *Characterization of tree with equal domination and independent domination numbers*, Congr. Numer. **55** (1986) 121–150.

[4] D. Ma, X. Chen and L. Sun, *On total restrained domination in graphs*, Czechoslovak Math. J. **55** (2005) 165–173.

[5] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi and L.R. Markus, *Restrained domination in trees*, Discrete Math. **211** (2000) 1–9.

[6] E.J. Cockayne, C.M. Mynhardt and B. Yu, *Total dominating functions in trees: minimality and convexity*, J. Graph Theory **19** (1995) 83–92.

Received 22 September 2006
Revised 24 January 2007
Accepted 24 January 2007