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COMPETITION HYPERGRAPHS OF DIGRAPHS WITH CERTAIN PROPERTIES I STRONG CONNECTEDNESS

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Abstract

If $D = (V, A)$ is a digraph, its *competition hypergraph* $C\mathcal{H}(D)$ has the vertex set V and $e \subseteq V$ is an edge of $C\mathcal{H}(D)$ iff $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V | (w, v) \in A\}$. We tackle the problem to minimize the number of strong components in D without changing the competition hypergraph $C\mathcal{H}(D)$. The results are closely related to the corresponding investigations for competition graphs in Fraughnaugh *et al.* [3].

Keywords: hypergraph, competition graph, strong component.

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1. Introduction and Definitions

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, graphs $G = (V(G), E(G))$ and digraphs $D = (V(D), A(D))$ considered here may have isolated vertices but no multiple edges and no loops.

In 1968 Cohen [2] introduced the *competition graph* $C(D)$ associated with a digraph $D = (V, A)$ representing a food web of an ecosystem. $C(D) = (V, E)$ is the graph with the same vertex set as D (corresponding to the species) and

$$E = \{\{u, v\} \mid u \neq v \ \wedge \ \exists \, w \in V : (u, w) \in A \ \wedge \ (v, w) \in A\},$$

i.e., $\{u, v\} \in E$ iff u and v compete for a common prey $w \in V$.

Surveys of the large literature around competition graphs can be found in Roberts [6], Kim [4] and Lundgren [5].

In [8] it is shown that in many cases competition hypergraphs yield a more detailed description of the predation relations among the species in $D = (V, A)$ than competition graphs. If $D = (V, A)$ is a digraph its *competition hypergraph* $C\mathcal{H}(D) = (V, \mathcal{E})$ has the vertex set V and $e \subseteq V$ is an edge of $C\mathcal{H}(D)$ iff $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V | (w, v) \in A\}$. In this case we say $v \in V = V(D)$ *corresponds to* $e \in \mathcal{E}$ and vice versa.

In our investigations, we make intensive use of the fact that in the digraphs under consideration no vertex is a hunter of itself. Moreover, it is obvious that loops play no role for the connectedness of digraphs.

In standard terminology concerning digraphs we follow Bang-Jensen and Gutin [1]. With $d_D^-(v), d_D^+(v), N_D^-(v)$ and $N_D^+(v)$ we denote the *in-degree, out-degree, in-neighbourhood* and *out-neighbourhood* of a vertex v in a digraph D, respectively. A set of t isolated vertices is denoted as I_t, and $i(G)$ is the number of isolated vertices in G, where G is a graph or a hypergraph. For a subset $\widehat{V}$ of vertices let $D[\widehat{V}]$ be the subdigraph of D generated by $\widehat{V}$. For a graph G, let us use $\widehat{m}(G)$ to denote the *edge clique cover number of* G, i.e., the smallest number of cliques covering all the edges of G.

2. Results

Competition graphs of strongly connected digraphs are investigated in Fraughnaugh *et al.* [3]. The most interesting result is the following characterization.

Theorem 1 ([3]). *A graph G with $n \geq 3$ vertices is the competition graph of a strongly connected digraph if and only if $\widehat{m}(G) + i(G) \leq n$.*

Consider the edge clique cover of $C(D)$ where each clique is formed by the hunters of a prey $v \in V(D)$. Then these cliques correspond to the edges of the competition hypergraph $C\mathcal{H}(D)$ (cf. Sonntag and Teichert [8]). However, Theorem 1 cannot be generalized to competition hypergraphs; only one direction can be shown.

Lemma 2. *If $\mathcal{H}$ is a competition hypergraph of a strongly connected digraph with n vertices, then $|\mathcal{E}(\mathcal{H})| + i(\mathcal{H}) \leq n$.*

Proof. Let $\mathcal{H} = C\mathcal{H}(D)$ where D is strongly connected. Then each edge in $\mathcal{H}$ corresponds to $N_D^-(v)$ for some $v \in V(D)$ with $|N_D^-(v)| \geq 2$, hence $|\mathcal{E}(\mathcal{H})| \leq n$. Because D is strongly connected, for each isolated vertex $\widehat{v}$ in $\mathcal{H}$ there is a vertex $w \neq \widehat{v}$ with $(\widehat{v}, w) \in A(D)$ and $N_D^-(w) = \{\widehat{v}\}$. Therefore no edge of $\mathcal{H}$ corresponds to $N_D^-(w)$ and we obtain $|\mathcal{E}(\mathcal{H})| + i(\mathcal{H}) \leq n$. ■

In the following we give an example of an infinite family of competition hypergraphs $C\mathcal{H}(D)$ which fulfill the inequality in Lemma 2 but are not competition hypergraphs of a strongly connected digraph D. After that we tackle the problem to minimize the number of strong components in D without changing the competition hypergraph.

Following Bang-Jensen and Gutin [1] for a digraph D with strong components $D_1, \ldots, D_k$ we define the *strong component digraph* $SC(D)$ as follows:

$$V(SC(D)) = \{w_1, \ldots, w_k\},$$
$$A(SC(D)) = \{(w_i, w_j) \mid i \neq j \wedge \exists\, x \in V(D_i)\ \exists\, y \in V(D_j) : (x, y) \in A(D)\}.$$

Because $SC(D)$ is acyclic, it is possible to arrange the strong components $D_1, \ldots, D_k$ of D in an *acyclic ordering* (cf. [1]), i.e., they are denoted such that

$$\forall\, i, j \in \{1, \ldots, k\}\ \forall\, x \in V(D_i)\ \forall\, y \in V(D_j) : i \neq j \wedge (x, y) \in A(D) \Rightarrow i < j.$$

In the first instance we restrict our investigations to digraphs having no trivial strong components, i.e., $\forall i \in \{1, 2, \ldots, k\} : |V(D_i)| > 1$. We denote such digraphs as *digraphs with* (*nontrivial*) *strong components* $D_1, \ldots, D_k$.

At the end of the paper we will discuss some problems which can be caused by trivial strong components.

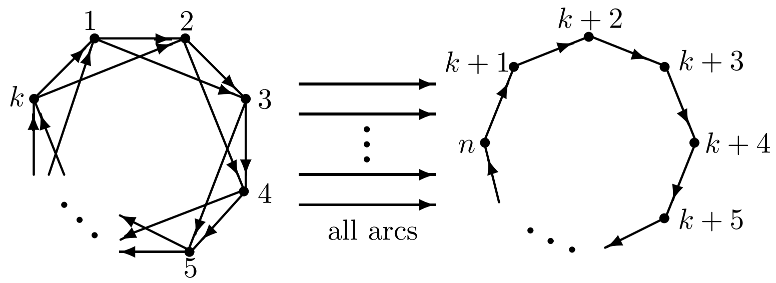
A graph G with n vertices is a competition graph of some digraph D, if and only if $G \neq K_2$ and $\widehat{m}(G) \leq n$ (cf. Roberts and Steif [7]). From Theorem 1 it follows that every competition graph G without isolated vertices is even a competition graph of a strongly connected digraph.

A characterization of hypergraphs which are competition hypergraphs $C\mathcal{H}(D)$ of digraphs D (without loops) is given in Sonntag and Teichert [8]. The question arises whether or not — analogously to graphs — every such hypergraph $C\mathcal{H}(D)$ without isolated vertices is also a competition hypergraph of a strongly connected digraph $\widetilde{D}$. The answer is no, as the following class of examples will show. Later in this section we discuss characteristic structures appearing in the digraphs D of these examples, and this will be the starting point for our further investigations.

Example. Let $n \geq 5$ und $k \in \{3, 4, \ldots, n-2\}$. Then $D_1 = D_1(k, n)$ has the vertices $V(D_1) = \{1, \ldots, n\}$ and the arcs

$$\begin{aligned} A(D_1) = \; & \{(i, i+1) \mid i \in \{1, \ldots, k-1\}\} \cup \{(k, 1)\} \cup \\ & \{(i, i+2) \mid i \in \{1, \ldots, k-2\}\} \cup \{(k-1, 1), (k, 2)\} \cup \\ & \{(i, i+1) \mid i \in \{k+1, \ldots, n-1\}\} \cup \{(n, k+1)\} \cup \\ & \{(i, j) \mid i \in \{1, \ldots, k\} \;\wedge\; j \in \{k+1, \ldots, n\}\}. \end{aligned}$$

The digraph D_1 and its competition hypergraph $\mathcal{H}_1 = C\mathcal{H}(D_1)$ are shown in Figure 1.



$D_1 = D_1(k, n)$

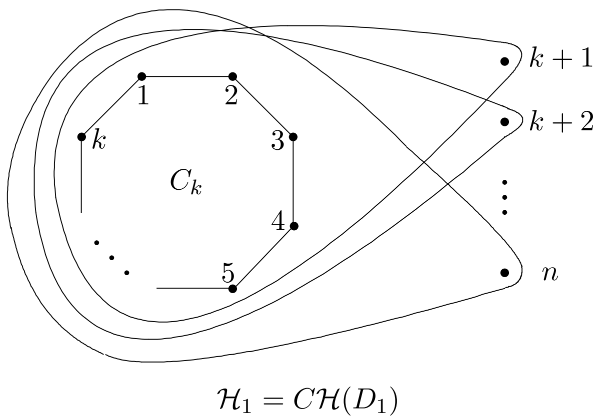


Figure 1. The digraph $D_1 = D_1(k,n)$ and its competition hypergraph $\mathcal{H}_1 = C\mathcal{H}(D_1)$.

Clearly, D_1 is not strongly connected; consequently we obtain

Lemma 3. *Let $n \geq 5, k \in \{3,4,\ldots,n-2\}$ and $D_1 = D_1(k,n)$. Then there is no strongly connected digraph $\widetilde{D}_1$ with $\mathcal{H}_1 = C\mathcal{H}(D_1) = C\mathcal{H}(\widetilde{D}_1)$.*

Proof. Let $\widetilde{D} = (V,A)$ be a digraph with $\mathcal{H}_1 = C\mathcal{H}(\widetilde{D})$ and $V = V_1 \cup V_2$, where $V_1 = \{1,\ldots,k\}, V_2 = \{k+1,\ldots,n\}$. Then $\mathcal{H}_1$ has k edges with 2 vertices (α-edges) and $(n-k)$ edges with $(k+1)$ vertices (β-edges). Because each β-edge $\tilde{e}$ contains V_1, for the vertex $\tilde{v}$ corresponding to $\tilde{e}$ it holds $\tilde{v} \in V_2$. Hence the existence of $(n-k)$ β-edges implies that each $v \in V_2$ corresponds to one of these β-edges, and it follows $\{(i,j) \mid i \in V_1 \wedge j \in V_2\} \subseteq A$.

The considerations above imply that $\widehat{v} \in V_1$ for each vertex $\widehat{v}$ corresponding to an α-edge. Now assume there is an arc $(v_2,v_1) \in A$ with $v_i \in V_i$; $i = 1,2$. Since $|N^-_{\widetilde{D}}(v_1)| \geq 2$ we obtain $N^-_{\widetilde{D}}(v_1) \in \mathcal{E}(\mathcal{H}_1)$, but $N^-_{\widetilde{D}}(v_1)$ is neither an α-edge (because of $v_2 \in V_2$ and $v_2 \in N^-_{\widetilde{D}}(v_1)$) nor a β-edge (because of $v_1 \in V_1$ and $v_1 \notin N^-_{\widetilde{D}}(v_1)$), a contradiction. Hence there are no arcs from V_2 to V_1 in $\widetilde{D}$, i.e., $\widetilde{D}$ is not strongly connected. ∎

Thus, for each $n \geq 5$, there exists a connected competition hypergraph $C\mathcal{H}(D)$ (of a digraph D with n vertices) being not the competition hypergraph of any strongly connected digraph D'.

The question arises, for what reasons there is no strongly connected digraph $\widetilde{D}_1$ with $\mathcal{H}_1 = C\mathcal{H}(D_1) = C\mathcal{H}(\widetilde{D}_1)$; in other words: why does

$D_1 = D_1(k, n)$ have no strongly connected *"competition equivalent"* digraph $\widetilde{D}_1$?

As we will see the three reasons are

- the existence of all arcs from the "left" strong component $D_1[\{1, 2, \ldots, k\}]$ to the "right" strong component $D_1[\{k+1, k+2, \ldots, n\}]$ of $D_1 = (V, E)$;
- different vertices $v \neq v'$ have different sets of predecessors $N^-(v) \neq N^-(v')$ and
- every vertex $v \in V$ has at least two predecessors.

These three properties can even be used to characterize the digraphs having no competition equivalent strong digraphs.

Definition. A digraph $D = (V, A)$ with (nontrivial) strong components $D_1, \ldots, D_k$ (in acyclic ordering) is an *mcce-digraph* iff $k = 1$ (i.e., D is strongly connected) or $k > 1$ and

(a) $\forall\, i, j \in \{1, 2, \ldots, k\}\ \forall\, v \in V(D_i)\, \forall\, v' \in V(D_j) : i < j \Rightarrow (v, v') \in A$;

(b) $\forall\, v \in V : |N^-(v)| \geq 2$;

(c) $\forall\, v, v' \in V : v \neq v' \Rightarrow N^-(v) \neq N^-(v')$.

The abbreviation **mcce**-digraph comes from **m**aximal **c**onnected with respect to **c**ompetition **e**quivalence.

The main results of this section are the following two theorems, which will be proved in Section 4.

Theorem 4. *For every digraph $D = (V, A)$ (with nontrivial strong components) there exists an mcce-digraph D' with $C\mathcal{H}(D) = C\mathcal{H}(D')$.*

In the following section we will give a constructive proof of Theorem 4 using an algorithm (Algorithm MCCE). Algorithm MCCE will be able to construct a competition equivalent mcce-digraph D' to a given digraph D such that the connectedness of D' is "best possible" in the sense of

Theorem 5. *A competition hypergraph $C\mathcal{H}(D)$ of a digraph $D = (V, A)$ (with nontrivial strong components) is the competition hypergraph of a strongly connected digraph iff every competition equivalent mcce-digraph D' of D is strongly connected.*

3. Algorithm

In Algorithm MCCE we will need three basic operations closely related to the defining properties of mcce-digraphs. For this end let $D = (V, A)$ be a digraph with the (nontrivial) strong components $D_1, \ldots, D_k$ (in acyclic ordering). Operations A, B and C modify D and generate a new digraph $D' = (V, A')$ as described below.

Operation A: Interchange of in-neighbourhoods.
Let $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$ and $v \in V(D_i)$, $v' \in V(D_j)$ be two non-adjacent vertices. We obtain D' from D by interchanging the in-neighbourhoods of v and v', i.e.,:

$$N^-_{D'}(v) := N^-_D(v') \qquad \text{and} \qquad N^-_{D'}(v') := N^-_D(v).$$

Operation B: Vertices of in-degree 1.
Let $i, j \in \{1, 2, \ldots, k\}$, $i < j$, $v \in V(D_i)$ with $|N^-_D(v)| = 1$ and $\forall\, v_i \in V(D_i) \forall\, v_j \in V(D_j) : (v_i, v_j) \in A(D)$. Delete the incoming arc of v, add an arc (v', v) for an arbitrarily chosen $v' \in V(D_j)$, i.e., the only difference between D and D' is that in D' the vertex v has $N^-_{D'}(v) = \{v'\}$.

Operation C: Separation of in-neighbourhoods.
Let $v_1, v_2, \ldots, v_s \in V$ with $N^-_D(v_1) = N^-_D(v_2) = \cdots = N^-_D(v_s)$. Delete the incoming arcs of $v_2, v_3, \ldots, v_s$ and add the arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{s-1}, v_s)$, i.e., in D' we have $N^-_{D'}(v_1) = N^-_D(v_1), N^-_{D'}(v_2) = \{v_1\}, N^-_{D'}(v_3) = \{v_2\}, \ldots,$ $N^-_{D'}(v_s) = \{v_{s-1}\}$.

Now we discuss some important properties of the described operations.

Lemma 6. *Let $D = (V, A)$ be a digraph having only nontrivial strong components. Let $D' = (V, A')$ be the digraph constructed from D*

(A) *by applying Operation A to $v \in V(D_i)$ and $v' \in V(D_j)$ or*

(B) *by applying Operation B to $v \in V(D_i)$ and $v' \in V(D_j)$, where $i < j$, $|N^-_D(v)| = 1$ and $\forall\, v_i \in V(D_i) \forall\, v_j \in V(D_j) : (v_i, v_j) \in A$ hold.*

Then we obtain:

(1) *$V(D_i) \cup V(D_j)$ is contained in a strong component D'_l of D'.*

(2) *If $D_i = D_{i_1}, D_{i_2}, \ldots, D_{i_t} = D_j$ induce a path in $SC(D)$, then $V(D_{i_1}) \cup V(D_{i_2}) \cup \ldots \cup V(D_{i_t})$ is contained in a strong component D'_l of D'.*

(3) *$C\mathcal{H}(D) = C\mathcal{H}(D')$.*

Proof. (A): First, we consider Operation A.

(1) For $u, u' \in V(D_i) \cup V(D_j)$ we have to demonstrate the existence of a (u, u')-path $w_{D'}^{u,u'}$ in D'.

The existence of $w_{D'}^{u,u'}$ is evident if $u, u' \in V(D_i)$ or $u, u' \in V(D_j)$ or $i < j \wedge u \in V(D_i) \wedge u' \in V(D_j)$ and in D there is a (u, u')-path $w_D^{u,u'}$ which does not contain incoming arcs of v and v'. In this case we choose $w_{D'}^{u,u'} = w_D^{u,u'}$.

Now let $u, u' \in V(D_i)$ and $w_D^{u,u'} = (u, \ldots, v^-, v, \ldots, u')$ with $v^- \in N_D^-(v)$. In D' we have $v^- \in N_{D'}^-(v')$ and we find a $v'^- \in N_D^-(v') = N_{D'}^-(v)$ such that there is a (v', v'^-)-path $w_{D'}^{v',v'^-} := w_D^{v',v'^-}$, where w_D^{v',v'^-} obviously does not contain an incoming arc of v'. We modify $w_D^{u,u'}$ and obtain a (u, u')-path $w_{D'}^{u,u'}$ in D' in the following way:

$$w_{D'}^{u,u'} = (u, \ldots, v^-, w_{D'}^{v',v'^-}, v, \ldots, u').$$

The case $u, u' \in V(D_j)$ and $w_D^{u,u'} = (u, \ldots, v'^-, v', \ldots, u')$ with $v'^- \in N_D^-(v')$ can be considered analogously.

If $u \in V(D_i) \wedge u' \in V(D_j)$, then there is a $v^- \in N_D^-(v) \cap V(D_i) \subseteq N_{D'}^-(v')$ such that in $D'[V(D_i)]$ (as well as in $D[V(D_i)]$) there exists a (u, v^-)-path $w_{D'}^{u,v^-} = w_D^{u,v^-}$ not containing any incoming arc of v (and, obviously, of v'). Moreover, in $D'[V(D_j)]$ (as well as in $D[V(D_j)]$) there is a (v', u')-path $w_{D'}^{v',u'} = w_D^{v',u'}$, not containing any incoming arc of v' (and, obviously, of v). Consequently, we have $w_{D'}^{u,u'} = (w_{D'}^{u,v^-}, w_{D'}^{v',u'})$. The case $u \in V(D_j) \wedge u' \in V(D_i)$ follows analogously.

(2) is an immediate conclusion of (1).

(3) is obvious, because the interchange of in-neighbourhoods does not influence the set $\{N_D^-(v) | v \in V \wedge |N_D^-(v)| > 1\} = \mathcal{E}(C\mathcal{H}(D))$, i.e., $C\mathcal{H}(D)$ remains unchanged.

(B): Now, we investigate Operation B.

(1) Again, for $u, u' \in V(D_i) \cup V(D_j)$ we need a (u, u')-path $w_{D'}^{u,u'}$ in D'. If $u \in V(D_i) \wedge u' \in V(D_j)$, because of $(u, u') \in A$ we have the path $w_{D'}^{u,u'} = (u, u')$. In the cases $u, u' \in V(D_j)$ or $u, u' \in V(D_i) \wedge v \notin V(w_D^{u,u'}) \setminus \{u\}$, we choose $w_{D'}^{u,u'} = w_D^{u,u'}$. Let $u, u' \in V(D_i) \wedge v \in V(w_D^{u,u'}) \setminus \{u\}$ and $w_D^{u,u'} = (u, \ldots, v^-, v, \ldots, u')$. Owing to $(v^-, v'), (v', v) \in A'$ we obtain the wanted path by $w_{D'}^{u,u'} = (u, \ldots, v^-, v', v, \ldots, u')$, where the vertex v' was inserted in $w_D^{u,u'}$ between v^- and v.

Now $u \in V(D_j) \wedge u' \in V(D_i)$. Since D_j and D_i are strong components of D, there is a (u,v')-path $w_D^{u,v'}$ in D_j and a (v,u')-path $w_D^{v,u'}$ in D_i. It follows easily that both paths are also paths in D', consequently we can choose $w_{D'}^{u,u'} = (w_D^{u,v'}, w_D^{v,u'})$.

Again, (2) is a direct conclusion from (1).

(3) Operation B does not change $C\mathcal{H}(D)$, because we manipulate only a vertex v having in-degree 1 in D as well as in D' and therefore v does not correspond to any edge in the competition hypergraph. ∎

Proposition 7. *For every digraph $D = (V,A)$ (with nontrivial strong components) there is a digraph $D' = (V,A')$ with the nontrivial strong components $D'_1,\ldots,D'_k$ (in acyclic ordering) and $C\mathcal{H}(D) = C\mathcal{H}(D')$, such that $SC(D')$ is a transitive tournament and*

$$\forall\, i,j \in \{1,2,\ldots,k\}\ \forall\, v \in V(D'_i)\, \forall\, v' \in V(D'_j):\ i<j \Rightarrow (v,v') \in A'.$$

Proof. Starting with D, the iterated application of Operation A to pairs (u,u') of non-adjacent vertices $u \in V(D_i)$ and $u' \in V(D_j)$, where $i \neq j$, leads to D'. ∎

Lemma 8. *Let $D = (V,A)$ be a digraph having only nontrivial strong components and $D' = (V,A')$ be the digraph constructed from D by applying Operation C to $v_1,v_2,\ldots,v_s \in V$ with $N_D^-(v_1) = N_D^-(v_2) = \ldots = N_D^-(v_s)$.*

Then we obtain:

(1) *There is a strong component D_j of D such that $v_1,v_2,\ldots,v_s \in V(D_j)$.*

(2) *$D'[V(D_j)]$ is a strong component of D'.*

(3) *$C\mathcal{H}(D) = C\mathcal{H}(D')$.*

Proof. (1) Assume, $v_x \in V(D_{x'})$ and $v_y \in V(D_{y'})$, where $x,y \in \{1,2,\ldots,s\}$, $x \neq y$, $x',y' \in \{1,2,\ldots,k\}$ and $x' \neq y'$. Without loss of generality let $x' < y'$, i.e., because of the acyclic ordering of the strong components of D there is no arc from $D_{y'}$ to $D_{x'}$. Since $D_{y'}$ is strongly connected, v_y must have a predecessor v_y^- in $D_{y'}$. Due to $N_D^-(v_1) = N_D^-(v_2) = \cdots = N_D^-(v_s)$ the vertex $v_y^- \in V(D_{y'})$ is also a predecessor of $v_x \in V(D_{x'})$, i.e., (v_y^-, v_x) is an arc from $D_{y'}$ to $D_{x'}$, a contradiction.

(2) Let $u,u' \in V(D_j)$ and $u \neq u'$. If in D there is a (u,u')-path $w_D^{u,u'}$ with $\{v_1,v_2,\ldots,v_s\} \cap V(w_D^{u,u'}) = \emptyset$, then $w_D^{u,u'}$ is also a (u,u')-path $w_{D'}^{u,u'}$ in D'.

Otherwise, let $v_x, v_y \in \{v_1, v_2, \ldots, v_s\}$ such that in $w_D^{u,u'}$ v_x is the first and v_y is the last of these vertices appearing in $w_D^{u,u'} = (u = u_0, u_1, \ldots, u_{p-1}, u_p = v_x, \ldots, u_q = v_y, u_{q+1}, \ldots, u_t = u')$. In D' we substitute $u_p = v_x, \ldots, u_q = v_y$ in $w_D^{u,u'}$ by $v_1, v_2, \ldots, v_y$ and obtain a (u, u')-path $w_{D'}^{u,u'} = (u = u_0, u_1, \ldots, u_{p-1}, v_1, v_2, \ldots, v_y, u_{q+1}, \ldots, u_t = u')$.

(3) Operation C does not change $C\mathcal{H}(D)$, since in $C\mathcal{H}(D)$ the vertices $\{v_1, v_2, \ldots, v_s\}$ correspond to one and the same hyperedge $e = N_D^-(v_1)$. In D' the vertex v_1 corresponds to this hyperedge e and the remaining vertices $\{v_2, v_3, \ldots, v_s\}$ have in-degree 1, i.e., they do not correspond to any edge of the competition hypergraph $C\mathcal{H}(D')$. ■

Now we give

Algorithm MCCE.

Let $D = (V, A)$ be a digraph with (nontrivial) strong components $D_1, \ldots, D_k$ (in acyclic ordering).

1. **Apply** Operation A as long as possible and
   obtain a digraph $D' = (V, A')$ with k' strong components.
2. **Let** $k := k'$, $D := D'$ and $D_1, \ldots, D_k$ be the (new) strong components of $D = D'$ (in acyclic ordering).
3. **If** D is strongly connected (i.e., $k = 1$),
   then goto 6.
4. **If** $\exists\, v : |N_D^-(v)| = 1$ (obviously, in this case $v \in V(D_1)$ must be valid),
   then choose $v' \in V(D_k)$,
   - **apply** Operation B to v and v',
   - **obtain** a strongly connected digraph $D := D' = (V, A')$ and
   - **goto** 6.
5. **If** $\exists\, i\, \exists\, u, v \in V(D_i) : N_D^-(u) = N_D^-(v)$,
   then apply Operation C to u and v and
   - **obtain** a digraph $D := D' = (V, A')$, where $|N_{D'}^-(v)| = 1$;
   - **if** $i < k$,
   - **then choose** $v' \in V(D_k)$,
     - **apply** Operation B to v and v',
     - **obtain** a digraph $D' = (V, A')$ with strong components
       $D'_1, \ldots, D'_{k'}$ (in acyclic ordering; note that
       $D'_{k'} = D'[V(D_i) \cup V(D_{i+1}) \cup \ldots \cup V(D_k)]$) and

let $k := k'$, $D := D'$ and $D_1, \ldots, D_k$ be the (new) strong components of $D = D'$ (in acyclic ordering);

if $k > 1$ (note that in this case because of $v \in V(D_k)$ and $|N_D^-(v)| = 1$ no vertex $u \in V(D_1)$ is a predecessor of v),

then choose $u \in V(D_1)$,

apply Operation A to u and v and

obtain a strongly connected digraph $D := D' = (V, A')$.

6. **Stop.**

4. Proofs and Concluding Remarks

In this section we prove Theorems 4 and 5. Obviously, to show Theorem 4 it suffices to verify Algorithm MCCE.

***Proof of Theorem* 4.** At first we verify the feasibility of Algorithm MCCE. Obviously, it suffices to investigate steps 4 and 5.

In step 4 we have $k > 1$ and Operation A cannot be applied to D. Consequently the validity of condition (a) of the definition of an mcce-digraph follows. In particular this means that there are all possible arcs from vertices of D_1 to vertices of D_k. Therefore, for a vertex $v \in V(D_1)$ with in-degree 1 and an arbitrary vertex $v' \in V(D_k)$ Operation B can be applied.

Besides we remark that after steps 1–3 the strong component digraph $SC(D)$ is a transitive tournament (cf. Proposition 7). Hence, $D_1, D_2, \ldots, D_k$ induce a path in $SC(D)$ and Operation B evidently provides a strongly connected digraph D' (cf. Lemma 6(2)).

In step 5 it is trivial that Operation C is feasible. Since in case $i < k$ there are all arcs from $V(D_i)$ to $V(D_k)$, Operation B can be applied (note that Operation C before influenced only incoming arcs of D_i or arcs inside the strong component D_i) to $v \in V(D_i)$ (with in-degree 1) and an arbitrary vertex $v' \in V(D_k)$.

Again, since $D_i, D_{i+1}, \ldots, D_k$ induce a path in $SC(D)$, Lemma 6(2) provides that Operation B results in a strongly connected subdigraph $D'[V(D_i) \cup V(D_{i+1}) \cup \cdots \cup V(D_k)]$. Because there are no arcs in D' from $V(D_i) \cup V(D_{i+1}) \cup \cdots \cup V(D_k)$ to $V(D_1) \cup V(D_2) \cup \cdots \cup V(D_{i-1})$ we obtain $D'_{k'} = D'[V(D_i) \cup V(D_{i+1}) \cup \cdots \cup V(D_k)]$.

After updating k, D and $D_1, D_2, \ldots, D_k$ the vertex v (with in-degree 1) is now in the strong component D_k. Therefore the only predecessor of v is in D_k and in case $k > 1$ Operation A can be applied to $u \in V(D_1)$ and v.

Consequently, Algorithm MCCE is feasible.

Now we verify that Algorithm MCCE results in an mcce-digraph having the same competition hypergraph as the initial digraph.

Starting from the initial digraph $D = (V, A)$, Algorithm MCCE uses Operations A, B and C to construct a new digraph. Owing to Lemma 6 and Lemma 8 this procedure does not change the competition hypergraph $C\mathcal{H}(D)$.

Let D be the new digraph constructed in Algorithm MCCE. Steps 3, 4 and 5 lead to a strongly connected digraph D, i.e., in these cases D is an mcce-digraph.

Now let D result from steps 1 and 2, where $k > 1$. Since Operation A cannot be applied any longer, in D there are no non-adjacent vertices $v \in V(D_i)$ and $v' \in V(D_j)$ with $i \neq j$. Therefore condition (a) of the definition of an mcce-digraph holds.

Because D was computed in steps 1 and 2, neither the premise of step 4 nor the premise of step 5 can be fulfilled. But this is equivalent to property (b) and property (c) of an mcce-digraph, respectively. ■

***Proof of Theorem* 5.** It suffices to show that if the competition hypergraph $C\mathcal{H}(D)$ of the digraph $D = (V, A)$ (with nontrivial strong components) is the competition hypergraph of a strongly connected digraph, then every competition equivalent mcce-digraph D' of D is strongly connected. Let $\widetilde{D} = (V, \widetilde{A})$ be strongly connected and competition equivalent to D.

Assume, $D' = (V, A')$ is a competition equivalent mcce-digraph of the digraph $D = (V, A)$ and $D_1, \ldots, D_k$ are the strong components of D' in acyclic ordering, where $k > 1$.

Then, obviously, D' and $\widetilde{D}$ are competition equivalent and we set $\mathcal{H} = (V, \mathcal{E}) := C\mathcal{H}(D') = C\mathcal{H}(\widetilde{D})$.

Since $\widetilde{D}$ is strongly connected, we obtain

$$(1) \qquad \exists\, v' \in V(D_k)\, \exists\, v \in U := \bigcup_{i=1}^{k-1} V(D_i) : (v', v) \in \widetilde{A}.$$

Property (b) in the definition of mcce-digraph implies $\forall\, w \in V : N^-_{D'}(w) \in \mathcal{E}$ and property (c) yields $\forall\, w, w' \in V : w \neq w' \Rightarrow N^-_{D'}(w) \neq N^-_{D'}(w')$. I.e., every vertex $w \in V$ corresponds to a hyperedge $N^-_{D'}(w) \in \mathcal{E}$ and all these hyperedges are pairwise distinct. Because of $\mathcal{E}(C\mathcal{H}(D')) = \mathcal{E}(C\mathcal{H}(\widetilde{D})) = \mathcal{E}$ the same holds for $\widetilde{D}$, i.e., for the hyperedges $N^-_{\widetilde{D}}(w) \in \mathcal{E}$.

Since in D' there is no arc from $V(D_k)$ to $U = \bigcup_{i=1}^{k-1} V(D_i)$ and we have no loops, it follows $\forall\, u \in U : N^-_{D'}(u) \subset U$. On the other hand, the acyclic ordering of the strong components $D_1, \ldots, D_k$ of D' and property (a) in the definition of mcce-digraph provide

$$\forall e \in \mathcal{E}(C\mathcal{H}(D')) = \mathcal{E} : e \cap V(D_k) \neq \emptyset \Rightarrow U \subset e. \tag{2}$$

Owing to the competition equivalence of D' and $\widetilde{D}$ for every vertex $w \in V(\widetilde{D}) = V$ there is a vertex $w^* \in V(D') = V$ such that $N^-_{\widetilde{D}}(w) = N^-_{D'}(w^*)$, and vice versa.

Let us consider the vertex v from (1) and let $v^* \in V$ with $N^-_{\widetilde{D}}(v) = N^-_{D'}(v^*)$. Because there are no loops in $\widetilde{D}$ we have $v \notin N^-_{\widetilde{D}}(v) = N^-_{D'}(v^*)$, and consequently $U \not\subseteq N^-_{\widetilde{D}}(v)$. Property (2) implies $N^-_{\widetilde{D}}(v) \cap V(D_k) = \emptyset$. This contradicts (1). ■

Up to now, we excluded trivial strong components from our considerations. One reason is that algorithm MCCE could handle such components only under special assumptions.

The following example shows an infinite family of digraphs $D_2(k, n)$ having one trivial strong component, where Operation A fails, if we try to apply it to the trivial strong component.

Example. Let $n \geq 6$ and $k \in \{3, 4, \ldots, n-3\}$. Then $D_2 = D_2(k, n)$ has the vertices $V(D_2) = \{1, \ldots, n\}$ and the arcs

$$\begin{aligned} A(D_2) = \; & \{(i, i+1) \mid i \in \{1, \ldots, k-1\}\} \cup \{(k, 1)\} \cup \\ & \{(i, i+2) \mid i \in \{1, \ldots, k-2\}\} \cup \{(k-1, 1), (k, 2)\} \cup \\ & \{(i, i+1) \mid i \in \{k+2, \ldots, n-1\}\} \cup \{(n, k+2)\} \cup \\ & \{(i, j) \mid i \in \{1, \ldots, k\} \;\wedge\; j \in \{k+2, \ldots, n\}\} \cup \\ & \{(i, k+1) \mid i \in \{1, \ldots, k\}\} \cup \{(k+1, j) | j \in \{k+3, \ldots, n\}\}. \end{aligned}$$

Note that $(k+1, k+2) \notin A(D_2)$. The digraph $D_2 = D_2(k,n)$ and its competition hypergraph $\mathcal{H}_2 = C\mathcal{H}(D_2)$ are shown in Figure 2.

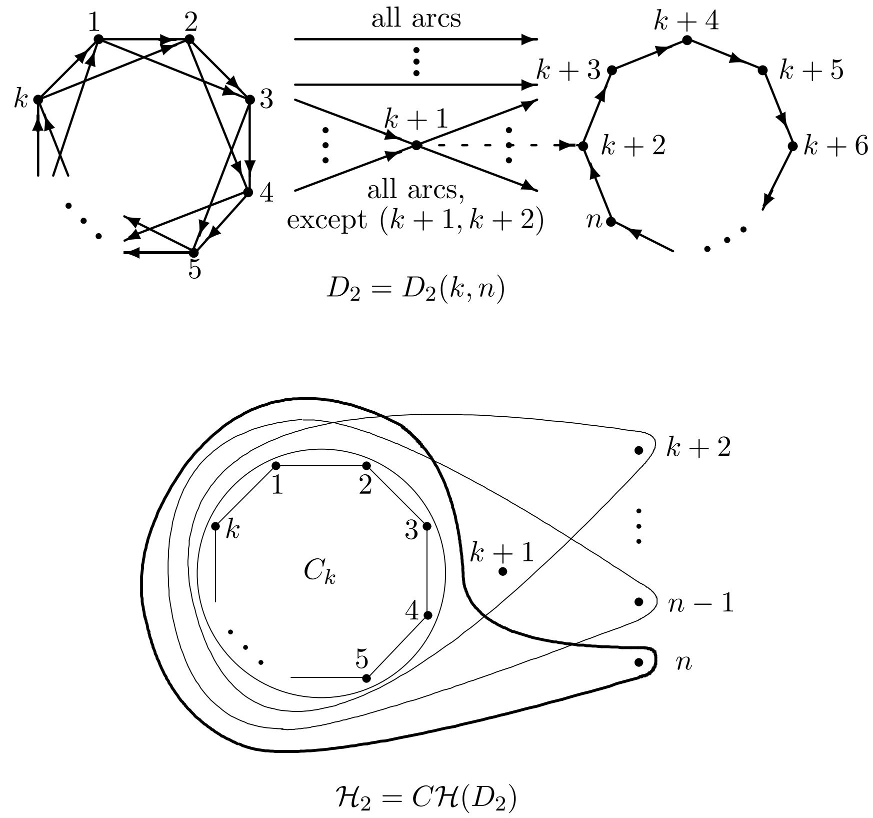


Figure 2. The digraph $D_2 = D_2(k,n)$ and its competition hypergraph $\mathcal{H}_2 = C\mathcal{H}(D_2)$.

The digraph $D_2(k,n)$ is not strongly connected and we obtain

Lemma 9. *Let $n \geq 6, k \in \{3, 4, \ldots, n-3\}$ and $D_2 = D_2(k,n)$. Then every digraph $\widetilde{D}$ being competition equivalent to D_2 has at least three strong components.*

Proof. Let $\widetilde{D} = (V, A)$ be a digraph with $\mathcal{H} := C\mathcal{H}(\widetilde{D}) = C\mathcal{H}(D_2)$ and $V = V_1 \cup V_2 \cup \{k+1\}$ with $V_1 = \{1, \ldots, k\}, V_2 = \{k+2, \ldots, n\}$. There are four types of edges in $\mathcal{H}$:

$(n-k-2)$	α-edges	of the form $V_1 \cup \{k+1, k+t\}, t \in \{2, \ldots, n-k-1\}$,
1	β-edge	$V_1 \cup \{n\}$ (thick lined edge),
1	γ-edge	V_1,
k	δ-edges	of the form $\{k, 1\}$ or $\{i, i+1\}, i \in \{1, \ldots, k-1\}$.

Because $V_1 \cup \{k+1\}$ is contained in each α-edge, the corresponding vertices to these α-edges belong to $\{k+2, \ldots, n\}$, i.e., exactly one of these vertices, say $(k+2)$, is still available as the corresponding vertex to another edge. Either $(k+2)$ corresponds to the β-edge, then $(k+1)$ corresponds to the γ-edge (case 1) or vice versa (case 2). Hence the vertices $1, \ldots, k$ correspond to the δ-edges.

In case 1 we obtain that there is no arc from V_2 to $V_1 \cup \{k+1\}$ and no arc from $k+1$ to V_1. Consequently, each pair of vertices x, y from different vertex sets out of $V_1, \{k+1\}$ and V_2 has the property that x and y have to be in different strong components of $\widetilde{D}$. Therefore, $\widetilde{D}$ consists of at least three strong components. Changing the roles of $k+1$ and $k+2$, case 2 can be considered analogously. ■

By Lemma 9 it follows that Operation A has to fail if we try to apply this operation to the non-adjacent vertices $k+1, k+2 \in V(D_2)$. In detail we see that if we would change the sets of predecessors $N^-_{D_2}(k+1) = \{1, \ldots, k\}$ and $N^-_{D_2}(k+2) = \{1, \ldots, k, n\}$ (cf. Operation A), we would obtain a digraph D'_2 with the same competition hypergraph $C\mathcal{H}(D_2) = C\mathcal{H}(D'_2)$, but the vertices $k+1$ and $k+2$ would still belong to different strong components. So in D'_2 we have $N^-_{D'_2}(k+2) = \{1, \ldots, k\}$, i.e., there is no path from any vertex of $\{k+1, k+3, k+4, \ldots, n\}$ to $k+2$. Moreover, it is obvious that there exists no path from $\{k+1, k+2, \ldots, n\}$ to $\{1, \ldots, k\}$.

Note that Operations A, B, C can be also applied to digraphs with trivial strong components if we assume that all vertices explicitly mentioned in these operations are contained in nontrivial components; results analogous to Lemma 6 and Lemma 8 can be verified.

In special cases trivial strong components of the digraph $D = (V, A)$ can be integrated into other strong components without changing the competition hypergraph:

Remark 10. Let $D = (V, A)$ be a connected digraph with the strong components $D_1, \ldots, D_k$ (in acyclic ordering) and let D_l be a trivial component, e.g. $V(D_l) = \{v\}$. If one of the following conditions is fulfilled, then there is a digraph $\widetilde{D}$ competition equivalent to D with the strong components $D'_1, \ldots, D'_{k'}$ (in acyclic ordering), such that $v \in V(D'_{l'})$ with $|V(D'_{l'})| > 1$ and $l' \in \{1, \ldots, k'\}$:

(a) $\exists\, u \in V - \{v\}\ \exists\, w_D^{u,v} : N_D^-(u) = \emptyset \wedge w_D^{u,v}$ is a (u, v)-path in D;

(b) $|N_D^-(v)| \leq 1 \wedge N_D^+(v) \neq \emptyset$;

(c) $\exists\, i \in \{1, \ldots, l-1\}\ \exists\, j \in \{l+1, \ldots, k\}\ \exists\, x \in V(D_i)\ \exists\, y \in V(D_j)\ \exists\, w_D^{x,y}$: D_i, D_j nontrivial $\wedge\ w_D^{x,y}$ is an (x, y)-path in D containing $v\ \wedge$

$$\Big(\text{(i)}\ (\exists\, x_i \in V(D_i)\ \exists\, y_j \in V(D_j) : (x_i, y_j) \notin A)\ \vee$$
$$\text{(ii)}\ (\exists\, x_i \in V(D_i) : |N_D^-(x_i)| \leq 1)\ \vee$$
$$\text{(iii)}\ (\exists\, x_i, y_i \in V(D_i) : N_D^-(x_i) = N_D^-(y_i)) \Big).$$

Proof. *Case* (a). If we add a new arc (v, u) to $A(D)$, we obtain a competition equivalent digraph $\widetilde{D}$, where all strong components of D containing a vertex $x \in V(w_D^{u,v})$ are included in one strong component $D'_{l'}$ of $\widetilde{D}$.

Case (b). We delete the incoming arc of v and add an arc (u, v), where $u \in N_D^+(v)$. Then the vertices v and u are in one strong component $D'_{l'}$ of $\widetilde{D}$. Note that in this case $\widetilde{D}$ may be even disconnected; this can be avoided by using (a) instead of (b) (if possible) or by the application of Operation A after deleting the incoming arc of v and adding (u, v).

Case (c). Since D_i and D_j are nontrivial strong components connected by $w_D^{x,y}$, where v is contained in $w_D^{x,y}$, we can proceed as follows (similarly to Algorithm MCCE):

If (i) is fulfilled Operation A (cf. Lemma 6(2)) yields the desired result.

If (i) is not valid but (ii) is true, then Operation B can be used (cf. Lemma 6(2)) and we are done.

If neither (i) nor (ii) is fulfilled, it remains to consider (iii). We apply Operation C (cf. Lemma 8) and obtain $z \in \{x_i, y_i\} \subseteq V(D_i)$ with indegree 1. Now Operation B (cf. Lemma 6(2)) completes the proof. ■

It seems to be very difficult to generalize Algorithm MCCE and Theorems 4 and 5 such that trivial strong components can be included without stint.

One reason is the more complicated structure of the digraphs under consideration. On the other hand, Operations A, B and C do not work for a lot of configurations, where trivial strong components occur. It seems to be hopeless to search for modifications of Operations A, B and C or for new operations in order to obtain a complete description of the competition hypergraphs of strongly connected digraphs in analogy to Theorem 5.

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