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FRACTIONAL DOMINATION IN PRISMS

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Abstract

Mynhardt has conjectured that if G is a graph such that $\gamma(G) = \gamma(\pi G)$ for all generalized prisms πG then G is edgeless. The fractional analogue of this conjecture is established and proved by showing that, if G is a graph with edges, then $\gamma_f(G \times K_2) > \gamma_f(G)$.

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Throughout let us assume that graphs are finite and simple; our notation concurs with [3]. Let $G = (V, E)$ be a graph; the (*closed*) *neighbourhood* $N[v]$ of a vertex $v \in V$ consists of v itself and all vertices $u \in V$ such that $u \sim v$. A set $S \subseteq V$ is *independent* if no two members of S are adjacent; S is *dominating* if $\cup_{v \in S} N[v] = V$. The size of a smallest dominating set in G is denoted by $\gamma(G)$ and termed the *domination number* of G.

By generalizing "set" to "fuzzy set" in the definition of domination, one can define the concept of fractional domination. A function $f : V \to [0, 1]$ is a *fractional dominating function* precisely when $\sum_{u \in N[v]} f(u) \geq 1$ for all $v \in V$. If one defines the size of a fractional dominating function f by $|f| = \sum_{v \in V} f(v)$ then one can talk about the minimum size of a fractional dominating function of G; this is the *fractional domination number* of G and denoted by $\gamma_f(G)$. Since the characteristic function of a dominating set in G is clearly a fractional dominating function of G, $\gamma_f(G) \leq \gamma(G)$.

(Notation will sometimes be abused in the following standard fashions: if S is a set of vertices, then $f(S) = \sum_{v \in S} f(v)$. Thus, $|f| = f(V)$. In the particular case where the set in question is the closed neighbourhood $N[v]$ of the vertex v, the notation is further condensed to $f[v] = f(N[v])$.)

An *equitable partition* $P_1, \ldots, P_k$ of the vertices of a graph G is a partition with the properties that every induced graph $G[P_i]$ is regular, and every induced bipartite graph between two cells P_i, P_j is biregular. The following result can be found in [5].

Theorem 1. *If G is a graph that admits an equitable partition $\{P_i\}_{i=1}^k$, then there exists a minimum fractional dominating function of G that is constant on each cell P_i, $i = 1, \ldots, k$.*

Suppose that G is a graph and π a permutation on its vertex set V. The *generalized prism* πG is the graph with vertex set $V_\pi = V \times \{0, 1\}$, with $(u, i) \sim (v, j)$ when either $i = j$ and $u \sim v$ in G, or else $i \neq j$ and $v = \pi(u)$. When $\pi = 1$, the identity permutation, then the graph $1G = G \times K_2$ is called the *prism* of G.

The following result from [1] is easily shown.

Lemma 2. *For any graph G and any permutation π of its vertex set, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$.*

A graph G for which $\gamma(G) = \gamma(\pi G)$ for any permutation π is a *universal γ-fixer*; if $2\gamma(G) = \gamma(\pi G)$ for all π, then G is a *universal γ-doubler*. In [4] it is conjectured that the only universal γ-fixers are graphs without edges.

This paper is concerned with the fractional analogue of the conjecture mentioned above. To develop this, some elementary tools are needed.

As discussed in [2], for a function $f : V \to [0, 1]$ define the sets $B_f = \{v \in V : f[v] = 1\}$ and $P_f = \{v \in V : f(v) > 0\}$.

Lemma 3 [2]**.** *A dominating function f is a minimal dominating function if and only if B_f dominates P_f.*

If f is a fractional dominating function of the prism πG, then define the *condensation* $f_\pi : V(G) \to [0, 1]$ of f by

$$f_\pi(v) = \min\{1, f((v, 0)) + f((\pi(v), 1))\}$$

for all $v \in V(G)$.

Lemma 4. *If f is a fractional dominating function on πG, then its condensation f_π is a fractional dominating function on G with $|f_\pi| \leq |f|$.*

Proof. Let $v \in V(G)$ and consider $\sum_{u \in N_G[v]} f_\pi(u)$. If $f_\pi(v) = 1$ then clearly this sum exceeds 1; otherwise, for each $u \in N_G(v)$ we have that $f_\pi(u) \geq f((u,0))$, and $f_\pi(v) = f((v,0)) + f((\pi(v),1))$. Hence

$$\begin{aligned}\sum_{u \in N_G[v]} f_\pi(u) &= f_\pi(v) + \sum_{u \in N_G(v)} f_\pi(u) \\ &\geq f((v,0)) + f((\pi(v),1)) + \sum_{u \in N_G(v)} f((u,0)) \\ &= \sum_{x \in N_{\pi G}[(v,0)]} f(x) \\ &\geq 1.\end{aligned}$$

A similar calculation shows that $|f_\pi| \leq |f|$. ■

Corollary 5. *For any graph G and any permutation π of its vertex set, $\gamma_f(G) \leq \gamma_f(\pi G) \leq 2\gamma_f(G)$, and these bounds are sharp.*

Proof. The lower bound follows from Lemma 4. To show the upper bound, let f be a minimum fractional dominating function of G. Then the function $f' : V(\pi G) \to [0,1]$ defined by $f'((u,i)) = f(u)$ is fractional dominating with $|f'| = 2|f|$.

An example of the lower bound occurs when G contains no edges and π is an arbitrary permutation: $\gamma_f(G) = \gamma_f(\pi G) = |V(G)|$. For the upper bound, let $G = K_{1,n}$ for $n \geq 2$ and let π be any automorphism of G; then $\gamma_f(G) = 1$ and $\gamma_f(\pi G) = 2$. ■

The fractional version of Mynhardt's question is then: *For which graphs G is it true that, for any permutation π of $V(G)$, $\gamma_f(\pi G) = \gamma_f(G)$?* Such a graph would naturally be termed a universal γ_f-fixer. As it turns out, this question can be answered without considering any permutations other than the identity.

Lemma 6. *Let f be fractional dominating on $1G$ with condensation f_1 such that $|f_1| = |f|$. Then for any vertex $v \in V(G)$, $f_1[v] = f[(v,0)] + f[(v,1)] - f_1(v)$.*

Proof. Since $|f_1| = |f|$ is follows that $f_1(v) = f((v,0)) + f((v,1))$ for all vertices v. The result then follows from a simple computation using the fact that $f[(v,i)] = f(\{(u,i) : u \in N_G[v]\} + f((v, 1-i))$ for $i = 0, 1$. ■

Lemma 7. *Let* $1G$ *be the prism of a simple graph* G *with vertex set* $V = \{v_1, \ldots, v_n\}$*. Then the collection of sets* $\{(v_i, 0), (v_i, 1)\}_{i=1}^{n}$ *forms an equitable partition of the vertices of* $1G$*.*

Proof. Let P_i denote the set containing the images of v_i in the prism. Each $1G[P_i]$ consists of a single edge (and is thus 1-regular); the bipartite graph between P_i and P_j will either be edgeless (if v_i and v_j are not adjacent) or 1-regular. ■

Theorem 8. *Let* G *be a graph such that* $\gamma_f(1G) = \gamma_f(G)$*. Then* $G = \overline{K_n}$ *for some positive integer* n*.*

Proof. Let G be a graph such that $\gamma_f(1G) = \gamma_f(G)$, and suppose that f is a minimum fractional dominating function of $1G$ with condensation f_1. Let us assume (by Theorem 1 and Lemma 7) that for any $v \in V(G)$, $f((v,0)) = f((v,1))$. By Lemma 4 f_1 is a fractional dominating function of G with $|f_1| \leq \gamma_f(1G) = \gamma_f(G)$, and hence f_1 is in fact a minimum fractional dominating function of G. Further, by this equality we know that $f((v,0)) + f((v,1)) \leq 1$, and hence that $f(x) \leq \frac{1}{2}$ for any vertex $x \in V(1G)$.

Suppose that v is a vertex in G such that $f_1(v) = 0$. Then by Lemma 6, $f_1(N[v]) = f(N[(v,0)]) + f(N[(v,1)])$, and since f is fractional dominating in $1G$ the two right-hand terms are each at least 1; hence, $f_1(N[v]) \geq 2$ for any vertex v receiving a weight of 0.

Let $v^* \in V(G)$ be such that $f_1[v^*] = 1$; such a vertex exists from Lemma 3. It follows that $f[(v^*, 0)] = \frac{1}{2} f_1[v^*] + \frac{1}{2} f_1(v^*) = \frac{1}{2} + \frac{1}{2} f_1(v^*) \geq 1$ since f is dominating; hence $f_1(v^*) \geq 1$ so $f_1(v^*) = 1$. Moreover, $f_1(u) = 0$ for all $u \sim v^*$.

By Lemma 3, if $f_1(w) > 0$ then there exists $v^* \in N[w]$ such that $f_1[v^*] = 1$; if $v^* \in N(w)$ then $f_1(w) = 0$, contradicting our premise, and hence $w = v^*$ and so $f_1(w) = 1$. Therefore, f_1 is the characteristic function of an independent 2-dominating set of G. (A 2-dominating set S is one where, for every vertex $u \notin S$, $|N(u) \cap S| \geq 2$. The 2-domination comes from the fact that f only takes the values 0 and $\frac{1}{2}$; any vertex in $1G$ which receives a weight of 0 must therefore be adjacent to two vertices in the support of f, and this carries over into the condensation.)

So let $d = d_G(v^*)$ for some vertex v^* such that $f_1(v^*) = 1$, and suppose that $d > 0$. Pick some $w \in V(G)$ that is distance 2 from v such that $f_1(w) > 0$; this exists by fact that the support of f_1 is 2-dominating.

Define the function $f^* : V(G) \to [0,1]$ as follows:

$$f^*(v) = \begin{cases} 0 & \text{if } v = v^*, \\ \frac{1}{d} & \text{if } v \sim v^*, \\ 1 - \frac{1}{d} & \text{if } v = w, \\ f_1(v) & \text{otherwise.} \end{cases}$$

f^* is a fractional dominating function of G: If v is a vertex such that $f_1(v) = 1$, then clearly $f^*(v) = 1$. Otherwise $f_1(v) = 0$ and hence $f_1[v] \geq 2$ as v has at least two neighbours with weight 1. If its only two such neighbours are v^* and w, then $f^*[v] = f^*(v) + f^*(w) = 1$; otherwise, it is clear that $f^*[v] \geq 1$.

But $|f^*| < |f_1|$, so the latter is not minimum, and hence $\gamma_f(G) < \gamma_f(1G)$. This fails only when there is no v^* with neighbouring vertices, and hence only when G contains no edges. ∎

Corollary 9. *The only universal γ_f-fixers are the edgeless graphs.*

One consequence of this result to the original conjecture is that if G is a γ-fixer with respect to the identity permutation and not empty then it must be the case that $\gamma_f(G) < \gamma(G)$, and hence this must be true of any universal γ-fixer.

Much of the power in the proof of Theorem 8 comes from the fact that the equitable partition in $1G$ guaranteed by Lemma 7 allows us to restrict our choice of fractional dominating functions significantly. This can be exploited for more general permutations π.

Theorem 10. *Let G be a graph that admits the equitable partition $P_1, \ldots, P_k$, and let π be a permutation of $V(G)$ that fixes each P_i setwise. Then $\gamma_f(G) = \gamma_f(\pi G)$ if and only if G is edgeless.*

Proof. The images $\{(v, j) : v \in P_i, j \in \{0, 1\}\}$ of the partition cells P_i form an equitable partition in πG, so we find a minimum fractional dominating function f of πG that is constant on each of these sets. Using this, we can show (analogously to Lemma 6) that if f_π is the condensation of f to G, then $f_\pi(N[v]) = f(N[(v, 0)]) + f(N[(\pi v, 1)]) - f_\pi(v)$. The proof then echoes that of Theorem 8. ∎

Finally, here is a construction for γ_f-fixers with respect to restricted classes of permutations. Construct the corona $\text{cor}(G)$ of a graph G by adjoining a pendant vertex to every node of G.

Theorem 11. *For any graph G, let $V = V(G)$ and $V^* = V(\text{cor}(G)) - V$. Let π be any permutation of $V(\text{cor}(G))$ such that $\pi(V) = V^*$. Then $\gamma_f(\text{cor}(G)) = \gamma_f(\pi\,\text{cor}(G))$.*

Proof. Since the closed neighbourhoods of pendant vertices in $\text{cor}(G)$ are disjoint, $\gamma_f(\text{cor}(G)) = |V|$. Define f on $V(\pi\,\text{cor}(G))$ by

$$f((v,i)) = \begin{cases} \frac{1}{2} & \text{if } v \in V, \\ 0 & \text{if } v \in V^*. \end{cases}$$

Then f is fractional dominating, and $|f| = |V|$. ■

An example of this construction is shown in Figure 1, with $P_4 = \text{cor}(P_2)$.

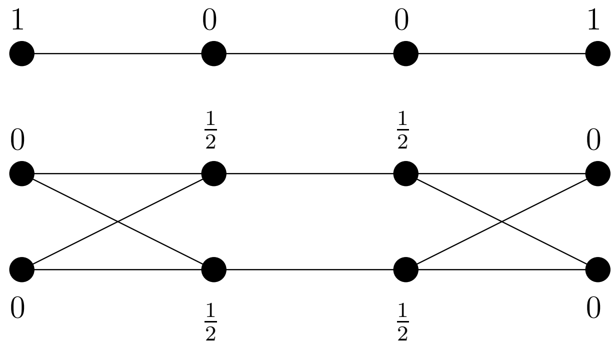

Figure 1. P_4 and its prism πP_4, where $\pi = (12)(34)$, with minimum fractional dominating functions.

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