Discussiones Mathematicae
Graph Theory 27 (2007) 457–470

ON (k,l)-KERNELS IN D-JOIN OF DIGRAPHS

Waldemar Szumny, Andrzej Włoch and Iwona Włoch

Faculty of Mathematics and Applied Physics
Rzeszów University of Technology
W. Pola 2, 35–959 Rzeszów, Poland

e-mail: awloch@prz.edu.pl
e-mail: iwloch@prz.edu.pl

Abstract

In [5] the necessary and sufficient conditions for the existence of (k,l)-kernels in a D-join of digraphs were given if the digraph D is without circuits of length less than k. In this paper we generalize these results for an arbitrary digraph D. Moreover, we give the total number of (k,l)-kernels, k-independent sets and l-dominating sets in a D-join of digraphs.

Keywords: (k,l)-kernel, k-independent set, l-dominating set, D-join, counting.

2000 Mathematics Subject Classification: 05C20.

1. Introduction

For concepts not defined here see [2]. Let D be a finite, directed graph (for short: a digraph) without loops and multiple arcs, where $V(D)$ is the set of vertices and $A(D)$ is the set of arcs of D. By a *path* from a vertex x_1 to a vertex x_n, $n \geqslant 2$, we mean a sequence of vertices $x_1,\ldots,x_n$ and arcs $(x_i,x_{i+1}) \in A(D)$ for $i = 1,2,\ldots,n-1$ and for simplicity we denote it by $x_1 \ldots x_n$. A *circuit* is a path with $x_1 = x_n$. By $d_D(x_i,x_j)$ we denote the length of the shortest path from x_i to x_j in D. If there does not exist a path from x_i to x_j in D, then we put $d_D(x_i,x_j) = \infty$. For any $X \subseteq V(D)$ and $x \in (V(D) \setminus X)$ we put $d_D(x,X) = \min_{y\in X} d_D(x,y)$. By $\mathcal{C}_D^{\eta\leqslant d\leqslant\mu}(x_i)$

we denote the family of all circuits in D containing the vertex x_i of length d, where $\eta \leqslant d \leqslant \mu$.

We say that a subset $J \subset V(D)$ is a (k,l)-*kernel* of D if

(1) for each $x_i, x_j \in J$ and $i \neq j$, $d_D(x_i, x_j) \geqslant k$ and

(2) for each $x_i \notin J$ there exists $x_j \in J$ such that $d_D(x_i, x_j) \leqslant l$.

If the set J satisfies the condition in (1) or in (2), then we shall call it a k-*independent* set of D (also called a k-*stable* set of D) or an l-*dominating* set of D, respectively. We notice that a 2-independent set is an independent set and a 1-dominating set is a dominating set of D. In addition, we assume that a subset containing only one vertex and an empty set is also meant as a k-independent set. The set $V(D)$ is an l-dominating set of D. If an l-dominating set of D has exactly one vertex, then this vertex we shall call an l-dominating vertex of D. Moreover, the l-dominating vertex of D is also a (k,l)-kernel of D for every $k \geqslant 2$. A digraph D whose every induced subdigraph has a (k,l)-kernel is called a (k,l)-*kernel perfect* digraph. Sufficient conditions for the existence of kernels and (k,l)-kernels in digraphs have been investigated, for instance in [1, 3, 4, 5]. By $NkI(D)$, $NlD(D)$ and $NklK(D)$ we mean the number of all k-independent sets, l-dominating sets and (k,l)-kernels of the digraph D, respectively. Moreover, by $Nld(D)$ we will denote the number of all l-dominating vertices of D. The total number of k-independent sets and (k,l)-kernels in graphs and in some their products were studied in [6] and [8].

Let D be a digraph with $V(D) = \{x_1, \ldots, x_n\}$, $n \geqslant 2$ and $\alpha = (D_i)_{i \in \{1,\ldots,n\}}$ be a sequence of vertex disjoint digraphs on $V(D_i) = \{y_1^i, \ldots, y_{p_i}^i\}$, $p_i \geqslant 1$, $i = 1, \ldots, n$. The D-*join* of the digraph D and the sequence α is a digraph $\sigma(\alpha, D)$ such that $V(\sigma(\alpha, D)) = \bigcup_{i=1}^n (\{x_i\} \times V(D_i))$ and $A(\sigma(\alpha, D)) = \{((x_s, y_j^s), (x_q, y_t^q)) : x_s = x_q$ and $(y_j^s, y_t^s) \in A(D_s)$ or $(x_s, x_q) \in A(D)\}$. By D_i^c we mean a copy of the digraph D_i in $\sigma(\alpha, D)$.

It may be noted that if all digraphs from the sequence α have the same vertex set, then from the D-join we obtain the *generalized lexicographic product* of the digraph D and the sequence of the digraphs D_i, i.e., $\sigma(\alpha, D) = D[D_1, \ldots, D_n]$. If all digraphs from the sequence α are isomorphic to the same digraph H, then from the D-join we obtain the *composition* $D[H]$ of the digraphs D and H.

The existence of (k,l)-kernels in the lexicographic product $D[D_1, \ldots, D_n]$ was studied in [7]. Moreover, in [8] the total number of k-independent sets of a lexicographic product of graphs were determined using the concept of the Fibonacci polynomial of graphs. In [5] the necessary and sufficient conditions for the existence of (k,l)-kernels in D-join were given, where D is a digraph without circuits of length less than k. It was proved:

Theorem 1 [5]. *Let D be a digraph without circuits of length less than k. A subset $S^* \subset V(\sigma(\alpha, D))$ is a k-independent set of $\sigma(\alpha, D)$ if and only if there exists a k-independent set $S \subset V(D)$ such that $S^* = \bigcup_{i\in\mathcal{I}} S_i$, where $\mathcal{I} = \{i; x_i \in S\}$, $S_i \subseteq V(D_i^c)$ and S_i is a k-independent set of D_i^c for every $i \in \mathcal{I}$.*

Theorem 2 [5]. *Let $Q \subseteq V(D)$, $\mathcal{I} = \{i : x_i \in Q\}$ and $Q_i \subseteq V(D_i)$. If Q is an l-dominating set of D and Q_i is an l-dominating set of D_i^c for every $i \in \mathcal{I}$, then $Q^* = \bigcup_{i\in\mathcal{I}} Q_i$ is an l-dominating set of $\sigma(\alpha, D)$.*

Theorem 3 [5]. *Let $k \geqslant 2$, $l \leqslant k-1$ be integers. Let D be a digraph without circuits of length less than k. The subset J^* is a (k,l)-kernel of the $\sigma(\alpha, D)$ if and only if there exists a (k,l)-kernel $J \subseteq V(D)$ of the digraph D such that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and J_i is a (k,l)-kernel of D_i^c for every $i \in \mathcal{I}$.*

In this paper, we generalize these results for an arbitrary digraph D. Moreover, we determine the total number of k-independent sets, l-dominating sets and (k,l)-kernels in $\sigma(\alpha, D)$.

2. The Existence of (k,l)-Kernels in D-Join

In this section, we give the necessary and sufficient conditions for the existence of (k,l)-kernels in D-join if D is an arbitrary digraph on n, $n \geqslant 2$ vertices and $\alpha = (D_i)_{i\in\{1,\ldots,n\}}$ is an arbitrary sequence of vertex disjoint digraphs on p_i, $p_i \geqslant 1$ vertices.

Theorem 4. *Let $(x_i, y_p^i), (x_j, y_q^j) \in V(\sigma(\alpha, D))$. Then*

$$d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) = \begin{cases} d_D(x_i, x_j) & for \quad i \neq j, \\ \min\{d_{D_i}(y_p^i, y_q^i), d_D(x_i)\} & for \quad i = j, \end{cases}$$

where $d_D(x_i)$ denotes the length of the shortest circuit containing the vertex x_i in D.

Proof. Assume that $(x_i, y_p^i), (x_j, y_q^j)$ are two different vertices of $V(\sigma(\alpha, D))$ and distinguish two possible cases:

1. $i \neq j$. Then the theorem follows immediately from the definition of $\sigma(\alpha, D)$.

2. $i = j$. Using the definition of $\sigma(\alpha, D)$ we have that there exists a path from (x_i, y_p^i) to (x_i, y_q^i) in $\sigma(\alpha, D)$ of the same length as the path from y_p to y_q in D_i. Moreover, if there exists a circuit in D which includes a vertex x_i, then by the definition of $\sigma(\alpha, D)$ it follows that there also exists a path from (x_i, y_p^i) to (x_i, y_q^i) of length $d_D(x_i)$ equal to the length of the shortest circuit in D, which includes a vertex x_i. Otherwise, if there does not exist a circuit in D which includes a vertex x_i, then we put $d_D(x_i) = \infty$. Evidently $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) = \min\{d_{D_i}(y_p, y_q), d_D(x_i)\}$.

Thus the theorem is proved. ∎

Theorem 5. *A subset $S^* \subset V(\sigma(\alpha, D))$ is a k-independent set of $\sigma(\alpha, D)$ if and only if $S \subset V(D)$ is a k-independent set of D such that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i : x_i \in S\}$, $S_i \subseteq V(D_i^c)$ and for every $i \in \mathcal{I}$*

(a) *S_i is a k-independent set of D_i^c if $\mathcal{C}_D^{d \leqslant k-1}(x_i) = \emptyset$ or*

(b) *S_i is 1-element set containing an arbitrary vertex from $V(D_i^c)$, otherwise.*

Proof. I. Let S^* be a k-independent set of the D-join $\sigma(\alpha, D)$. Denote $S = \{x_i \in V(D) : S^* \cap V(D_i^c) \neq \emptyset\}$. First, we shall prove that S is a k-independent set of D. Let $x_i, x_j \in S$ be two different vertices. Then by the definition of the set S there exist $1 \leqslant r \leqslant p_i$ and $1 \leqslant s \leqslant p_j$ such that $(x_i, y_r^i), (x_j, y_s^j) \in S^*$. By Theorem 4 and from the assumption of the set S^* we obtain that $d_D(x_i, x_j) = d_{\sigma(\alpha,D)}((x_i, y_r^i), (x_j, y_s^j)) \geqslant k$. The definition of the set S implies that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i : x_i \in S\}$. We consider the following cases.

I.1. Let $\mathcal{C}_D^{d \leqslant k-1}(x_i) = \emptyset$.
Because S^* is k-independent so by the definition of $\sigma(\alpha, D)$ and by assumption it follows immediately that S_i is a k-independent set of D_i^c.

I.2. Let $\mathcal{C}_D^{d \leqslant k-1}(x_i) \neq \emptyset$.
We shall prove that S_i contains exactly one arbitrary vertex from $V(D_i^c)$. By Theorem 4 we obtain that for arbitrary two vertices from $V(D_i^c)$ the distance between them in $\sigma(\alpha, D)$ is less than k. Consequently, S_i must contain exactly one arbitrary vertex from $V(D_i^c)$.

Hence from the above cases we obtain that S_i is a k-independent set of D_i^c if there does not exist in D a circuit containing x_i of length less than k or S_i contains exactly one arbitrary vertex from $V(D_i^c)$, otherwise.

II. Let $S \subset V(D)$ be a k-independent set of the digraph D. Let $\mathcal{I} = \{i : x_i \in S\}$ and let S_i be as in the assumption. We shall prove that $S^* = \bigcup_{i\in\mathcal{I}} S_i$ is a k-independent set of the D-join $\sigma(\alpha, D)$. Let $(x_i, y_p^i), (x_j, y_q^j) \in S^*$ be two distinct vertices. Consider the possible cases:

II.1. $(x_i, y_p^i) \in S_i$ and $(x_j, y_q^j) \in S_j$, where $i \neq j$.
Since S is k-independent in D, so by Theorem 4 it follows that $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) = d_D(x_i, x_j) \geqslant k$.

II.2. $(x_i, y_p^i), (x_i, y_q^i) \in S_i$, where $p \neq q$ for some $i \in \mathcal{I}$.
Since S_i contains at least two vertices, so by the assumption, S_i is k-independent of D_i^c and $\mathcal{C}_D^{d\leqslant k-1}(x_i) = \emptyset$. To prove that S^* is a k-independent set of $\sigma(\alpha, D)$ assume on the contrary that $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_i, y_q^i)) < k$. If $k = 2$, then a contradiction with the independence of S_i in D_i^c. Let $k \geqslant 3$. This means that there exists a path $(x_i, y_p^i) \ldots (x_i, y_q^i)$ in $\sigma(\alpha, D)$ of length less than k such that at least one inner vertex of this path does not belong to $V(D_i^c)$. Hence there exists in D a circuit containing the vertex x_i of length less than k, a contradiction to the assumption.

Taking the two above cases into considerations we obtain that for distinct $(x_i, y_p^i), (x_j, y_q^j) \in S^*$ there holds $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) \geqslant k$, hence S^* is a k-independent set of $\sigma(\alpha, D)$.

Thus the theorem is proved. ■

If D is a digraph without circuits of length less than k, then we obtain Theorem 1.

Theorem 6. *A subset $Q^* \subseteq V(\sigma(\alpha, D))$ is an l-dominating set of $\sigma(\alpha, D)$ if and only if $Q \subseteq V(D)$ is an l-dominating set of D such that $Q^* = \bigcup_{i\in\mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$, $Q_i \subseteq V(D_i^c)$ and for every $i \in \mathcal{I}$*

(a) *Q_i is an l-dominating set of D_i^c if $\mathcal{C}_D^{d\leqslant l}(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$, there holds $d_D(x_i, x_j) > l$ or*

(b) *Q_i is an arbitrary nonempty subset of $V(D_i^c)$, otherwise.*

Proof. I. Let Q^* be an l-dominating set of the D-join $\sigma(\alpha, D)$. Denote $Q = \{x_i \in V(D); Q^* \cap V(D_i^c) \neq \emptyset\}$. First, we shall prove that Q is an

l-dominating set of D. Let $x_j \notin Q$. By the definition of the set Q we have that for each $1 \leqslant r \leqslant p_j$ there holds $(x_j, y_r^j) \notin Q^*$. Since Q^* is l-dominating so there exists $(x_i, y_s^i) \in Q^*$, where $i \neq j$ such that $d_{\sigma(\alpha,D)}((x_j, y_r^j), (x_i, y_s^i)) \leqslant l$. Evidently, $x_i \in Q$, so using Theorem 4 we obtain that $d_D(x_j, x_i) = d_{\sigma(\alpha,D)}((x_j, y_r^j), (x_i, y_s^i)) \leqslant l$. The definition of the set Q implies that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$. Consider the following cases:

I.1. Assume that $\mathcal{C}_D^{d \leqslant l}(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) > l$.
Since Q^* is l-dominating so from the definition of $\sigma(\alpha, D)$ and by our assumptions immediately follows that Q_i is an l-dominating set of D_i^c.

I.2. Assume that case I.1 does not hold.
We shall prove that Q_i is an arbitrary nonempty subset of D_i^c. If $\mathcal{C}_D^{d \leqslant l}(x_i) \neq \emptyset$, then there exists in D a circuit which includes the vertex x_i of length less than or equal to l. So for arbitrary two vertices $(x_i, y_q^i), (x_i, y_p^i) \in V(D_i^c)$ there holds $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_i, y_q^i)) \leqslant l$. If there exists $j \in \mathcal{I}$ and $j \neq i$ such that there exists in D a path $x_i \dots x_j$ of length less than or equal to l, then for an arbitrary vertex $(x_i, y_p^i) \in V(D_i^c)$ holds $d_{\sigma(\alpha,D)}((x_i, y_p^i), Q_j) \leqslant l$. Hence $d_{\sigma(\alpha,D)}((x_i, y_p^i), Q^*) \leqslant l$. All this implies that Q_i is an arbitrary nonempty subset of $V(D_i^c)$.

II. Let $Q \subseteq V(D)$ be an l-dominating set of the digraph D, where $\mathcal{I} = \{i : x_i \in Q\}$ and let Q_i be as in the theorem. We shall prove that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$ is an l-dominating set of the D-join. We distinguish the following cases:

II.1. Let $(x_j, y_p^j) \notin Q^*$ and $j \notin \mathcal{I}$.
Then by the definition of the set Q we have that $x_j \notin Q$. Since Q is an l-dominating set of D, so there exists $i \in \mathcal{I}$ such that $x_i \in Q$ and $d_D(x_j, x_i) \leqslant l$. Hence there is $1 \leqslant q \leqslant p_i$ such that $(x_i, y_q^i) \in Q^*$. By Theorem 4 we obtain that $d_{\sigma(\alpha,D)}((x_j, y_p^j), Q^*) \leqslant l$.

II.2. Let $(x_j, y_p^j) \notin Q^*$ and $j \in \mathcal{I}$.
If Q_j is an l-dominating set of D_j^c, then $d_{D_i^c}((x_j, y_p^j), Q_j) \leqslant l$. So $d_{\sigma(\alpha,D)}((x_j, y_p^j), Q^*) \leqslant l$. If Q_j is a nonempty subset of $V(D_j^c)$, then from the assumption of the theorem we have that there exists $t \in \mathcal{I}$ and $t \neq j$ such that there exists a path $x_j \dots x_t$ in D of length less than or equal to l or $\mathcal{C}_D^{d \leqslant l}(x_j) \neq \emptyset$. Consequently, $d_{\sigma(\alpha,D)}((x_j, y_p^j), Q_t) \leqslant l$ or

$d_{\sigma(\alpha,D)}((x_j, y^j_p), Q_j) \leqslant l$, respectively. Hence $d_{\sigma(\alpha,D)}((x_j, y^j_p), Q^*) \leqslant l$, so Q^* is an l-dominating set of $\sigma(\alpha, D)$.

Thus the theorem is proved. ■

Theorem 7. *Let $k \geqslant 2$, $1 \leqslant l \leqslant k-1$ be integers. The subset $J^* \subset V(\sigma(\alpha, D))$ is a (k,l)-kernel of the D-join $\sigma(\alpha, D)$ if and only if there exists a (k,l)-kernel $J \subset V(D)$ such that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D^c_i)$ and for every $i \in \mathcal{I}$*

(a) *J_i is a (k,l)-kernel of D^c_i if $\mathcal{C}^{d\leqslant k-1}_D(x_i) = \emptyset$ or*

(b) *J_i is 1-element set containing an arbitrary vertex of $V(D^c_i)$ if $\mathcal{C}^{d\leqslant l}_D(x_i) \neq \emptyset$ or*

(c) *J_i is 1-element set containing an l-dominating vertex of D^c_i, otherwise.*

Proof. I. Let $k \geqslant 2$, $1 \leqslant l \leqslant k-1$ be integers. Let J^* be a (k,l)-kernel of the D-join $\sigma(\alpha, D)$. Denote $J = \{x_i \in V(D); J^* \cap V(D^c_i) \neq \emptyset\}$. First, we shall prove that J is a (k,l)-kernel of D. Let $x_i, x_j \in J$ and $i \neq j$. Then from the definition of the set J we have that there exists $1 \leqslant p \leqslant p_i$ and $1 \leqslant q \leqslant p_j$ such that $(x_i, y^i_p), (x_j, y^j_q) \in J^*$. By Theorem 4 we have that $d_D(x_i, x_j) = d_{\sigma(\alpha,D)}((x_i, y^i_p), (x_j, y^i_q)) \geqslant k$. So, J is a k-independent set of D. Now, we will show that J is an l-dominating set of D. Let $x_j \notin J$. Using the definition of the set J for each $1 \leqslant r \leqslant p_j$ holds $(x_j, y^j_r) \notin J^*$. Since J^* is l-dominating, hence there exists $(x_i, y^i_s) \in J^*$, where $j \neq i$ such that $d_{\sigma(\alpha,D)}((x_j, y^j_r), (x_i, y^i_s)) \leqslant l$.

From the definition of the set J we have that $x_i \in J$, so by Theorem 4 there holds $d_D(x_j, x_i) = d_{\sigma(\alpha,D)}((x_j, y^j_r), (x_i, y^i_s)) \leqslant l$. Consequently, J is an l-dominating set of D, hence J is a (k,l)-kernel of D. The definition of the set J implies that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$. Consider the possible cases:

I.1. Let $\mathcal{C}^{d\leqslant k-1}_D(x_i) = \emptyset$.
We shall prove that J_i is a (k,l)-kernel of D^c_i in this case. From Theorem 5(a) we obtain that J_i is a k-independent set of D^c_i. Next we shall show that J_i is l-dominating. Since J is a k-independent set of D and $l \leqslant k-1$, then for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) \geqslant k \geqslant l+1$. So, there does not exist in D a path $x_i \ldots x_j$ of length less than or equal to l. Moreover, $\mathcal{C}^{d\leqslant k-1}_D(x_i) = \emptyset$ and $l \leqslant k-1$, hence $\mathcal{C}^{d\leqslant l}_D(x_i) = \emptyset$.

From the above and by Theorem 6(a) we obtain that J_i is an l-dominating set of D^c_i. Consequently, J_i is a (k,l)-kernel of D^c_i in this case.

I.2. Let $\mathcal{C}_D^{d\leqslant k-1}(x_i) \neq \emptyset$.
Then by Theorem 5(b) the set J_i contains exactly one arbitrary vertex from $V(D_i^c)$. So J_i is a k-independent set of D_i^c. Because $l \leqslant k-1$, then for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) \geqslant k \geqslant l+1$. Hence there does not exists in D a path $x_i \dots x_j$ of length $d_D(x_i, x_j) \leqslant l$. From the assumption there exists in D a circuit containing the vertex x_i of length less than k. We distinguish the following possibilities:

I.2.1. $\mathcal{C}_D^{d\leqslant l}(x_i) \neq \emptyset$.
Then by Theorem 6(b) it follows immediately that J_i is a 1-element set containing an arbitrary vertex of $V(D_i^c)$.

I.2.2. $\mathcal{C}_D^{d\leqslant l}(x_i) = \emptyset$ and $\mathcal{C}_D^{l+1\leqslant d\leqslant k-1}(x_i) \neq \emptyset$.
We will show that J_i is a 1-element set containing an l-dominating vertex of D_i^c. Using Theorem 6(a) we obtain that J_i is an l-dominating set of D_i^c. Because J_i contains exactly one vertex, so $J_i = \{(x_i, y_t^i)\}$, where (x_i, y_t^i) is an l-dominating vertex of D_i^c.

II. Let $J \subset V(D)$ be a (k,l)-kernel of the digraph D. Let $\mathcal{I} = \{i : x_i \in J\}$ and J_i be as in the statements of the theorem. We shall prove that $J^* = \bigcup_{i\in\mathcal{I}} J_i$ is a (k,l)-kernel of $\sigma(\alpha, D)$. Firstly we will prove that J^* is a k-independent set of the D-join $\sigma(\alpha, D)$. Let $(x_i, y_p^i), (x_j, y_q^j) \in J^*$ be two different vertices. Consider the following cases:

II.1. $(x_i, y_p^i) \in J_i$ and $(x_i, y_q^j) \in J_j$, where $i \neq j$.
Evidently, $x_i, x_j \in J$ and because J is k-independent so by Theorem 4 we have that $d_D(x_i, x_j) = d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) \geqslant k$.

II.2. $(x_i, y_p^i), (x_i, y_q^i) \in J_i$ for some $i \in \mathcal{I}$.
Since J_i contains at least two vertices, so by assumption J_i is a (k,l)-kernel of D_i^c. Hence $d_{D_i^c}((x_i, y_p^i), (x_i, y_q^i)) \geqslant k$. Assume on the contrary that $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_i, y_q^i)) < k$. If $k = 2$, then there is a contradiction with the independence of J_i in D_i^c. Let $k \geqslant 3$. This means that there exists a path $(x_i, y_p^i) \dots (x_i, y_q^i)$ in $\sigma(\alpha, D)$ of length less than k such that at least one inner vertex of this path does not belong to $V(D_i^c)$. Hence there exists in D a circuit containing the vertex x_i of length less than k and by Theorem 5(b) the set J_i contains exactly one vertex from $V(D_i^c)$, a contradiction to $(x_i, y_p^i), (x_i, y_q^i) \in J_i$.

Taking the two above cases into consideration we obtain that for distinct $(x_i, y_p^i), (x_j, y_q^j) \in J^*$ there holds $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) \geqslant k$. Hence J^* is a k-independent set of $\sigma(\alpha, D)$.

Now we shall prove that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, is an l-dominating set of the D-join $\sigma(\alpha, D)$. Consider the possible cases:

II.3. Let $(x_j, y_p^j) \notin J^*$ and $j \notin \mathcal{I}$.
Then by the definition of the set J we have that $x_j \notin J$. Since J is l-dominating in D, so there exists $i \in \mathcal{I}$ such that $x_i \in J$ and $d_D(x_j, x_i) \leqslant l$. Consequently, there exists $1 \leqslant q \leqslant p_i$ such that $(x_i, y_q^i) \in J^*$ and by Theorem 4 we obtain that $d_{\sigma(\alpha,D)}((x_j, y_p^j), (x_i, y_q^i)) \leqslant l$.

II.4. Let $(x_j, y_p^j) \notin J^*$ and $j \in \mathcal{I}$.
If J_j is a (k,l)-kernel of D_j^c, then J_j is an l-dominating set of D_j^c, so $d_{D_j^c}((x_j, y_q^j), J_j) \leqslant l$. Hence $d_{\sigma(\alpha,D)}((x_j, y_q^j), J^*) \leqslant l$. If J_j contains exactly one arbitrary vertex of $V(D_j^c)$, then by assumption of the theorem there exists in D a circuit containing the vertex x_j of length less than or equal to l. So there exists in $\sigma(\alpha, D)$ a path from (x_j, y_p^j) to J_j and $d_{\sigma(\alpha,D)}((x_j, y_p^j), J_j) \leqslant l$. Hence $d_{\sigma(\alpha,D)}((x_j, y_p^j), J^*) \leqslant l$. If J_j is a 1-element set containing an l-dominating vertex of D_j^c, then by the definition of the l-dominating vertex $d_{D_j^c}((x_j, y_p^j), J_j) \leqslant l$. Hence $d_{\sigma(\alpha,D)}((x_j, y_p^j), J^*) \leqslant l$. Thus it follows that J^* is an l-dominating set of $\sigma(\alpha, D)$.

Taking the above cases into consideration we obtain that J^* is a (k,l)-kernel of $\sigma(\alpha, D)$.

Thus the theorem is proved. ■

If the digraph D is without circuits of length less than k, then we obtain Theorem 3.

Theorem 8. *Let $k \geqslant 2$, $l \geqslant k$ be integers. The subset $J^* \subset V(\sigma(\alpha, D))$ is a (k,l)-kernel of the D-join $\sigma(\alpha, D)$ if and only if there exists a (k,l)-kernel $J \subset V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and for every $i \in \mathcal{I}$*

- (a) *J_i is a (k,l)-kernel of D_i^c if $\mathcal{C}_D^{d \leqslant l}(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) > l$ or*
- (b) *J_i is a 1-element set containing an arbitrary vertex from $V(D_i^c)$ if $\mathcal{C}_D^{d \leqslant k-1}(x_i) \neq \emptyset$ or*
- (c) *J_i is an arbitrary nonempty k-independent set of D_i^c, otherwise.*

Proof. I. Let $k \geqslant 2$, $l \geqslant k$ be integers. Let J^* be a (k,l)-kernel of the D-join $\sigma(\alpha, D)$. Denote $J = \{x_i \in V(D) : J^* \cap V(D_i^c) \neq \emptyset\}$. Proving analogously as in Theorem 7 we obtain that J is a (k,l)-kernel of the digraph D. Of course, the definition of the set J implies that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$. Consider the following cases:

I.1. Let $\mathcal{C}_D^{d\leqslant l}(x_i) = \emptyset$.
Since $l \geqslant k$, so there does not exist in D a circuit containing the vertex x_i of length less than k. Then from Theorem 5(a) we obtain that J_i is a k-independent set of D_i^c. By our assumption $l \geqslant k$, so to establish sets J_i we consider the following possibilities:

I.1.1. There exists $j \in \mathcal{I}$ and $j \neq i$ such that $d_D(x_i, x_j) \leqslant l$.
By Theorem 6 (b) an arbitrary nonempty subset of $V(D_i^c)$ is l-dominating in D_i^c, so J_i is an arbitrary k-independent set of D_i^c.

I.1.2. For each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) > l$.
Then by Theorem 6(a) we obtain that J_i is an l-dominating set of D_i^c. Consequently, J_i is a (k,l)-kernel in this case.

I.2. Let $\mathcal{C}_D^{d\leqslant l}(x_i) \neq \emptyset$.
Because $l \geqslant k$, we consider the following possibilities:

I.2.1. $\mathcal{C}_D^{k\leqslant d\leqslant l}(x_i) \neq \emptyset$ and $\mathcal{C}_D^{d\leqslant k-1}(x_i) = \emptyset$.
Then by Theorem 5(a) and Theorem 6(b) we obtain that the set J_i is an arbitrary k-independent set of $V(D_i^c)$.

I.2.2. $\mathcal{C}_D^{d\leqslant k-1}(x_i) \neq \emptyset$.
We shall prove that J_i is a 1-element set containing an arbitrary vertex from $V(D_i^c)$. By Theorem 5(b) the set J_i contains exactly one vertex from $V(D_i^c)$. Because $l \geqslant k$, so there exists in D a circuit containing the vertex x_i of length less than or equal to l. Hence by Theorem 6(b) we obtain that J_i contains exactly one arbitrary vertex from $V(D_i^c)$.

II. Let $J \subset V(D)$ be a (k,l)-kernel of the digraph D and let $\mathcal{I} = \{i : x_i \in J\}$. Proving analogously as in Theorem 7 we can show that $J^* = \bigcup_{i\in\mathcal{I}} J_i$ is a (k,l)-kernel of the D-join $\sigma(\alpha, D)$ where J_i, $i \in \mathcal{I}$ satisfy the assumption of the theorem.

Thus the theorem is proved. ■

3. On (k,l)-Kernel Perfectness of the D-Join

From the definition of the $\sigma(\alpha, D)$ it follows immediately:

Proposition 1. *Every induced subdigraph of* $\sigma(\alpha, D)$ *is*

(a) *a digraph of the form* $\sigma(\widetilde{\alpha}, \widetilde{D})$, *where* $\widetilde{D}$ *is an induced subdigraph of* D *with* $V(\widetilde{D}) = \{x_t : t \in \widetilde{\mathcal{I}}\}$, $|\widetilde{\mathcal{I}}| > 1$, $\widetilde{\mathcal{I}} \subsetneq \{1, \ldots, n\}$ *and* $\widetilde{\alpha}$ *is a family of induced subdigraphs of* D_t, *where* $t \in \widetilde{\mathcal{I}}$ *or*

(b) *an induced subdigraph of* D_i *for some* $1 \leqslant i \leqslant n$ *or*

(c) *the union of the digraphs from* (a) *and* (b).

From the definition of the (k, l)-kernel perfect digraph and by Proposition 1 it follows immediately:

Proposition 2. *If* $\sigma(\alpha, D)$ *is* (k, l)*-kernel perfect, then* D *and* D_i, $i = 1, \ldots, n$ *are* (k, l)*-kernel perfect.*

In [5] it has been proved:

Theorem 9 [5]**.** *Let* D *be a digraph without circuits of length less than* k *and let* $\alpha = (D_i)_{i \in \{1,\ldots,n\}}$ *be a sequence of vertex disjoint digraphs. The* D*-join* $\sigma(\alpha, D)$ *is a* (k, l)*-kernel perfect digraph if and only if the digraph* D *and the digraphs* D_i, $i = 1, \ldots, n$ *are* (k, l)*-kernel perfect digraphs.*

In this section, we generalize this result for an arbitrary digraph D.

Theorem 10. *Let* D *be a* (k, l)*-kernel perfect digraph. Let* D_i, $i = 1, \ldots, n$ *be a* (k, l)*-kernel perfect digraph if* $\mathcal{C}_D^{d \leqslant k-1}(x_i) = \emptyset$ *or every subdigraph of* D_i *has an* l*-dominating vertex, otherwise. Then* $\sigma(\alpha, D)$ *is a* (k, l)*-kernel perfect digraph.*

Proof. Assume that D and D_i, $i = 1, \ldots, n$ are as in the statements of the theorem. We shall show that $\sigma(\alpha, D)$ is a (k, l)-kernel perfect digraph. From Proposition 1 it follows that we need only to prove that $\sigma(\alpha, D)$ has a (k, l)-kernel. By Theorem 7, Theorem 8 and from our assumptions there exists a (k, l)-kernel $J \subset V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a (k, l)-kernel of the D-join, where $\mathcal{I} = \{i; x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and J_i is a (k, l)-kernel of D_i^c if $\mathcal{C}_D^{d \leqslant k-1}(x_i) = \emptyset$ or J_i is a 1-element set containing an l-dominating vertex of D_i^c.

Thus the theorem is proved. ■

4. The Total Number of (k, l)-Kernels of the D-Join

In this section, we calculate the number of all k-independent sets, l-dominating sets and (k, l)-kernels of the D-join $\sigma(\alpha, D)$.

Theorem 11. *Let $k \geqslant 2$, $n \geq 2$ be integers. Let $\sigma(\alpha, D)$ be a D-join of the digraph D on n vertices and α be a sequence of vertex disjoint digraphs $(D_i)_{i\in\{1,\ldots,n\}}$ on p_i vertices, $p_i \geqslant 1$. Let $\mathcal{S} = \{S_1, \ldots, S_j\}$, $j \geqslant 1$ be a family of all nonempty k-independent sets of the digraph D and let $\mathcal{S} \ni S_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subset \{1, \ldots, n\}$. Then $NkI(\sigma(\alpha, D)) = 1 + \sum_{r=1}^{j} \prod_{i\in\mathcal{I}_r} \varphi(D_i)$, where*

$$\varphi(D_i) = \begin{cases} NkI(D_i) - 1 & \text{if } \mathcal{C}_D^{d\leqslant k-1}(x_i) = \emptyset, \\ p_i & \text{otherwise.} \end{cases}$$

Proof. Let D be a given digraph on n-vertices, $n \geqslant 2$. Theorem 4 implies that to obtain a k-independent set of $\sigma(\alpha, D)$ first we have to choose a k-independent set of D. Let $\mathcal{S} = \{S_1, \ldots, S_j\}$, $j \geqslant 1$ be a family of all nonempty k-independent sets of the digraph D. Assume that $\mathcal{S} \ni S_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subset \{1, \ldots, n\}$. Next by Theorem 5 in each of the D_i^c, $i \in \mathcal{I}_r$ we have to choose an nonempty k-independent set if there does not exist in D a circuit containing the vertex x_i of length less than k or we choose an arbitrary vertex from $V(D_i^c)$, otherwise. Evidently we can do it on $NkI(D_i)-1$ or p_i ways, respectively. Hence from the fundamental combinatorial statement we have $\sum_{r=1}^{j} \prod_{i\in\mathcal{I}_r} \varphi(D_i)$ sets being k-independent, where

$$\varphi(D_i) = \begin{cases} NkI(D_i) - 1 & \text{if } \mathcal{C}_D^{d\leqslant k-1}(x_i) = \emptyset, \\ p_i & \text{otherwise.} \end{cases}$$

Moreover, the empty set also is a k-independent set of $\sigma(\alpha, D)$. Consequently, $NkI(\sigma(\alpha, D)) = 1 + \sum_{r=1}^{j} \prod_{i\in\mathcal{I}_r} \varphi(D_i)$.

Thus the theorem is proved. ■

Theorem 12. *Let $l \geqslant 1$, $n \geq 2$ be integers. Let $\sigma(\alpha, D)$ be a D-join of the digraph D on n vertices and α be a sequence of vertex disjoint digraphs $(D_i)_{i\in\{1,\ldots,n\}}$ on p_i vertices, $p_i \geqslant 1$. Let $\mathcal{Q} = \{Q_1, \ldots, Q_j\}$, $j \geqslant 1$ be a family of all l-dominating sets of the digraph D and let $\mathcal{Q} \ni Q_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subseteq \{1, \ldots, n\}$. Then $NlD(\sigma(\alpha, D)) = \sum_{r=1}^{j} \prod_{i\in\mathcal{I}_r} \psi(D_i)$, where*

$$\psi(D_i) = \begin{cases} NlD(D_i) & \textit{if } \mathcal{C}_D^{d\leqslant l}(x_i) = \emptyset \textit{ and for each } \mathcal{I} \ni j \neq i \\ & \textit{holds } d_D(x_i, x_j) > l, \\ 2^{p_i} - 1 & \textit{otherwise.} \end{cases}$$

Proof. Let D be a given digraph on n vertices, $n \geqslant 2$. By Theorem 4 we have that to obtain an l-dominating set of $\sigma(\alpha, D)$ first we have to choose an l-dominating set of D. Let $\mathcal{Q} = \{Q_1, \ldots, Q_j\}$, $j \geqslant 1$ be a family o all l-dominating sets of the digraph D. Assume that $\mathcal{Q} \ni Q_r = \{x_i : i \in \mathcal{I}_r\}$ and $\mathcal{I}_r \subseteq \{1, \ldots, n\}$. Next by Theorem 6 in each of the D_i^c, $i \in \mathcal{I}_r$ we have to choose an l-dominating set if for each $j \in \mathcal{I}_r$ and $j \neq i$ there does not exist a path $x_i \ldots x_j$ of length less than or equal to l and there does not exist in D a circuit containing the vertex x_i of length less than or equal to l or we have to choose in D_i^c an arbitrary nonempty subset of $V(D_i^c)$. Evidently, we can do it on $NlD(D_i)$ or $2^{p_i} - 1$ ways, respectively. Hence from the fundamental combinatorial statement we have $\sum_{r=1}^{j} \prod_{i \in \mathcal{I}_r} \psi(D_i)$ sets being l-dominating sets of $\sigma(\alpha, D)$, where

$$\psi(D_i) = \begin{cases} NlD(D_i) & \text{if } \mathcal{C}_D^{d\leqslant l}(x_i) = \emptyset \text{ and for each } \mathcal{I} \ni j \neq i \\ & \text{holds } d_D(x_i, x_j) > l, \\ 2^{p_i} - 1 & \text{otherwise.} \end{cases}$$

Thus the theorem is proved. ∎

Using the same method as in Theorems 11 and 12 we can prove:

Theorem 13. *Let $k \geqslant 2$, $1 \leqslant l \leqslant k-1$, $n \geqslant 2$ be integers. Let $\sigma(\alpha, D)$ be a D-join of the digraph D on n vertices and α be a sequence of vertex disjoint digraphs $(D_i)_{i \in \{1,\ldots,n\}}$ on p_i vertices, $p_i \geqslant 1$. Let $\mathcal{J} = \{J_1, \ldots, J_j\}$, $j \geqslant 1$ be a family of all (k,l)-kernels of the digraph D and let $\mathcal{J} \ni J_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subseteq \{1, \ldots, n\}$. Then $NklK(\sigma(\alpha, D)) = \sum_{r=1}^{j} \prod_{i \in \mathcal{I}_r} \mu(D_i)$, where*

$$\mu(D_i) = \begin{cases} NklK(D_i) & \textit{if } \mathcal{C}_D^{d\leqslant k-1}(x_i) = \emptyset, \\ p_i & \textit{if } \mathcal{C}_D^{d\leqslant l}(x_i) \neq \emptyset, \\ Nld(D_i) & \textit{otherwise.} \end{cases}$$

Theorem 14. *Let $k \geqslant 2$, $l \geqslant k$, $n \geqslant 2$ be integers. Let $\sigma(\alpha, D)$ be a D-join of the digraph D on n vertices and α be a sequence of vertex disjoint digraphs*

$(D_i)_{i\in\{1,\dots,n\}}$ *on* p_i *vertices,* $p_i \geqslant 1$*. Let* $\mathcal{J} = \{J_1, \dots, J_j\}$*,* $j \geqslant 1$ *be a family of all* (k,l)*-kernels of the digraph* D *and let* $\mathcal{J} \ni J_r = \{x_i : i \in \mathcal{I}_r\}$*, where* $\mathcal{I}_r \subseteq \{1, \dots, n\}$*. Then* $NklK(\sigma(\alpha, D)) = \sum_{r=1}^{j} \prod_{i\in\mathcal{I}_r} \eta(D_i)$*, where* $\eta(D_i) =$

$$\begin{cases} NklK(D_i) & \text{if } \mathcal{C}_D^{d\leqslant l}(x_i) = \emptyset \text{ and for each } \mathcal{I} \ni j \neq i \text{ holds } d_D(x_i, x_j) > l, \\ p_i & \text{if } \mathcal{C}_D^{d\leqslant k-1}(x_i) \neq \emptyset, \\ NkI(D_i) - 1 & \text{otherwise.} \end{cases}$$

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Received 29 April 2006
Revised 18 May 2007
Accepted 18 May 2007