

THE STRUCTURE AND EXISTENCE OF 2-FACTORS IN ITERATED LINE GRAPHS

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Abstract

We prove several results about the structure of 2-factors in iterated line graphs. Specifically, we give degree conditions on G that ensure $L^2(G)$ contains a 2-factor with every possible number of cycles, and we give a sufficient condition for the existence of a 2-factor in $L^2(G)$ with all cycle lengths specified. We also give a characterization of the graphs G where $L^k(G)$ contains a 2-factor.

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1. INTRODUCTION

A spanning cycle in a graph G is called a *hamiltonian cycle* and if such a cycle exists, we say that G is *hamiltonian*. Similarly, a *2-factor* in a graph G is a 2-regular spanning subgraph, or equivalently a partition of $V(G)$ into cycles. Hamiltonicity and the existence of 2-factors in graphs have been widely studied. A good reference for the current state of such problems is [4].

We say that two edges in a graph G are *adjacent* if they share an end-vertex. The *line graph* of G , denoted $L(G)$ is the graph with $V(L(G)) = E(G)$ and $E(L(G)) = \{e_i e_j \mid e_i \text{ and } e_j \text{ are adjacent in } G\}$. We define the i^{th} iterated line graph of G recursively with $L^1(G) = L(G)$ and $L^{i+1}(G) = L(L^i(G))$.

Chartrand [2] was one of the first to study properties of iterated line graphs, proving that for every graph G (with a few trivial exceptions), $L^k(G)$ is hamiltonian for k sufficiently large. Since this first paper, many cycle-structural properties of iterated line graphs have been studied, including when $L^k(G)$ is k -ordered ([10]), pancyclic ([12]), k -ordered hamiltonian ([8]), and characterizations of G when $L^k(G)$ is hamiltonian ([11]).

In this paper, we extend Chartrand's result by giving degree conditions on G to ensure that $L^2(G)$ contains a 2-factor with every possible number of cycles. We also give a sufficient condition for the existence of a 2-factor in $L^2(G)$ with all cycle lengths specified. Finally in section 5, we give a characterization of the graphs G where $L^k(G)$ contains a 2-factor.

2. PRELIMINARIES

In the majority of this paper, we consider only undirected, loopless graphs without multiple edges. The main results in the last section will consider multigraphs as well. We will use n_k to denote the number of vertices in the k^{th} iteration of the line graph, and δ_k to denote $\delta(L^k(G))$.

A connected graph is *prolific* if it is not isomorphic to a path, cycle or the claw $K_{1,3}$. For the remainder of this paper, we will consider only prolific graphs.

For any terms not defined here, consult [13].

2.1. Hamiltonian line graphs

A *dominating circuit* of G is a circuit in G such that each edge of G is either in the circuit or adjacent to an edge of the circuit. Harary and Nash-

Williams [6] used the notion of a dominating circuit to characterize the graphs G such that $L(G)$ is hamiltonian.

Theorem 1 (Harary and Nash-Williams [6]). *Let G be a connected graph. Then $L(G)$ is hamiltonian if and only if G contains a dominating circuit.*

Theorem 2 (Chartrand [2]). *For every prolific graph G , there exists an integer T_h such that for all $t \geq T_h$, $L^t(G)$ is hamiltonian.*

Theorem 3 (Chartrand [2]). *If G is a prolific graph with $\delta(G) \geq 3$, then $L^2(G)$ is hamiltonian.*

In the proof of Theorem 3, Chartrand uses an important description of the structure of line graphs. If $H = L(G)$ for some G , then H can be decomposed into maximal edge-disjoint cliques that intersect in at most one vertex. Each vertex v in G can be associated with a unique clique K_v in this decomposition of $L(G)$, in that the vertices of K_v in H are precisely those edges adjacent to v in G . As $\delta(G) \geq 3$, each clique in the decomposition has order at least three, and is therefore hamiltonian. It is then simple to construct a dominating circuit in $L(G)$ by selecting a hamiltonian cycle from each clique and considering the subgraph of $L(G)$ induced by the edges of these cycles.

2.2. 2-factors in line graphs

Gould and Hynds [5] generalized Theorem 1 to 2-factors with more than one cycle.

Definition. *A k -system that dominates* is a collection \mathcal{C} of k edge-disjoint circuits and stars ($K_{1,s}$ with $s \geq 3$) in G such that each edge e of G is either in one of the circuits or stars of \mathcal{C} , e is adjacent to an edge of a circuit of \mathcal{C} , or e is incident to the center of a star of \mathcal{C} .

Theorem 4 (Gould and Hynds [5]). *$L(G)$ contains a 2-factor with exactly k cycles if and only if G contains a k -system that dominates.*

Several of the results in this paper will use a technique similar to the one given in the proof of Theorem 3 to construct a k -system that dominates for various values of k . The following well-known result will be useful.

Theorem 5. *Let $k \geq 1$ be a positive integer. Then K_{2k+1} can be decomposed into k edge-disjoint hamiltonian cycles and K_{2k+2} can be decomposed into k edge-disjoint hamiltonian cycles and a matching of size $k + 1$.*

3. 2-FACTORS WITH SPECIFIED NUMBER OF CYCLES

The following lemma was used by Knor and Niepel [9] for determining the distance-independent domination number of iterated line graphs.

Lemma 6 (Knor and Niepel [9]). *Let G be a connected graph. If $\delta(G) \geq 5$, then $L(G)$ contains $\lfloor n_2/3 \rfloor$ edge-disjoint copies of a claw $K_{1,3}$.*

As $\lfloor n_2/3 \rfloor = \lfloor |E(L(G))|/3 \rfloor$, this lemma implies that if $\delta(G) \geq 5$, then $L^2(G)$ contains the maximum possible number of edge-disjoint triangles. However, this does not imply that $L^2(G)$ contains a 2-factor with the maximum possible number of cycles, since if n_1 is not divisible by 3, then there may be one or two extra edges not covered by a claw and not incident to a center of a claw. In that case, the set of claws from Lemma 6 do not form an $\lfloor n_2/3 \rfloor$ -system that dominates. We present a strengthening of Lemma 6 that addresses this issue.

Lemma 7. *Let G be a connected graph. If $\delta(G) \geq 6$, then $E(L(G))$ can be decomposed into $\lfloor n_2/3 \rfloor$ edge-disjoint copies of the claw $K_{1,3}$, the star $K_{1,4}$, or the star $K_{1,5}$.*

Proof. In Knor and Niepel's proof of Lemma 6, they prove that there exists an orientation D of $L(G)$ such that the indegree of every vertex except possibly one is a multiple of 3. If the indegree of every vertex is a multiple of 3, then the claw decomposition \mathcal{C} is formed by partitioning the incoming edges at each vertex into sets of size 3. Otherwise, let v be the sole vertex whose indegree is not a multiple of 3. To form the set \mathcal{C} of $\lfloor n_2/3 \rfloor$ claws, partition the incoming edges at every vertex except v into sets of size 3; at v discard $n_2 \bmod 3$ incoming edges, and partition the remaining incoming edges into sets of size 3.

If the indegree of v is at least 3, then all the incoming edges incident on v can be partitioned into $\lfloor \deg_{L(G)}^-(v)/3 \rfloor$ sets of size 3, 4, and 5.* Along

*Note that if $\deg_{L(G)}^-(v)/3 \equiv 1 \pmod{3}$, then there will be exactly one set of size 4 and no set of size 5, while if $\deg_{L(G)}^-(v)/3 \equiv 2 \pmod{3}$, then there will be either exactly two sets of size 4 and no set of size 5, or exactly one set of size 5 and no sets of size 4.

with the partitions at the other vertices, we obtain a decomposition \mathcal{C}' of the edges of $L(G)$ into $\lfloor n_2/3 \rfloor$ edge-disjoint copies of the claw $K_{1,3}$, the star $K_{1,4}$, or the star $K_{1,5}$.

We now consider when the indegree of v is less than 3 by distinguishing two cases. First, suppose that $\deg_{L(G)}^-(v) = 1$, and let \overrightarrow{uv} be the sole incoming edge. If the indegree of u is positive, then we can reverse the edge \overrightarrow{uv} and obtain a decomposition \mathcal{C}' as above, with u playing the role of v . Thus, we may assume that the indegree of u is 0. Since $\delta(G) \geq 6$, \overrightarrow{uv} is in a clique Q of size at least 6 in $L(G)$, and hence there exists a vertex y in $Q - \{u, v\}$ such that $\deg_{Q - \{u, v\}}^-(y) \geq 2$. Let \overrightarrow{xy} be an edge in Q . Since u has indegree 0 and v has indegree 1, the directed edges \overrightarrow{ux} , \overrightarrow{uy} , \overrightarrow{vx} , and \overrightarrow{vy} are present in D . Form a new orientation D' of $L(G)$ from D by reversing the edges \overrightarrow{uv} , \overrightarrow{uy} , \overrightarrow{ux} , and \overrightarrow{xy} , as depicted in Figure 1. Notice that the indegree of v becomes 0, the indegree of u becomes 3, the indegree of x is unchanged, and the indegree of y is reduced by 2. Prior to any edge reversals, $\deg_Q^-(y) \geq 4$ and $\deg_{L(G)}^-(y) \equiv 0 \pmod 3$ by assumption. Hence, after reversing these edges, the indegree of y is at least 4, and the indegree of all other vertices is $0 \pmod 3$, completing the proof in this case.

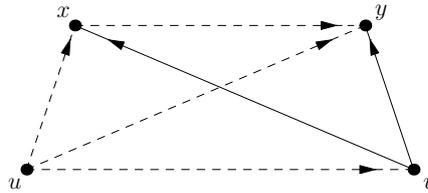


Figure 1. The edges to reverse in the case when $\deg_{L(G)}^-(v) = 1$ are shown with dashed edges.

Next, suppose that $\deg_{L(G)}^-(v) = 2$, and let $\overrightarrow{u_1v}$ and $\overrightarrow{u_2v}$ be the two incoming edges. If both u_1 and u_2 have positive indegree, then by reversing both $\overrightarrow{u_1v}$ and $\overrightarrow{u_2v}$ we can obtain a decomposition \mathcal{C}' as above. If the two cliques Q_1 and Q_2 that contain $\overrightarrow{u_1v}$ and $\overrightarrow{u_2v}$ are edge-disjoint, then we may also apply the same proof separately to each clique that we did when $\deg_{L(G)}^-(v) = 1$. Hence we may assume that both $\overrightarrow{u_1v}$ and $\overrightarrow{u_2v}$ lie in the same clique Q . Without loss of generality, assume that the edge $\overrightarrow{u_1u_2}$ is present in Q , implying that $\deg_{L(G)}^-(u_2) \geq 3$. Since $Q - \{u_1, v\}$ has at least 4 vertices, there exists a vertex y in $Q - \{u_1, v\}$ with $\deg_{Q - \{u_1, v\}}^-(y) \geq 2$.

Assume first that $y \neq u_2$, and note that, as above, there exists a vertex x in $Q - \{u_1, u_2, v\}$ such that \overrightarrow{xy} is an edge. As above, we reverse the edges $\overrightarrow{u_1v}$, $\overrightarrow{u_1y}$, $\overrightarrow{u_1x}$, and \overrightarrow{xy} , so that the indegree of y is at least 4 and $\equiv 1 \pmod 3$ and the indegrees of x and u_1 are $0 \pmod 3$. We will then reverse the orientation of the edge $\overrightarrow{u_2v}$ so that the degree of u_2 is at least 4 and $\equiv 1 \pmod 3$ and the indegree of $v \equiv 0 \pmod 3$.

If $y = u_2$, then we may assume that u_2 is the only vertex in $Q - \{u_1, v\}$ with $\deg_{Q - \{u_1, v\}}^-(y) \geq 2$. Hence there are exactly 3 vertices in $Q - \{u_1, u_2, v\}$ which are oriented as a directed cycle, and all three have incident edges oriented towards u_2 .

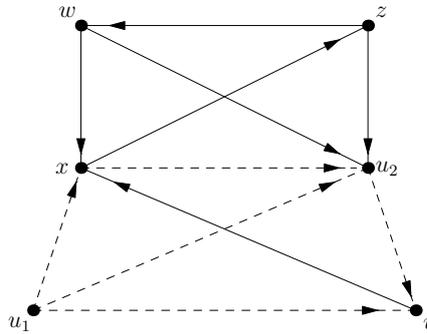


Figure 2. The subcase where $y = u_2$. Note that the edges $\overrightarrow{u_1w}$, $\overrightarrow{u_1z}$, \overrightarrow{vw} , and \overrightarrow{vz} are not shown for clarity. The edges to reverse in this case are shown with dashed edges.

Figure 2 depicts Q . Let $\overrightarrow{xu_2}$ be a directed edge in $Q - \{u_1, u_2, v\}$. Form a new orientation D' of $L(G)$ from D by reversing the edges $\overrightarrow{u_1v}$, $\overrightarrow{u_1u_2}$, $\overrightarrow{u_1x}$, $\overrightarrow{xu_2}$, and $\overrightarrow{vu_2}$. Notice that the indegree of v becomes 0, the indegree of u_1 becomes 3, the indegree of x is unchanged, and the indegree of u_2 is reduced by 1. By assumption $\deg_Q^-(u_2) \geq 4$, but $\deg_{L(G)}^-(u_2) \equiv 0 \pmod 3$. Hence the indegree of u_2 in D' is at least 5. Thus, we may obtain a decomposition \mathcal{C}' as above, with u_2 playing the role of v . ■

The following corollary is immediate.

Corollary 8. *Let G be a connected graph with $\delta(G) \geq 6$. Then $L^2(G)$ contains a 2-factor with $\lfloor n_2/3 \rfloor$ cycles.*

It is possible, depending on δ_1 , that many of the stars constructed in Lemma 7 will have common centers. We can therefore construct larger stars, and

hence different types of 2-factors, by joining stars with common center vertices.

Lemma 9. *If G has an edge decomposition into $\lfloor n_1/3 \rfloor$ stars, then $L(G)$ contains a 2-factor with k cycles, for $n_0 \leq k \leq \lfloor n_1/3 \rfloor$.*

Proof. Let \mathcal{C} be the edge decomposition of G into $\lfloor n_1/3 \rfloor$ stars. Let \mathcal{C}' be a subset of the stars in \mathcal{C} such that for every vertex v in G , at most one star of \mathcal{C}' is centered at v , and such that \mathcal{C}' is a $|\mathcal{C}'|$ -system that dominates. Note that $|\mathcal{C}'| \leq n_0$. For any $n_0 \leq k \leq \lfloor n_1/3 \rfloor$, let \mathcal{E} be any subset of $\mathcal{C} \setminus \mathcal{C}'$ such that $|\mathcal{E}| = k - |\mathcal{C}'|$. Then $\mathcal{C}' \cup \mathcal{E}$ is a k -system that dominates, and hence by Theorem 4, $L(G)$ contains a 2-factor with k cycles. ■

Lemma 10. *Let G be a graph containing a 2-factor with k cycles and containing an edge decomposition into $\lfloor n_1/3 \rfloor$ stars. Then $L(G)$ contains a 2-factor with j cycles, for $k \leq j \leq k + \lfloor n_1/3 \rfloor - n_0$.*

Proof. Let \mathcal{F} be a 2-factor in G with k components, and let \mathcal{C} be an edge decomposition of G into $\lfloor n_1/3 \rfloor$ stars. Let \mathcal{C}' be the set of stars in \mathcal{C} which are edge-disjoint from \mathcal{F} . Since \mathcal{F} has n_0 edges, $|\mathcal{C}'| \geq \lfloor n_1/3 \rfloor - n_0$. For any $k \leq j \leq k + \lfloor n_1/3 \rfloor - n_0$, let \mathcal{E} be any subset of \mathcal{C}' such that $|\mathcal{E}| = j - k$. Then $\mathcal{F} \cup \mathcal{E}$ is a j -system that dominates, and hence by Theorem 4, $L(G)$ contains a 2-factor with j cycles. ■

Definition. A graph G is *2-factor spectrum complete* if there exists a 2-factor in G with exactly k cycles for every $1 \leq k \leq \lfloor n_0/3 \rfloor$.

We are now able to show the following.

Proposition 11. *Let G be a prolific graph. If $\delta(G) \geq 6$, $\delta_1 \geq 12$, and $L(G)$ is hamiltonian, then $L^2(G)$ is 2-factor spectrum complete.*

Proof. Since $\delta(G) \geq 6$, Lemma 7 implies that $L(G)$ has an edge decomposition into $\lfloor n_1/3 \rfloor$ stars. By Lemma 9, $L^2(G)$ contains a 2-factor with k -cycles for $n_1 \leq k \leq \lfloor n_2/3 \rfloor$. By Lemma 10 and using the hamiltonian cycle in $L(G)$, $L^2(G)$ contains a 2-factor with k -cycles, for $1 \leq k \leq 1 + \lfloor n_2/3 \rfloor - n_1$. Since $\delta_1 \geq 12$, then $n_2 \geq 6n_1$, and therefore $1 + \lfloor n_2/3 \rfloor - n_1 \geq n_1$. Thus, $L^2(G)$ contains a 2-factor with k -cycles for $1 \leq k \leq \lfloor n_2/3 \rfloor$, and hence $L^2(G)$ is 2-factor spectrum complete. ■

Note that if $L^2(G)$ has a 2-factor with $\lfloor n_2/3 \rfloor$ cycles, and we assume that $\delta_0 \geq 4$, then $\lfloor n_2/3 \rfloor + \lfloor n_3/3 \rfloor - n_2$ just abuts with n_2 . Therefore any 2-factors in $L^2(G)$ with fewer than $\lfloor n_2/3 \rfloor$ cycles do not allow us to generate any new types of 2-factors. If we wish to strengthen Lemma 11, we would need a new method by which we could construct 2-factor with many cycles.

Corollary 12. *Let G be a prolific graph. If $\delta(G) \geq 5$, then $L^3(G)$ is 2-factor spectrum complete. If $\delta(G) \geq 4$, then $L^4(G)$ is 2-factor spectrum complete. If $\delta(G) \geq 3$, then $L^5(G)$ is 2-factor spectrum complete.*

Proof. The results follow from Proposition 11, Theorem 2, and the fact that $\delta(L(G)) \geq 2\delta(G) - 2$. ■

The following is a well-known fact about iterated line graphs (see, for example [7]).

Lemma 13. *For every prolific graph G , there exists an integer T_δ such that for all $t \geq T_\delta$, $\delta_t \geq 3$.*

Corollary 14. *For every prolific graph G , there exists an integer T_{tfsc} such that for all $t \geq T_{\text{tfsc}}$, $L^t(G)$ is 2-factor spectrum complete.*

Proof. By Lemma 13, there exists an integer T_δ such that $\delta_t = 3$ for all $t \geq T_\delta$. Applying Corollary 12, we have that $L^t(G)$ is 2-factor spectrum complete for all $t \geq T_\delta + 5$. ■

We strongly feel that the following is true, although we are unable to verify the conjecture.

Conjecture 15. *If G is a prolific graph with $\delta(G) \geq 3$, then $L^2(G)$ is 2-factor spectrum complete.*

4. 2-FACTORS WITH SPECIFIED CYCLE LENGTHS

We have shown that, given a sufficient minimum degree, $L^2(G)$ contains a 2-factor of every type. However we did not explicitly discuss the lengths of the cycles in many of these 2-factors. We will now give conditions that allow us to prescribe not only the number of cycles in a 2-factor, but also the lengths of these cycles.

4.1. Pancyclic and cycle complementary line graphs

A simple graph G is *pancyclic* if G contains a cycle of length k for every $3 \leq k \leq n_0$. Furthermore, G is said to be *vertex pancyclic* if every vertex in G lies on a cycle of each length. A simple graph G is *cycle complementary* if for any k between 3 and $n_0 - 3$, G contains a 2-factor with exactly two cycles of length k and $n_0 - k$.

Samodivkin [12] has determined the minimum t such that $L^t(G)$ is vertex pancyclic. The following can be shown using a technique similar to the proof of Theorem 3.

Proposition 16. *Let G be a connected graph. If $\delta(G) \geq 3$, then $L^2(G)$ is vertex pancyclic.*

Proof. We will show that each edge e in $L(G)$ lies on a circuit dominating k edges for any $3 \leq k \leq n_2$. Suppose that e lies in a maximal clique K of order t , then e lies on a cycles of length $k \leq t$ in $L(G)$. If this cycle, C , is hamiltonian, it dominates each edge in K , and hence implies the existence of cycles of length up to $\binom{t}{2}$ in $L^2(G)$. Choose any other clique $K' \cong K_{t_1}$ in $L(G)$ that shares a vertex v with K . Since v dominates $t_1 - 1$ edges in K' , we are able to identify cycles in $L^2(G)$ of length up to $\binom{t}{2} + t_1 - 1$ through the vertex corresponding to e . Choose any hamiltonian cycle in K' . This cycle forms a circuit with C that dominates between $\binom{t}{2} + t_1$ and $\binom{t}{2} + \binom{t_1}{2}$ edges in $L(G)$. We are able to extend this circuit by continuing to incorporate additional maximal cliques in $L(G)$. As $L(G)$ is connected, we may continue this process until we have dominated a sufficient number of edges. ■

Using a technique similar to that used in Proposition 16, we are able to show the following.

Proposition 17. *If $\delta(G) \geq 6$, then $L^2(G)$ is cycle complementary.*

Proof. Let k be an integer between 3 and $\lfloor \frac{n_2(G)}{2} \rfloor$. By Theorem 5 and the fact that $\delta(G) \geq 6$, each clique in $L(G)$ contains at least two edge-disjoint hamiltonian cycles. Additionally, it is not difficult to see that if $t \geq 6$, K_t contains a hamiltonian cycle and a copy of C_ℓ that are edge disjoint, for any $\ell \leq t$. Thus, if $k < 2n_1$ we can easily join hamiltonian cycles from each clique to form circuit C_1 that spans $L(G)$ and has $2n_1$ edges. We can also construct a circuit C_2 with k edges that is edge disjoint from C_1 using at

most an additional hamiltonian cycle from each clique. If C_2 is considered to dominate $E(C_2)$ and C_1 is considered to dominate all remaining edges in $L(G)$, we will obtain the desired pair of cycles in $L^2(G)$.

If $k \geq 2n_1$, we will construct a 2-system that dominates as follows. By theorem 5 and the fact that $\delta(G) \geq 6$ each clique in $L(G)$ contains at least two edge-disjoint hamiltonian cycles. We can form two edge-disjoint spanning circuits C_1 and C_2 in $L(G)$ by joining a hamiltonian cycle from each clique. As each of these circuits dominates each remaining edge in the graph, we will associate $k - 2n_1$ of these edges with C_1 and the remaining edges with C_2 , yielding the desired complementary cycles in $L^2(G)$. ■

4.2. The EZ_k property

The problem of determining when a graph has a 2-factor of a given type has been of considerable interest. One of the most well-known conjectures in this area is due to El-Zahar [3], who verified the conjecture when $t = 2$.

Conjecture 18 (El-Zahar, 1984). Let G be a graph on $n = \ell_1 + \ell_2 + \dots + \ell_t$ vertices. If

$$\delta(G) \geq \left\lceil \frac{\ell_1}{2} \right\rceil + \left\lceil \frac{\ell_2}{2} \right\rceil + \dots + \left\lceil \frac{\ell_t}{2} \right\rceil,$$

then G contains a 2-factor with cycles of length $\ell_1, \ell_2, \dots, \ell_t$.

This motivates the following definition.

Definition. Let k be an integer greater than 1. A simple graph G has the EZ_k property if G contains a 2-factor with k cycles of lengths $\ell_1, \ell_2, \dots, \ell_k$, for all possible sets of lengths ℓ_i where $3 \leq \ell_i \leq n_0$ for all $1 \leq i \leq k$ and $\sum_{i=1}^k \ell_i = n_0$.

We will prove that if G has sufficiently high minimum degree, then $L^2(G)$ has the EZ_k property. We begin with a result of Bondy and a lemma.

Theorem 19 (Bondy [1]). *Let G be a simple graph. If $\deg(u) + \deg(v) \geq n_0$ for all non-adjacent pairs of vertices u and v , then G is either pancyclic or the complete bipartite graph $K_{n/2, n/2}$. If G is $K_{n/2, n/2}$, then equality of the degree sum holds for all pairs of non-adjacent vertices.*

Lemma 20. *Let $n \geq 4k - 2$, and \mathcal{H} be a collection of $k - 1$ edge-disjoint subgraphs of K_n , each with maximum degree at most 2. Then $K_n - \cup \mathcal{H}$ is pancyclic.*

Proof. Let u and v be two nonadjacent vertices in $K_n - \cup \mathcal{H}$. Note that $\deg_{K_n}(u) = n - 1$ and that $\deg_{\cup \mathcal{H}}(u) \leq 2(k - 1)$. Then

$$\begin{aligned} \deg_{K_n - \cup \mathcal{H}}(u) + \deg_{K_n - \cup \mathcal{H}}(v) &\geq 2[(n - 1) - 2(k - 1)] \\ &= n + [n - (4k - 2)] \\ &\geq n, \end{aligned}$$

and by Theorem 19, $K_n - \cup \mathcal{H}$ is pancyclic. ■

Theorem 21. *If G is a prolific graph with $\delta(G) \geq 4k - 2$, then $L^2(G)$ has the EZ_k property.*

Proof. Order the vertices v_1, v_2, \dots, v_{n_0} of G such that for all $1 \leq r \leq n_0$, the subgraph $G[v_1, v_2, \dots, v_r]$ induced by the vertices v_1, v_2, \dots, v_r is connected. Let $\ell_1, \ell_2, \dots, \ell_k$ be specified cycle lengths for the 2-factor with k cycles in $L^2(G)$. Note that $\ell_k = n_2 - \sum_{i=1}^{k-1} \ell_i$. For each $1 \leq i \leq k - 1$, let s_i denote the minimum s such that $\sum_{r=1}^s \deg_G(v_r) \geq \ell_i$. Let Q_r denote the clique in $L(G)$ corresponding to the edges incident on v_r in G . For each $1 \leq i \leq k$ and $r < s_i$, let $H_{i,r}$ denote a hamiltonian cycle in Q_r such that $H_{i,r}$ is edge-disjoint from $H_{j,r}$ for $j < i$. For $r = s_i$, let $H_{i,r}$ denote a cycle of length $\ell_i - \sum_{r=1}^{s_i-1} \deg_G(v_r)$ in Q_r such that $H_{i,r}$ is edge-disjoint from $H_{j,r}$ for $j < i$. Such edge-disjoint cycles can be chosen in Q_r by Lemma 20. For $r > s_i$, let $H_{i,r}$ be an empty graph. For each i , let $\mathcal{H}_i = \bigcup_{r=1}^{n_0} H_{i,r}$. Partition the edges of $L(G) - \cup \mathcal{H}_i$ into sets $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_k$ such that $|E(\mathcal{H}_i) \cup \mathcal{E}_i| = \ell_i$. Then $\{\mathcal{H}_i \cup \mathcal{E}_i : 1 \leq i \leq k\}$ is a k -system that dominates $L(G)$. Hence by Lemma 4, $L^2(G)$ contains a 2-factor \mathcal{F} with k cycles. Moreover, $\mathcal{H}_i \cup \mathcal{E}_i$ corresponds to a cycle in \mathcal{F} of length ℓ_i . Thus \mathcal{F} is the desired 2-factor. ■

Throughout the preceding sections, we have repeatedly utilized the notion of adjoining cycles from maximal cliques to construct dominating circuits or components of k -systems that dominate. It is possible, with more in-depth analysis, to construct these subgraphs in a way that permits additional structure. For instance, note that if a clique is sufficiently large, then it will contain a hamiltonian cycle after the removal of a moderately large collection of edge-disjoint triangles. Choosing these triangles to contain specified edges gives rise to results akin to the following.

Proposition 22. *Given t vertices in $L^2(G)$, with $\delta(G) \geq 2t+2$, there exists a 2-factor with t cycles such that each vertex lies in a different cycle.*

5. A CHARACTERIZATION OF ITERATED LINE GRAPHS CONTAINING A 2-FACTOR

In this section, given a graph G , we determine exactly those values of k for which $L^k(G)$ has a 2-factor.

In this section, we will also consider graphs with multiple edges. We first introduce some additional notation that we will use in this section. For any $0 \leq i \leq \Delta(G)$, let $V_i(G)$ denote the set of vertices of G having degree i . For any subgraph H of a graph G , we will let $\bar{E}(H)$ denote those edges in G adjacent to at least one vertex in H . Given any subset S of $V(G)$, we shall write $G[S]$ to denote the subgraph of G induced by S . If G_1 and G_2 are two subgraphs of G , we will define $d_G(G_1, G_2)$, the distance between G_1 and G_2 in G , to be the minimum of the distances $d_G(x_1, x_2)$ where x_i is in G_i . If x is in $V(G')$, for some subgraph g' of G we let $d_{G'}(x)$ denote the degree of x in G' .

A *branch* b in G is a path of length at least 2 with internal vertices that lie in $V_2(G)$ and end-vertices that do not lie in $V_2(G)$. We will let $B(G)$ denote the collection of branches in G , and we will let $B_1(G)$ denote those branches having at least one end-vertex in $V_1(G)$. For any subgraph H of G , we let $B_H(G)$ denote those branches contained wholly within H .

5.1. $EU_k(G)$ and $F_k(G)$

Theorem 1 was extended by Liu and Xiong [11] as follows.

Definition. Let G be a graph, and let $EU_k(G)$ denote those subgraphs H of G with the following properties.

- (i) $d_H(x) \equiv 0 \pmod{2}$ for every $x \in V(H)$.
- (ii) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$.
- (iii) $d_G(H_1, H - H_1) \leq k - 1$ for any subgraph H_1 of H .
- (iv) $|E(b)| \leq k + 1$ for every branch b with $E(H) \cap E(b) = \emptyset$.
- (v) $|E(b)| \leq k$ for every branch b in $B_1(G)$.

With this definition of $EU_k(G)$ we present the following.

Theorem 23 (Xiong [11]). *For $k \geq 2$, $L^k(G)$ is hamiltonian if and only if $EU_k(G) \neq \emptyset$.*

We now wish to characterize when $L^k(G)$ has a 2-factor, and we proceed in a manner similar to [11].

Definition. Let $F_k(G)$ denote the subgraphs H of a graph G that satisfy the following conditions.

- (I) $d_H(x) \equiv 0 \pmod{2}$ for every $x \in V(H)$.
- (II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$.
- (III) $d_G(H_1, H - H_1) \leq k + 1$ for any subgraph H_1 of H .
- (IV) $|E(b)| \leq k + 1$ for every branch b with $E(b) \cap E(H) = \emptyset$.
- (V) $|E(b)| \leq k$ for every branch b in $B_1(G)$.

We will prove the following theorem.

Theorem 24. *Let G be a connected graph with at least three edges. For any $k \geq 2$, $L^k(G)$ has a 2-factor if and only if $F_k(G) \neq \emptyset$.*

Intuitively, we are searching for subgraphs which will converge into a t -system that dominates for some t as we iterate the line graph. The conditions describing the sets $EU_k(G)$ and $F_k(G)$ differ only in the condition describing the distance between components of their elements (conditions (iii) and (III)). This is because we do not require a t -system that dominates to be connected, unlike a dominating circuit. Theorem 24 follows immediately from induction on k using the following two theorems.

Theorem 25. *Let G be a connected graph with at least three edges and let $k \geq 1$ be an integer. Then $F_k(L(G)) \neq \emptyset$ if and only if $F_{k+1}(G) \neq \emptyset$.*

Theorem 26. *Let G be a connected graph with at least three edges. Then $L^2(G)$ has a 2-factor if and only if $F_2(G) \neq \emptyset$.*

Before we begin proving the above results, several lemmas are necessary. The first is a well-known fact about line graphs.

Lemma 27. *For any graph G , $L(G)$ does not contain an induced copy of $K_{1,3}$.*

Lemma 28 (Xiong [11]). *If H is a subgraph of G in $F_k(G)$ having a minimum number of components, then there exist no multiple edges in $\bar{E}(H_1) \cap \bar{E}(H_2)$ for any two components H_1 and H_2 of H .*

Lemma 29 (Xiong [11]). *Let $b = u_1u_2 \dots u_s$ ($s \geq 3$) be a path of G and let $e_i = u_iu_{i+1}$. Then b is in $B(G)$ if and only if $b' = e_1e_2 \dots e_{s-1}$ is in $B(L(G))$.*

We say a subgraph H of G is an *eulerian subgraph* if it is a circuit in G containing a cycle of length at least three.

Lemma 30 (Xiong [11]). *Let G be a connected graph and H be an eulerian subgraph of $L(G)$. Then there exists a subgraph C of G that satisfies the following:*

- (1) $d_C(x) \equiv 0 \pmod{2}$ for every $x \in V(C)$.
- (2) Every isolated vertex of C has degree at least 3 in G .
- (3) For any two components C^0, C^{00} of C , there exists a sequence of components $C^0 = C^1, C^2, \dots, C^s = C^{00}$ of C such that $d_G(C_i, C_{i+1}) \leq 1$ for all $1 \leq i \leq s - 1$.
- (4) $L(\bar{E}(C))$ contains H , and $V(H)$ contains all elements of $E(C)$.

We now prove our two key theorems.

Proof of Theorem 25. Suppose that $F_{k+1}(G) \neq \emptyset$, and choose an $H \in F_{k+1}(G)$ with a minimum number of components, C_1, C_2, \dots, C_t . Since each vertex in C_i has even degree, we can find a cycle C'_i in $L(G)$ that spans $\bar{E}(C_i)$. Let

$$H' = \bigcup_{i=1}^t C'_i.$$

We now show that H' is in $F_k(L(G))$.

Since H is in $F_{k+1}(G)$, we have that

$$\bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H).$$

As H' clearly has no isolated vertices,

$$V(H') = \bigcup_{j=1}^t \bar{E}(C_j)$$

we can see that H' satisfies (II).

Since $d_G(C_i, C_j) \geq 1$, Lemma 28 yields that any C'_i and C'_j are edge disjoint. Hence H' satisfies (I).

For any $T \subset \{1, \dots, t\}$, our choice of H assures that $d_G(H - \bigcup_T C_i, \bigcup_T C_i) \leq k + 2$. If P is a shortest path between $\bigcup_T C_i$ and $H - \bigcup_T C_i$ having at most $k + 2$ edges, then clearly $L(P)$ is a path between C'_i and $H' - C'_i$ having at most $k + 1$ edges. Thus H' satisfies (III).

We can immediately see that H' satisfies (IV) and (V) using Lemma 29. Conversely, assume that $F_k(L(G))$ is not empty, and choose any H therein having a minimum number of isolated vertices. We then claim that H has no isolated vertices. Indeed, any isolated vertex e in $V(H)$ has degree at least three in $L(G)$. Since $L(G)$ is claw-free, it must be that e lies on some triangle ee_1e_2 in $L(G)$. Construct H_0 as follows.

$$\begin{cases} H + \{ee_1, e_1e_2, e_2e\} & \text{if } e_1e_2 \notin E(H), \\ H + \{ee_1, ee_2\} - \{e_1e_2\} & \text{if } e_1e_2 \in H. \end{cases}$$

Clearly H_0 is in $F_k(L(G))$ and has fewer isolated vertices than H , verifying the claim.

Let H_1, H_2, \dots, H_m be the components of H , each of which is an eulerian subgraph of $L(G)$. Thus, by Lemma 30, for each H_i there exists a subgraph C_i of G satisfying the four given conditions. Set

$$C = \left(\bigcup_{i=3}^{\Delta(G)} V_i(G) \right) \cup \left(\bigcup_{i=1}^m C_i \right).$$

We now show that C is in $F_{k+1}(G)$.

Since each of the H_i are vertex disjoint, and $V(H_i)$ contains C_i for all i , each C_i is edge-disjoint. Thus, by property (1) in Lemma 30, C is an even subgraph. Each C_i also satisfies (2), and thus (II) holds trivially.

Since $\bigcup_{i=3}^{\Delta(L(G))} V_i(L(G)) \subseteq V(H)$ and H is a member of $F_k(L(G))$, $d_G(x, G[V(C) \setminus \{x\}]) \leq k + 2$ for every vertex x having degree zero in C .

Now choose some subset T of $\{1, \dots, m\}$ and note that $d_{L(G)}(H - \bigcup_T H_i, \bigcup_T H_i) \leq k + 1$ by our choice of H . Let $P = e_1 e_2 \dots e_s$ be a shortest path from $\bigcup_T H_i$ to $H - \bigcup_T H_i$, where $e_1 \in V(\bigcup_T H_i) \subseteq \bar{E}(\bigcup_T C_i)$ and $e_s \in V(H - \bigcup_T H_i) \subseteq \bar{E}(C - \bigcup_T C_i)$ and $s \leq k + 2$. As $G[e_1, \dots, e_s]$ is connected,

$$d_G\left(\bigcup_T C_i, C - \bigcup_T C_i\right) \leq |E(G[\{e_1, \dots, e_s\}])| \leq s \leq k + 2,$$

so C satisfies (III). As H satisfies (III) to (V), Lemma 29 yields that C satisfies (IV) and (V). Thus $C \in F_{k+1}(G)$. ■

Proof of Theorem 26. We suppose that $F_2(G) \neq \emptyset$ and choose an H in $F_2(G)$ having a minimum number of components, H_1, \dots, H_t . Since H is in $F_2(G)$, $|\bar{E}(H_i)| \geq 3$ for any i , so there exists a cycle C_i in $L(G)$ such that $V(C_i) = \bar{E}(H_i)$. Let

$$C = \bigcup_{i=1}^t C_i.$$

By Lemma 28, C_1, \dots, C_t are edge-disjoint cycles in $L(G)$ and hence C is even. Since $d_G(H_i, H - H_i) \leq 3$, we know that $d_{L(G)}(C_i, C - C_i) \leq 2$ for any i . Using Lemma 29, any branch in $B(L(G)) \setminus B_H(L(G))$ has length at most 2 and any branch in $B_1(L(G))$ has length at most 1. As H satisfies (II), we see that

$$\bigcup_{i=3}^{\Delta(L(G))} V_i(L(G)) \subseteq V(C).$$

Thus, $\bar{E}(C) = E(L(G))$ which implies that $L^2(G)$ has a 2-factor.

Conversely, suppose that $L^2(G)$ has a 2-factor. By Theorem 4, $L(G)$ has a k -system that dominates for some $k \geq 1$. Let S be such a system having a minimum number of stars and amongst such k -systems that dominate, a maximum number of vertices of degree three or more.

First we note that S includes no stars. Indeed, let S_i be any star in S , having center vertex e and pendant vertices including e_a, e_b and e_c . Since any line graph is claw-free, we may assume, without loss of generality, that $e_a e_b$ is an edge in $L(G)$. We discard S_i and modify S . If $e_a e_b$ is in some circuit in S , say C , we delete this edge from C and add the edges $e_a e$ and $e_b e$ creating a new circuit. If $e_a e_b$ lies in some star S_j , simply delete S_j from

S and add the triangle $e_a e_b e$ to S . Finally, in any other case, simply add triangle $e_a e_b e$ to S (which adds a new circuit to our system). In any case, it is easy to see that we have created a k -system that dominates in $L(G)$ having fewer stars than S , a contradiction.

One can now also see that S must contain all vertices of degree at least three in $L(G)$. If not, we may again use the fact that $L(G)$ is claw-free to reach a contradiction.

Let the components of S be S_1, \dots, S_k . As each S_i is a circuit, Lemma 30 gives that there are subgraphs H'_i in G for each S_i satisfying properties (1) to (4). Let $H = \bigcup_{i=1}^k H'_i$.

Claim 31. $d_G(x, H) \leq 1$ for any $x \in \bigcup_{i=3}^{\Delta(G)} V_i(G)$.

Proof. If G is either a star or a cycle, then the claim is trivial. If not, then for any vertex x having degree at least three, there is some edge e_x adjacent to x which has degree at least three as a vertex in $L(G)$. Now, via property (4) and the fact that C contains all vertices of degree three or more in $L(G)$, we can see that $e_x \in V(S) \subseteq \bar{E}(H)$. This implies that e_x has an endvertex in H , completing the proof of the claim.

Finally, we prove that

$$H' = H \cup \left(\bigcup_{i=3}^{\Delta(G)} V_i(G) \right)$$

is in $F_2(G)$. Let H_i be a component of H' . As $d_{L(G)}(S_i, S - S_i) \leq 2$ for any component S_i of S , we can see that $d_G(H' - H_i, H_i) \leq 3$ satisfying (III). It follows immediately from Lemma 29 and the fact that $\bar{E}(S) = E(L(G))$ that (IV) and (V) hold for H' . Thus $H' \in F_2(G)$. ■

The proof of Theorem 26 is now complete. ■

5.2. 2-factors with a given number of cycles

We now examine 2-factors with a given number of cycles in iterated line graphs. To this end, we restrict our interest to those elements of $F_k(G)$ having a particular decomposition.

Definition. Let $F_k^{(t)}(G)$ denote those elements H of $F_k(G)$ such that H can be partitioned into edge-disjoint subgraphs H_1, \dots, H_t such that the following hold.

- (a) $d_{H_i}(x) \equiv 0 \pmod{2}$ for all i and all $x \in V(H_i)$.
 (b) $\bar{E}(H)$ can be partitioned into disjoint sets E_1, \dots, E_t of at least three edges such that for each i ,

$$E(H_i) \subseteq E_i \subseteq \bar{E}(H_i).$$

The above definition allows us to state a useful theorem.

Theorem 32. *For any $k \geq 2$ and $t \geq 2$, if $F_k^{(t)}(G) \neq \emptyset$, then $L^k(G)$ has a 2-factor with exactly t cycles.*

Proof. We proceed as above by proving two claims.

Claim 33. For any $k \geq 2$, $F_{k+1}^{(t)}(G) \neq \emptyset \Rightarrow F_k^{(t)}(L(G)) \neq \emptyset$.

Proof. Let H be in $F_{k+1}^{(t)}(G)$ and let E_1, \dots, E_t be as given in the definition above. Then, as H_i is even for all i , there exist cycles C_i , $1 \leq i \leq t$, in $L(G)$ such that $V(C_i) = E_i$. As in the proof of Theorem 25, the union of these C_i form a subgraph H' in $F_k(L(G))$.

It remains to show that H' is in $F_k^{(t)}(L(G))$. Each C_i is an even subgraph so we take $H'_i = C_i$ for each i . Thus we satisfy (a), and since each H'_i has at least three edges we may distribute those edges in

$$\bigcup_{i=1}^t \bar{E}(H'_i) \setminus E(H'_i)$$

any way we wish to construct sets E'_1, \dots, E'_t that satisfy (b). ■

Claim 34. If $F_2^{(t)}(G) \neq \emptyset$ then $L^2(G)$ has a 2-factor with exactly t cycles.

Proof. Let H be in $F_2^{(t)}(G)$ and let H_i, E_i be as in (a) and (b) respectively. As in the previous claim, we can find cycles C_1, \dots, C_t in $L(G)$ such that $V(C_i) = E_i$. It is easy to see that these cycles have distance at most 2 and thus comprise a t -system that dominates in $L(G)$, implying that $L^2(G)$ has a 2-factor with exactly t cycles, as desired. ■

The proof of Theorem 32 now follows by induction. ■

It is important to note that the converse of Theorem 32 is not true. Consider any prolific graph. Then there exists some $k \geq 2$ such that $L^k(G)$ has a 2-factor and minimum degree at least 5. Let F be such a 2-factor, and assume that F has exactly t cycles. Clearly, F forms a t -system that dominates in $L^k(G)$. However, if we let v be a vertex in $L^k(G)$, our minimum degree assumption assures us that there are at least three edges adjacent to v that do not lie on any cycle in F . Thus, F , together with the star formed by these edges comprise a $(t + 1)$ -system that dominates. Hence, for any $t > 0$ there exists some k such that for any $k' > k$, $L^{k'}(G)$ has a 2-factor with at least t cycles. However, if the converse of Theorem 32 was true, there would then be an arbitrarily large number of edge-disjoint sets in G , each containing at least 3 edges. This is clearly impossible, and indicates why we do not use Theorem 32 to prove, for instance, Corollary 14.

6. CONCLUSION

The results of this paper show that iterated line graphs have very nice cycle-structural properties, including hamiltonicity and containing 2-factors with specified features. Here, however, we present our first negative result: a cycle-structural property that iterated line graphs do not have, regardless of how many iterations.

Definition. For any simple graph G , the power G^ℓ of G is the simple graph with vertex set $V(G)$ and where u and v are adjacent in G^ℓ if and only if $\text{dist}_G(u, v) \leq \ell$. The square of G is G^2 , and the cube of G is G^3 .

Using techniques from [8], it can be shown that the iterated line graph of a prolific graph contains the square of a hamiltonian cycle, for a sufficiently large iteration. However, the *cube* of a hamiltonian cycle *never* exists in iterated line graphs.

Proposition 35. *For any simple graph G that is not isomorphic to a triangle or a star, $L(G)$ does not contain the cube of a hamiltonian cycle.*

Proof. Suppose that $L(G)$ contains the cube H^3 of a hamiltonian cycle H . Let x, y, z , and w be consecutive vertices along H such that the common vertex of x and y in G is different than the common vertex of y and z . Since x and z are adjacent in H^3 , x and z share a distinct vertex in G , and hence

x , y , and z are the edges of a triangle in G . Since w is adjacent to all three of x , y , and z in H^3 , w must share a vertex with all three in G . However, this is clearly impossible. ■

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