

## EFFICIENT $(j, k)$ -DOMINATION

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### Abstract

A dominating set  $S$  of a graph  $G$  is called *efficient* if  $|N[v] \cap S| = 1$  for every vertex  $v \in V(G)$ . That is, a dominating set  $S$  is efficient if and only if every vertex is dominated exactly once. In this paper, we investigate efficient multiple domination. There are several types of multiple domination defined in the literature:  $k$ -tuple domination,  $\{k\}$ -domination, and  $k$ -domination. We investigate efficient versions of the first two as well as a new type of multiple domination.

**Keywords:** efficient domination, multiple domination.

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### 1. INTRODUCTION

A vertex  $v$  in  $V(G)$  has open neighborhood  $N(v)$  in the graph  $G = (V, E)$  consisting of the set of vertices adjacent to  $v$ , and the closed neighborhood of  $v$  is  $N[v] = N(v) \cup \{v\}$ . Vertex  $v$  is said to dominate each vertex in  $N[v]$ , including itself, and  $S \subseteq V(G)$  is a dominating set if  $\cup_{s \in S} N[s] = V(G)$ , that is, if every vertex in  $V(G)$  is dominated by at least one vertex in  $S$ . A dominating set  $S$  of a graph  $G$  with the smallest cardinality is called

a minimum dominating set and its size, the domination number, is denoted by  $\gamma(G)$ .

If every vertex is dominated exactly once by  $S \subseteq V(G)$ , that is, for every vertex  $w \in V(G)$  we have  $|N[w] \cap S| = 1$ , then  $S$  is called a perfect code in Biggs [5] or an efficient dominating set in Bange, Barkauskas, and Slater [1, 2, 3, 4]. Most graphs do not have an efficient dominating set (for example, the four-cycle), and Bange *et al.* [1, 2] introduced the following efficiency measure for a graph  $G$ . The *efficient domination number* of a graph, denoted  $F(G)$ , is the maximum number of vertices that can be dominated by a set  $S$  that dominates each vertex at most once. A graph  $G$  of order  $n = |V(G)|$  has an efficient dominating set if and only if  $F(G) = n$ . For the tree  $T_1$  of order  $n = 5$  in Figure 1, we have  $F(T_1) = 4$ . Note that the cardinality of  $S$  does not matter, so we can use  $S_1 = \{v_3\}$  or  $S_2 = \{v_1, v_4\}$  to achieve  $F(T_1) = 4$ .

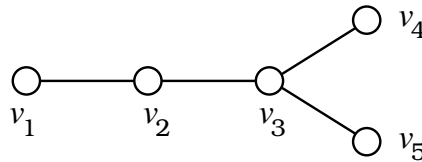


Figure 1. Tree  $T_1$  with  $F(T_1) = 4$

A vertex  $v$  of degree  $\deg v = |N(v)|$  dominates  $|N[v]| = 1 + \deg v$  vertices. Grinstead and Slater [12] defined the *influence* of a set of vertices  $S$  to be  $I(S) = \sum_{s \in S} (1 + \deg s)$ , the total amount of domination being done by  $S$ . Because  $S$  does not dominate any vertex more than once if and only if any two vertices in  $S$  are at distance at least three (that is,  $S$  is a packing), we have  $F(G) = \max\{I(S) : S \text{ is a packing}\}$ . On the other hand, if every vertex must be dominated at least once, the *redundance*  $R(G)$  defined in [12] equals the minimum possible amount of domination possible,  $R(G) = \min\{I(S) : S \text{ is a dominating set}\}$ .

As introduced by Harary and Haynes [13], a  $k$ -tuple dominating set  $D$  is a set  $D \subseteq V(G)$  for which  $|N[w] \cap D| \geq k$  for every  $w \in V(G)$ . Note that we must have the minimum degree  $\delta(G) \geq k - 1$  for a  $k$ -tuple dominating set to exist. The  $k$ -tuple domination number  $\gamma_{\times k}$  is the minimum cardinality of a  $k$ -tuple dominating set. We define a graph  $G$  to be *efficiently  $k$ -tuple dominatable* if it has a vertex set  $D$  with  $|N[w] \cap D| = k$  for every  $w \in V(G)$ . The  $k$ -tuple *efficient domination number* is  $F_{1,k}(G) = \max\{I(S) :$

$|N[w] \cap S| \leq k$  for all  $w \in V(G)$ }, the maximum amount of domination done by a set  $S$  that dominates no vertex more than  $k$  times, and the  $k$ -tuple redundancy number is  $R_{1,k}(G) = \min\{I(D) : D \text{ is a } k\text{-tuple dominating set}\}$ .

For  $k = 2$ , efficient double-dominatable graphs have been studied by Chellali, Khelladi, and Maffray [6]. A set  $S$  is a double dominating set if  $|N[u] \cap S| \geq 2$  for each  $u \in V(G)$ . A graph  $G$  has an efficient double-dominating set  $S$  if and only if for each  $u \in V(G)$  we have  $|N[u] \cap S| = 2$ . We define the *efficient double-domination number*  $F_{1,2}(G) = \max\{I(S) : |N[u] \cap S| \leq 2 \text{ for each } u \in V(G)\}$ , the maximum possible amount of domination done by a set  $S$  that does not dominate any vertex more than twice. Likewise, we define the 2-redundance number  $R_{1,2}(G) = \min\{I(S) : S \text{ is a double dominating set}\}$ .

One can consider the characteristic function  $f_S$  associated with each  $S \subseteq V(G)$ , where  $f_S(v) = 1$  if  $v \in S$  and  $f_S(v) = 0$  if  $v \in V(G) - S$ . For example, Farber [8] investigated the problem of determining when the linear programming formulation of the domination problem would provide an integer solution. Let  $f : V(G) \rightarrow [0, 1]$ , then the weight of  $f$  is  $w(f) = f(V(G)) = \sum_{v \in V(G)} f(v)$ , and  $f$  is a fractional dominating function if  $f(N[w]) = \sum_{x \in N[w]} f(x) \geq 1$  for every  $w \in V(G)$ . The fractional domination number  $\gamma_f(G)$  is the minimum weight of a fractional dominating function. For another example (see [7]), a function  $g : V(G) \rightarrow \{0, 1, 2, \dots, k\}$  is called a  $\{k\}$ -dominating function if for every  $v \in V(G)$  we have  $g(N[v]) \geq k$ . The minimum weight of such a function is denoted by  $\gamma_{\{k\}}(G)$ . Note that  $\{k\}$ -dominating functions dominate each vertex at least  $k$  times, while taking on integer values from 0 to  $k$ , as opposed to  $k$ -dominating functions, which take only the values of 0 or 1. In  $k$ -domination, introduced by Fink and Jacobson [9], only the vertices  $w \in V(G) - D$  must be dominated at least  $k$  times, however, in this paper, we do not focus on this type of domination and instead focus on all vertices being dominated  $k$  times.

Bange, Barkauskas, Host and Slater [4] and Goddard and Henning [10] extended the study of dominating functions to functions with values in an arbitrary subset  $Y$  of the real numbers  $\mathbb{R}$ . In general, if  $f : V(G) \rightarrow Y \subseteq \mathbb{R}$  then the weight of  $f$  is  $w(f) = \sum_{v \in V(G)} f(v)$ , and  $f$  is called a  $Y$ -valued dominating function if  $\sum_{x \in N[w]} f(x) \geq 1$  for every  $w \in V(G)$ . As defined by Bange, *et al.* [4], a dominating function  $f : V(G) \rightarrow Y$  is an efficient  $Y$ -valued dominating function if  $\sum_{v \in N[w]} f(v) = 1$  for every  $w \in V(G)$ . As noted in Bange *et al.* [4], for the tree  $T_k$  in Figure 2 there is a unique efficient dominating function, namely,  $f(v_1) = f(v_2) = \dots = f(v_k) = 1$ ,

$f(v_{k+3}) = 0$ ,  $f(v_{k+2}) = 1 - k$ , and  $f(v_{k+1}) = k$ . Letting  $Y_t = \{1 - t, 2 - t, 3 - t, \dots, t - 1, t\}$ , note that  $T_k$  has an efficient  $Y_k$ -valued dominating function but not an efficient  $Y_{k-1}$ -valued dominating function. In particular, the class of efficiently  $Y_k$ -valued dominatable graphs properly contains the class of efficiently  $Y_{k-1}$ -valued dominatable graphs. See [4] for examples of graphs that are not efficiently  $\mathbb{R}$ -valued dominatable.

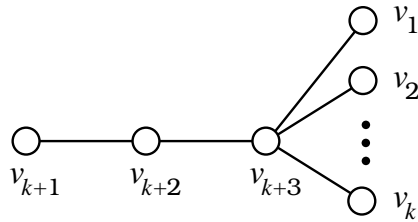


Figure 2. Efficiently  $Y_k$ -valued dominatable tree  $T_k$

Here we define a graph  $G$  to be *efficiently  $Y$ -valued  $k$ -tuple dominatable* if there is a function  $f : V(G) \rightarrow Y$  such that  $\sum_{v \in N[w]} f(v) = k$  for every  $w \in V(G)$ . The efficient  $Y$ -valued  $k$ -tuple domination number is  $F_{Y,k}(G) = \max\{I(f) \mid f(N[w]) \leq k \text{ for all } w \in V(G), \text{ where } f : V(G) \rightarrow Y\}$ , where the *influence of  $f$*  is  $I(f) = \sum_{v \in V(G)} (1 + \text{deg } v) \cdot f(v)$ . Note that  $G$  is efficiently  $Y$ -valued  $k$ -tuple dominatable if and only if  $F_{Y,k}(G) = k|V(G)|$ , and we always have  $F_{Y,k}(G) \leq k|V(G)|$ . Likewise,  $R_{Y,k}(G) = \min\{I(f) : f \text{ is a } Y\text{-valued, } k\text{-tuple dominating function}\}$ , and  $R_{Y,k}(G) \geq k|V(G)|$ .

In this paper, we introduce a new type of domination,  $(j, k)$ -domination. There is a natural connection between  $k$ -tuple and  $\{k\}$ -domination (or  $\{0, 1, \dots, k\}$ -valued  $k$ -tuple domination). If we require every vertex to be dominated at least  $k$  times (that is,  $f(N[v]) \geq k$  for all  $v \in V$ ), and let the function  $f$  take on values in the set  $\{1, 2, \dots, j\}$  where  $1 \leq j \leq k$ , then  $f$  is a  $(j, k)$ -dominating function. The minimum weight of such a function is the  $(j, k)$ -domination number, denoted by  $\gamma_{j,k}(G)$ . If we let  $j = 1$ , then  $(1, k)$ -domination is  $k$ -tuple domination, thus for any graph  $G$ ,  $\gamma_{1,k}(G) = \gamma_{\times k}(G)$ . If we let  $j = k$ , then  $(k, k)$ -domination is  $\{k\}$ -domination, thus for any graph  $G$ ,  $\gamma_{k,k}(G) = \gamma_{\{k\}}(G)$ . Ordinary domination can be obtained by letting  $k = 1$  ( $\gamma_{1,1}(G) = \gamma(G)$ ). As mentioned above, for  $(j, k) = (1, 2)$ , we have double domination and for  $(j, k) = (2, 2)$ , we have  $\{2\}$ -domination. We must take  $k \geq 3$  for new variations on the theme of multiple domination.

Figure 3a-c depicts minimum  $(1, 3)$ ,  $(2, 3)$ , and  $(3, 3)$ -dominating functions, respectively, of the wheel graph on seven vertices,  $W_7$ .

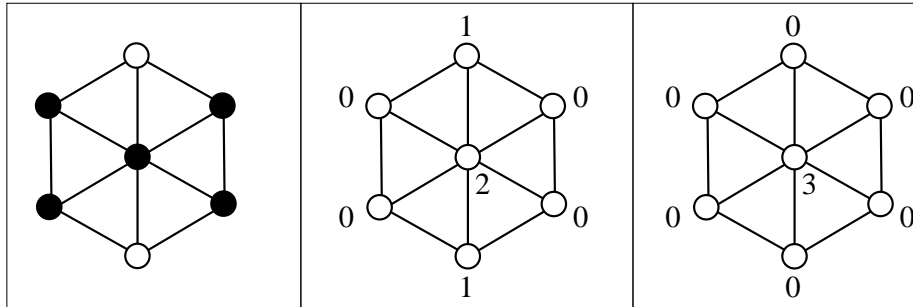


Figure 3.  $\gamma_{1,3}(W_7) = 5$        $\gamma_{2,3}(W_7) = 4$        $\gamma_{3,3}(W_7) = 3$

In this paper we also introduce the notion of efficiently  $\{0, 1, \dots, j\}$ -valued  $k$ -tuple dominatable graphs. The efficient  $(j, k)$ -domination number,  $F_{j,k}(G) = \max\{I(f) \mid f(N[w]) \leq k \text{ for all } w \in V(G), \text{ where } f : V(G) \rightarrow \{0, 1, 2, \dots, j\}\}$ . A graph  $G$  is efficiently  $(j, k)$ -dominatable if and only if  $F_{j,k}(G) = k \cdot |V(G)| = k \cdot n$ , and we always have  $F_{j,k}(G) \leq k \cdot n$ . Likewise,  $R_{j,k}(G) = \min\{I(f) : f \text{ is a } (j, k)\text{-dominating function}\}$ , and  $R_{j,k}(G) \geq k|V(G)|$ . Note, that  $F_{3,3}(W_7) = 21 = 3 \cdot 7 = R_{3,3}(W_7)$ , thus the  $(3, 3)$ -dominating function depicted in Figure 3c is efficient.

## 2. OBSERVATIONS

As indicated in Figure 4, the graph  $G3$  has an efficient double dominating set,  $F_{1,2}(G3) = 2n$ , but it does not have an efficient dominating set. In fact,  $F(G3) = F_{1,1}(G3) = 14 < 15$ . Note that the star  $K_{1,t}$  of order  $n = 1 + t$  with  $t \geq 2$  has an efficient dominating set, but does not have an efficient double dominating set. In fact,  $F_{1,2}(K_{1,t}) = t + 3 < 2n$ .

For  $r \geq 1$ , the open  $r$  neighborhood of a vertex  $v \in V(G)$ ,  $N_r(v)$  is the set of vertices in  $V(G)$  different from  $v$  at distance at most  $r$  from  $v \in V(G)$ , that is,  $N_r(v) = \{u \in V(G) - v : d(u, v) \leq r\}$ .

**Observation 1.** *If  $f : V(G) \rightarrow \{0, 1, \dots, k\}$  is an efficient  $(k, k)$ -dominating function, and for some vertex  $v$ , we have  $f(v) = k$ , then for every vertex  $z \in N_2(v)$ ,  $f(z) = 0$ .*

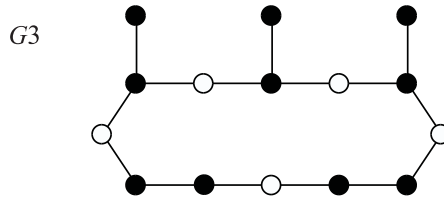


Figure 4. The unicyclic, bipartite graph  $G3$  with an efficient double dominating set and no efficient dominating set.

Generalizing results in [4, 16], we get the following obvious results. As usual,  $\delta(G)$  denotes the minimum degree of a vertex in  $G$ .

**Observation 2.** For  $Y \subseteq \mathbb{R}$ , graph  $G$  has a  $(Y, k)$ -dominating function  $f : V(G) \rightarrow Y$  if and only if there is an  $x \in Y$  with  $x \geq k/(1 + \delta(G))$ .

Henceforth, we assume  $Y \cap (\frac{k}{1+\delta(G)}, \infty) \neq \emptyset$ .

**Proposition 3.** If  $G$  is efficiently  $(j, k)$ -dominatable, then  $k \leq j(1 + \delta(G))$ .

**Proposition 4.** If  $Y_1 \subseteq Y_2 \subseteq \mathbb{R}$ , then  $\gamma_{Y_1, k}(G) \geq \gamma_{Y_2, k}(G)$ ,  $F_{Y_1, k}(G) \leq F_{Y_2, k}(G)$  and  $R_{Y_1, k}(G) \geq R_{Y_2, k}(G)$ .

In particular, if  $G$  is efficiently  $(Y_1, k)$ -dominatable, then  $G$  is efficiently  $(Y_2, k)$ -dominatable.

**Corollary 5.** For any graph  $G$ , we have  $F_{j, k}(G) \leq F_{j+1, k}(G)$  for  $1 \leq j < k$ . If  $1 \leq j < k$  and  $j(1 + \delta(G)) \geq k$ , then  $\gamma_{j, k}(G) \geq \gamma_{j+1, k}(G)$  and  $R_{j, k}(G) \geq R_{j+1, k}(G)$ .

In particular, if  $G$  is efficiently  $(j, k)$ -dominatable, (with  $j < k$ ) then  $G$  is efficiently  $(j + 1, k)$ -dominatable, and hence efficiently  $(k, k)$ -dominatable.

**Proposition 6.** For every graph  $G$ ,  $t \cdot F_{j, k}(G) \leq F_{jt, kt}(G)$ .

**Proof.** Let  $f : V(G) \rightarrow \{0, 1, 2, \dots, j\}$  be an  $F_{j, k}(G)$ -function, and define  $g : V(G) \rightarrow \{0, 1, 2, \dots, jt\}$  by  $g(v) = t \cdot f(v)$ . Then for each  $w \in V(G)$ , we have  $\sum_{x \in N[w]} g(x) = t \cdot \sum_{x \in N[w]} f(x) \leq tk$ , and  $F_{jt, kt}(G) \geq w(g) = \sum_{v \in V(G)} g(v) = t \cdot \sum_{v \in V(G)} f(v) = t \cdot F_{j, k}(G)$ . ■

**Corollary 7.** *If there exists an efficient dominating set  $S$  on a graph  $G$ , then there exists an efficient  $(k, k)$ -dominating function.*

**Proof.** Let  $f_S$  be the characteristic function of the efficient dominating set which has the value of 1 at any vertex in the set  $S$  and the value of 0 otherwise. Then the function  $g$  which assigns  $g(v) = k \cdot f_S(v)$  is an efficient  $(k, k)$ -dominating function. ■

The graph  $G4$  of order  $n = 20$  has the efficient  $(2, 2)$ -dominating function as indicated in Figure 5, so  $F_{2,2}(G4) = 40 = 2n$ . However,  $F(G4) = F_{1,1}(G4) = 19 < n$ , thus  $G4$  does not have an efficient dominating set. Also  $F_{1,2}(G4) = 37 < 2n$ , thus  $G4$  does not have an efficient double dominating set.

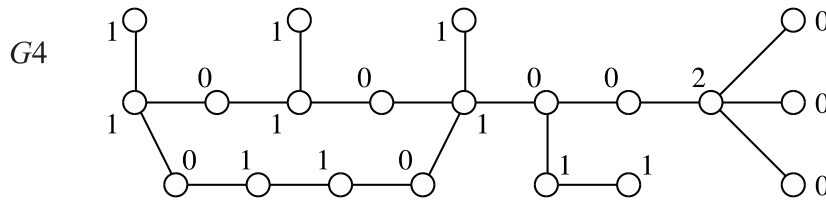


Figure 5. The graph  $G4$  has an efficient  $(2, 2)$ -dominating function and no efficient dominating set or efficient double dominating set.

As with the example in Figure 5, the converse to Corollary 7 does not hold in general. However, if the graph is a tree or union of trees, then the converse to Corollary 7 does hold.

**Theorem 8.** *If a forest  $T$  of order  $n$  has  $F_{j,k}(T) = k \cdot n$  using  $f : V(T) \rightarrow \{0, 1, \dots, j\}$  for some  $j \leq k$ ,  $k \geq 2$ , that is,  $T$  is efficiently  $(j, k)$ -dominatable, then  $T$  has an efficient dominating set  $S$  where  $S \subseteq \{z \in V(T) \mid f(z) \geq 1\}$ .*

**Proof.** Consider the cases where  $T$  is a tree of diameter at most three. The theorem is easily seen to hold affirmatively for paths  $P_1$ ,  $P_2$ ,  $P_3$ , and  $P_4$ , and for stars  $K_{1,t}$  for  $t \geq 3$ , and the double-stars  $S_{a,b}$  with  $1 \leq a \leq b$  and  $b \geq 2$  have  $F_{k,k}(S_{a,b}) < k \cdot n = k(a + b + 2)$  for all  $k \geq 1$ . We can prove the theorem by induction on the order  $n$  of the forest  $T$ , and by the above we can assume that each component of  $T$  has diameter at least four.

Select vertices  $u$  and  $y$  in the same component of  $T$  such that the distance  $d(u, y)$  is maximized, and let  $u, v, w, x, \dots, y$  be the  $u - y$  path in  $T$ .

*Case 1.* Assume  $f(u) = k$ . Then  $f(v) = 0$ , and if  $z \in N(v) - u$ , then  $f(z) = 0$  (that is, if  $z \in N_2(u)$  then  $f(z) = 0$ ). Let  $T^* = T - \{u, v\}$ , and let  $f^* : V(T^*) \rightarrow \{0, 1, \dots, k\}$  be the restriction of  $f$  to  $T^*$  (that is,  $f^* = f|_{T^*}$ ), so  $f^*(z) = f(z)$  for all  $z \in V(T^*)$ . Because  $f^*$  is an efficient  $(k, k)$ -dominating function for  $T^*$ , by induction  $T^*$  has an efficient dominating set  $S^* \subseteq \{z \in V(T^*) \mid f^*(z) \geq 1\}$ . In particular,  $S^* \cap N(v) = \emptyset$ . Thus  $S^* \cup \{u\}$  is an efficient dominating set for  $T$  with  $S^* \cup \{u\} \subseteq \{z \in V(T) \mid f(z) \geq 1\}$ .

*Case 2.* Assume  $f(u) = 0$ . Then  $f(v) = k$ , and if  $z \in N_2(v)$  then  $f(z) = 0$ . In particular, if  $z \in N[w] - v$ , then  $f(z) = 0$ . Consider the forest  $T^* = T - N[v]$  and  $f^* = f|_{T^*}$ . Because  $f^*$  is an efficient  $(j, k)$ -dominating function for  $T^*$ , by induction  $T^*$  has an efficient dominating set  $S^* \subseteq \{z \in V(T^*) \mid f^*(z) \geq 1\}$ . Now,  $S^* \cap N(w) = \emptyset$ , and  $S^* \cup \{v\}$  is an efficient dominating set for  $T$  with  $S^* \cup \{v\} \subseteq \{z \in V(T) \mid f(z) \geq 1\}$ .

*Case 3.* Assume  $1 \leq f(u) = h \leq k - 1$ . In this case we consider  $T$  to be rooted at  $y$ . Because  $f(N[u]) = f(N[v]) = k$  we have  $f(v) = k - h$  and  $f(w) = 0$ . Also, we must have  $\deg(v) = 2$ , that is,  $u$  is the only child of  $v$ . Let the children of  $w$  be  $v = v_1, \dots, v_p$ . It is easy to see that each  $v_i$  must have exactly one child, say  $u_i$  (with  $u_1 = u$ , as in Figure 6).

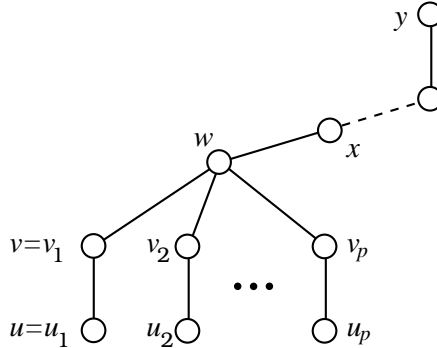


Figure 6. A rooted tree

Let  $f(u_i) = h_i$  (and note that  $h_i \neq 0$  or else  $f(v_i) = k$  and  $f(N[w]) \geq 2k - h > k$ ). We have  $f(v_i) = k - h_i$ , for  $1 \leq i \leq p$ , and  $f(x) = k - \sum_{i=1}^p f(v_i)$ . Let  $T^* = T - \{w, u_1, \dots, u_p, v_1, \dots, v_p\}$ , and again let  $f^* = f|_{T^*}$ . Because  $f^*$  is an efficient  $(j, k)$ -dominating function for  $T^*$ , we have an efficient dominating



set  $S^* \subseteq \{z \in V(T^*) \mid f^*(z) \geq 1\}$ . If  $x \in S^*$ , let  $S = S^* \cup \{u_1, \dots, u_p\}$ , and if  $x \notin S^*$  then let  $S = S^* \cup \{v_1, u_2, \dots, u_p\}$ . Then  $S$  is an efficient dominating set for  $T$  with  $S \subseteq \{z \in V(T) \mid f(z) \geq 1\}$ . ■

**Corollary 9.** *Every efficiently  $(j, k)$ -dominatable tree has an efficient dominating set. If there exists one  $t \geq 1$  such that the tree  $T$  is  $(t, t)$ -efficiently dominatable, then  $T$  is efficiently  $(k, k)$ -dominatable for every  $k \geq 1$ .*

### 3. REGULAR GRAPHS

For the 7-regular graph  $J$  in Figure 7, note that  $\{v_1, v_2, v_3, v_4\} = S$  is an efficient double dominating set and that  $V(G) - S$  is an efficient 6-tuple dominating set. This graph  $J$  has diameter two, so  $F_{1,1}(J) = 8 < 16$ , and  $J$  is not efficiently dominatable. It follows from the next theorem, that  $J$  is not efficiently 7-tuple dominatable.

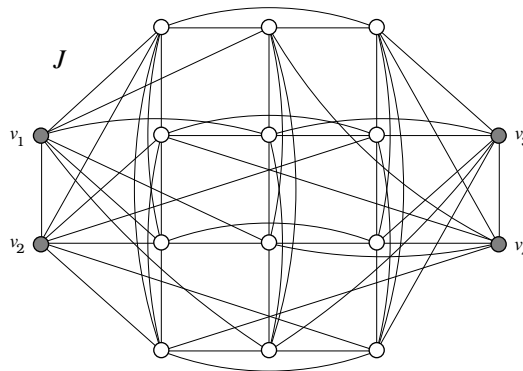


Figure 7. The 7-regular Jesse graph

**Theorem 10.** *For an  $r$ -regular graph  $G$  and  $1 \leq k \leq r$ ,  $G$  is efficiently  $(1, k)$ -dominatable (that is,  $F_{1,k}(G) = kn$ ) if and only if there exists a set  $S \subseteq V(G)$  such that the induced subgraph  $\langle S \rangle$  is regular of degree  $k - 1$  and the induced subgraph  $\langle V(G) - S \rangle$  is regular of degree  $r - k$ . Also, an  $r$ -regular graph  $G$  is efficiently  $(1, k)$ -dominatable if and only if  $G$  is efficiently  $(1, r - k + 1)$ -dominatable.*

**Proof.** Let  $S$  be an efficient  $(1, k)$ -dominating set. Then  $\langle S \rangle$  is regular of degree  $k - 1$ , and each vertex in  $V(G) - S$  is adjacent to exactly  $k$  vertices in  $S$ . Thus,  $\langle V(G) - S \rangle$  is regular of degree  $r - k$ , and each vertex in  $S$  is adjacent to exactly  $r - (k - 1)$  vertices in  $V(G) - S$ . It follows that  $V(G) - S$  is an efficient  $(1, r - k + 1)$ -dominating set. ■

#### 4. MINIMUM EFFICIENT $(j, k)$ -DOMINATING FUNCTIONS

**Theorem 11** (Bange, Barkauskas, Slater [3]). *If  $G$  has an efficient dominating set  $S \subseteq V$ , then  $|S| = \gamma(G)$ . In particular, all efficient dominating sets have the same cardinality.*

We first generalize Theorem 11 to vertex sets  $S$  that dominate every vertex exactly  $k$  times, graphs with  $F_{1,k}(G) = k \cdot n$ .

**Theorem 12.** *If  $G$  has an efficient  $k$ -tuple dominating set  $S \subseteq V$ , then  $|S| = \gamma_{1,k}(G)$ . In particular, all efficient  $k$ -tuple dominating sets have the same cardinality.*

**Proof.** Let  $S \subseteq V(G)$  be an efficient  $k$ -tuple dominating set. By definition, we have  $\gamma_{1,k}(G) \leq |S|$ . Let  $D \subseteq V(G)$  be a minimum cardinality  $k$ -tuple dominating set (a  $\gamma_{1,k}(G)$ -set). Each  $v \in S$  has  $|N(v) \cap S| = k - 1$ , and each  $v \in V(G) - S$  has  $|N(v) \cap S| = k$ . In particular, each  $v \in D - S$  has *exactly*  $k$  edges connecting it to  $S$ .

Let  $x \in S - D$ , and let  $d_1(x) = |N(x) \cap (S - D)|$  and  $d_2(x) = |N(x) \cap (S \cap D)|$ . Then  $d_1(x) + d_2(x) = k - 1$ . Because  $|N(x) \cap D| \geq k$  we have  $|N(x) \cap (D - S)| \geq 1 + d_1(x)$ . For each  $y \in D \cap S$  let  $d_2(y) = |N(y) \cap (S - D)|$ , and then we have  $|N(y) \cap (D - S)| \geq d_2(y)$ . Thus, letting  $(D - S, S)$  be the set of edges between  $D - S$  and  $S$ , we have  $|(D - S, S)| \geq \sum_{x \in S - D} (1 + d_1(x)) + \sum_{y \in S \cap D} d_2(y) = \sum_{x \in S - D} (1 + d_1(x)) + \sum_{x \in S \cap D} d_2(x) = \sum_{x \in S - D} (1 + d_1(x) + d_2(x)) = k \cdot |S - D|$ . Now each  $v \in D - S$  has  $|N(v) \cap S| = k$  and  $|(D - S, S)| \geq k \cdot |S - D|$  implies that  $|D - S| \geq |S - D|$ . Thus  $\gamma_{j,k}(G) = |D| \geq |S|$ . Consequently,  $|S| = \gamma_{1,k}(G)$ . ■

Using the above theorem, we can generalize to graphs  $G$  which are efficiently  $(j, k)$ -dominatable, that is, graphs  $G$  with  $F_{j,k}(G) = k \cdot n$ .

**Theorem 13.** *If  $G$  has an efficient  $(j, k)$ -dominating function  $f : V(G) \rightarrow \{0, 1, \dots, j\}$ , with  $f(N[v]) = k$  for all  $v \in V(G)$ , so that  $F_{j,k} = k \cdot n$ , then the weight of  $f$  satisfies  $w(f) = \gamma_{j,k}(G)$ . In particular, all efficient  $(j, k)$ -dominating functions have the same weight.*

**Proof.** Let  $f$  be an efficient  $(j, k)$ -dominating function and let  $g : V(G) \rightarrow \{0, 1, \dots, j\}$  be a  $\gamma_{j,k}(G)$ -function, so that  $g(N[v]) \geq k$  for all  $v \in V(G)$  and  $w(g) = \gamma_{j,k}(G)$ .

Let  $R$  be the set of vertices in  $V(G)$  with nonzero weight under the function  $f$ ,  $R = \{v \in V(G) : f(v) \geq 1\}$  and  $B = \{x \in V(G) : g(x) \geq 1\}$ , and let  $W = V(G) - (R \cup B)$ . We construct a graph  $H$  on  $w(f) + w(g) + |W|$  vertices as follows. Replace each  $v \in R - B$  by a clique on  $f(v)$  “red” vertices; replace each  $x \in B - R$  by a clique on  $g(x)$  “blue” vertices; replace each  $y \in R \cap B$  by a clique on  $f(y) + g(y)$  vertices with  $f(y)$  of them considered to be “red” and  $g(y)$  to be “blue”; and replace each  $w \in W$  by one vertex in  $H$ . So  $|V(H)| = w(f) + w(g) + |W|$ . Thus for each vertex  $z$  in  $G$  we have a corresponding clique  $K_z$  in  $H$  with  $|K_z| = f(z)$  if  $z \in R - B$ ,  $|K_z| = g(z)$  if  $z \in B - R$ ,  $|K_z| = f(z) + g(z)$  if  $z \in R \cap B$ , and  $|K_z| = 1$  if  $z \in W$ .

For each edge  $\{a, b\}$  in  $E(G)$ , we form the join  $K_a + K_b$  in  $H$ , that is, every vertex in  $K_a$  is made adjacent to every vertex in  $K_b$ . For each vertex  $a^*$  in  $K_a$ , observe that in  $H$  we have  $N_H[a^*]$  containing exactly  $\sum_{u \in N_G[a]} f(u) = k$  red vertices and  $N_H[a^*]$  containing exactly  $\sum_{u \in N_G[a]} g(u) \geq k$  blue vertices. That is, the set  $R^*$  of red vertices in  $H$  is an efficient  $k$ -tuple dominating set. Because the set  $B^*$  of blue vertices in  $H$  is a  $k$ -tuple dominating set for  $H$ , by Theorem 12, we have  $w(f) = |R^*| = \gamma_{1,k}(H) \leq |B^*| = w(g)$ . In  $G$ , the function  $f$  is a  $k$ -tuple dominating function, and so we have,  $w(g) = \gamma_{j,k}(G) \leq w(f)$ . Consequently,  $w(f) = w(g) = \gamma_{j,k}(G)$ . ■

## 5. FRACTIONAL CONNECTIONS

In Grinstead and Slater [12] and Rubalcaba and Walsh [14], efficient fractional dominating functions were investigated. A fractional dominating function  $f : V \rightarrow [0, 1]$  is efficient if  $f(N[v]) = \sum_{u \in N[v]} f(u) = 1$ , for every  $v \in V$ . Domke, Hedetniemi, Laskar, and Fricke in [7] showed that the fractional domination number can be defined as the following limit:  $\lim_{k \rightarrow \infty} \frac{\gamma_{\{k\}}(G)}{k} = \gamma_f(G)$ .

Moreover, since the fractional domination number is, in fact, a rational (optimal solutions to linear programs with integer coefficients are rational

numbers), there exists an integer  $k$  so that  $\frac{\gamma_{\{k\}}(G)}{k} = \frac{\gamma_{k,k}(G)}{k} = \gamma_f(G)$ . For example, for  $C_5$ , this integer is  $k = 3$ .

In the following,  $\mathbf{x} = [x(v_1), x(v_2), \dots, x(v_n)]^t$  is the column vector representation of the function  $x : V(G) \rightarrow [0, 1]$ . The closed neighborhood matrix, denoted by  $N$ , is  $N = A + I$  where  $A$  is the adjacency matrix and  $I$  is the  $n \times n$  identity matrix. A graph  $G$  has an efficient fractional dominating function if and only if  $N\mathbf{x} = \mathbf{1}$  has a solution where each  $x_i$  in  $[0, 1]$ . As shown in Bange *et al.* [4], if a non-negative solution to  $N\mathbf{x} = \mathbf{1}$  exists, then it is either unique or there are infinitely many solutions (depending on whether or not  $N$  is invertible).

**Theorem 14.** *A graph  $G$  has an efficient fractional dominating function if and only if there exist positive integers  $j, k$  for which  $G$  has an efficient  $(j, k)$ -dominating function. Furthermore, the weights of the efficient fractional dominating function can be expressed with common denominator  $k$  and greatest numerator at most  $j$ .*

**Proof.** Suppose  $g : V(G) \rightarrow \{0, \dots, j\}$  is an efficient  $(j, k)$ -dominating function of  $G$ . Then  $f(v) = \frac{g(v)}{k}$  satisfies  $f(N[v]) = 1$  for all  $v \in V(G)$ . Thus,  $f$  is a fractional efficient dominating function of  $G$ . Furthermore, each of the weights  $f(v)$  is a fraction with denominator  $k$  and each numerator is less than or equal to  $j$ . Let  $f : V(G) \rightarrow [0, 1]$  be an efficient fractional dominating function of  $G$ , whose weights have (common) denominator  $k$ . Let  $g : V(G) \rightarrow \{0, \dots, k\}$  be defined as  $g(v) = k \cdot f(v)$  for each  $v \in V(G)$ . Since  $g(N[v]) = k$  for every  $v \in V(G)$ ,  $g$  is an efficient  $(j, k)$ -dominating function. ■

Grinstead and Slater in [12] give a formula for finding an efficient fractional dominating function of any complete  $k$ -partite graph which gives constant weights on each partite set. For the complete 3-partite graph  $K_{4,6,9}$ , assign the weights  $\frac{40}{319}, \frac{24}{319}, \frac{15}{319}$  to each vertex in the partite set of sizes 4, 6, 9, respectively, to obtain an efficient fractional dominating function. Thus,  $K_{4,6,9}$  has an efficient  $(40, 319)$ -dominating function by assigning the weights of 40, 24, 15 to each vertex in the partite sets of sizes 4, 6, 9, respectively. Since the efficient fractional dominating function is unique, the only efficient  $(j, k)$ -dominating functions for  $K_{4,6,9}$  are with  $j \geq 40t$  and  $k = 319t$  (for  $t$  any positive integer). Thus,  $K_{4,6,9}$  has no efficient dominating set since  $k = 1$  is not an integral multiple of  $k = 319$ .

As another example, the Herschel graph depicted in Figure 8(a) and the  $2 \times 4$  grid graph depicted in Figure 8(b) both have efficient fractional dominating functions (with each weight as depicted, divided by 5). Thus, they both have efficient  $(2t, 5t)$ -dominating functions (for every positive integer  $t$ ). Since for each graph, the efficient fractional dominating functions are unique, neither has an efficient dominating set.

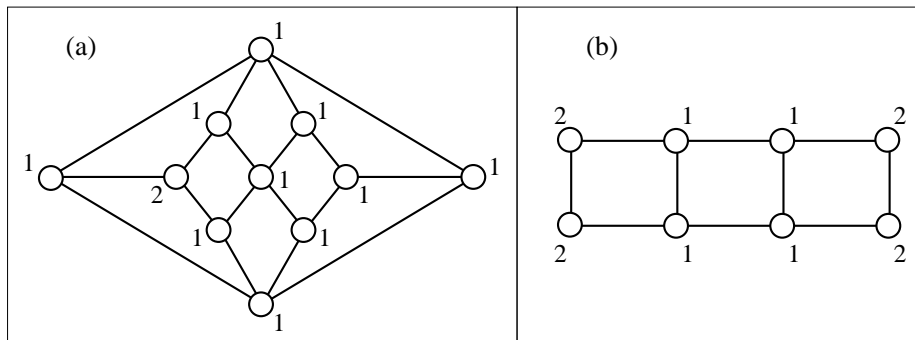


Figure 8. The Herschel graph (a) and the  $2 \times 4$  grid graph (b).

Determining whether or not a graph has an efficient dominating set is an NP-complete problem ([2]). However, if a unique solution to  $N\mathbf{x} = \mathbf{1}$  exists, then we can determine whether or not an efficient dominating set exists, merely by inspecting the unique efficient fractional dominating function  $f$ . If any of the weights satisfy  $0 < f_i = f(v_i) < 1$ , then no efficient dominating set exists. Note that this check can be done in polynomial time.

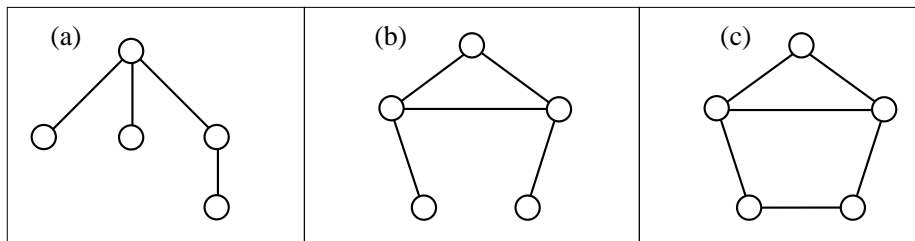


Figure 9. Graphs with no efficient  $(j, k)$ -dominating functions (for any choice of  $j$  and  $k$ ) and thus no efficient dominating set.

In Rubalcaba and Walsh [14], a graph was defined to be a member of Class null (denoted as Class  $\mathcal{N}$ ) if no minimum fractional dominating function was a maximum fractional closed neighborhood packing function. Consequently any graph in Class  $\mathcal{N}$  would have no efficient fractional dominating function (see Figure 9). Thus, from Theorem 14, any graph in Class  $\mathcal{N}$  would have no efficient  $(j, k)$ -dominating function, for any choice of  $j$  and  $k$ . In [14], several infinite families of graphs were found to be Class  $\mathcal{N}$ , such as incomplete or complete  $k$ -suns and generalized Hajós graphs.

We conclude with the following open questions.

- If for a graph  $G$ , there exists an efficient  $(k, k)$ -dominating function for all  $k \geq 2$ , then does  $G$  have an efficient dominating set?
- As shown in Figure 4, the graph  $G3$  is efficiently  $(1, 2)$ -dominatable. However, it is not efficiently  $(1, 1)$ -dominatable, nor is it efficiently  $(1, k)$ -dominatable for  $k \geq 3$ . That is,  $G3$  is efficiently  $(1, k)$ -dominatable if and only if  $k \in \{2\}$ . Note that the cycle  $C_{3k}$  is efficiently  $(1, k)$ -dominatable if and only if  $k \in \{1, 2, 3\}$ . For which subsets  $A$  of the positive integers do there exist graphs  $G_A$  so that  $G_A$  is efficiently  $(1, k)$ -dominatable if and only if  $k \in A$ ?

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