

## MONOCHROMATIC KERNEL-PERFECTNESS OF SPECIAL CLASSES OF DIGRAPHS

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### Abstract

In this paper, we introduce the concept of monochromatic kernel-perfect digraph, and we prove the following two results:

(1) If  $D$  is a digraph without monochromatic directed cycles, then  $D$  and each  $\alpha_v, v \in V(D)$  are monochromatic kernel-perfect digraphs if and only if the composition over  $D$  of  $(\alpha_v)_{v \in V(D)}$  is a monochromatic kernel-perfect digraph.

(2)  $D$  is a monochromatic kernel-perfect digraph if and only if for any  $B \subseteq V(D)$ , the duplication of  $D$  over  $B$ ,  $D^B$ , is a monochromatic kernel-perfect digraph.

**Keywords:** kernel, kernel by monochromatic paths, composition, duplication.

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## 1. Introduction

For general concepts we refer the reader to [1]. Let  $D$  be a digraph,  $V(D)$  and  $A(D)$  will denote the sets of vertices and arcs of  $D$  respectively. Let  $S_1, S_2 \subseteq V(D)$ , an arc  $(u_1, u_2)$  of  $D$  will be called an  $S_1S_2$ -arc whenever  $u_1 \in S_1$  and  $u_2 \in S_2$ ;  $D[S_1]$  will denote the subdigraph of  $D$  induced by  $S_1$ . A set  $I \subseteq V(D)$  is independent if  $A(D[I]) = \emptyset$ . A *kernel*  $N$  of  $D$  is an independent set of vertices such that for each  $z \in (V(D) - N)$  there exists a  $zN$ -arc in  $D$ . A digraph  $D$  is called a *kernel-perfect digraph* when every induced subdigraph of  $D$  has a kernel.

A digraph  $D$  is said to be an  $m$ -coloured digraph, if its arcs are coloured with  $m$  colours without loss of generality  $\{1, 2, \dots, m\}$ . A directed path (or a directed cycle) is called *monochromatic* if all of its arcs are coloured alike.

A set  $N \subseteq V(D)$  of vertices of  $D$  is said to be a *kernel by monochromatic paths* of the  $m$ -coloured digraph  $D$ , if it satisfies the two following properties, (1)  $N$  is independent by monochromatic paths; i.e., for any two different vertices  $x, y \in N$ , there is no monochromatic directed path between them, and (2)  $N$  is *absorbent* by monochromatic paths; i.e., for each  $u \in (V(D) - N)$  there exists a  $uv$ -monochromatic directed path, for some  $v \in N$ .

In this paper, we prove that if  $D$  is a digraph without monochromatic directed cycles, then (i)  $D$  has a kernel by monochromatic paths if and only if any composition over  $D$  of a family of digraphs  $(\alpha_v)_{v \in V(D)}$  each one of them having a kernel by monochromatic paths, has a kernel by monochromatic paths, and (ii)  $D$  has a kernel by monochromatic paths if and only if for any  $B \subseteq V(D)$  the duplication of  $D$  over  $B$ ,  $D^B$ , has a kernel by monochromatic paths.

As a consequence we obtain the two results mentioned in the abstract.

Clearly,  $D$  has a kernel if and only if the  $m$ -coloured digraph  $D$ , in which every two different arcs have different colours, has a kernel by monochromatic paths. Sufficient conditions for the existence of a kernel in a digraph have been investigated by several authors, namely Von Neumann and Morgenstern [16], Richardson [13], Duchet and Meyniel [5] and Galeana-Sánchez and Neumann-Lara [6]. The concept of a kernel is very useful in applications, and clearly, the concept of a kernel by monochromatic paths generalizes that of kernel. Sufficient conditions for the existence of kernels by monochromatic paths in  $m$ -coloured digraphs have also been investigated by several authors; see for example [7, 9, 10, 14, 15, 18].

**Definition 1.1.** Let  $D$  be an arc coloured digraph and  $\alpha = (\alpha_v)_{v \in V(D)}$  a family of pairwise vertex disjoint arc coloured digraphs. We define the *composition of  $\alpha$  over  $D$* , denoted  $\sigma(D, \alpha)$ , by the following conditions:

- (i)  $V(\sigma(D, \alpha)) = \bigcup_{v \in V(D)} V(\alpha_v)$ .
- (ii)  $A(\sigma(D, \alpha)) = \left( \bigcup_{v \in V(D)} A(\alpha_v) \right) \cup \{(x, y) \text{ coloured } i \mid x \in \alpha_u, y \in \alpha_v, (u, v) \in F(D) \text{ coloured } i\}$ .

The composition of a family of graphs  $\beta = (G_v)_{v \in V(G)}$  over a graph  $G$  was studied in [3] and its definition was extended to digraphs in [17]. The existence of kernels in the composition  $\sigma(D, \alpha)$  of a family of digraphs  $\alpha = (\alpha_v)_{v \in V(D)}$  over a digraph  $D$  was studied in [8], and the result was used to prove the existence of kernel-perfect digraphs with an arbitrarily large dichromatic number whose underlying graphs have no triangles.

In this paper, we study the existence of kernels by monochromatic paths in the composition  $\sigma(D, \alpha)$  of a family of arc coloured digraphs  $\alpha = (\alpha_v)_{v \in V(D)}$  over an arc coloured digraph  $D$ .

The duplication of a vertex of a graph was introduced in [4], and [11] gives the definition of the duplication of a subset of vertices of a graph as a generalization of the duplication of a vertex of a graph. This definition can be applied to arc coloured digraphs as follows:

**Definition 1.2.** Let  $D$  be an arc coloured digraph,  $B$  a proper subset of  $V(D)$  and let  $B'_D$  a digraph isomorphic to  $D[B]$  with  $V(B'_D) \cap V(D) = \emptyset$ . A vertex belonging to  $B'_D$  and corresponding to a vertex  $x \in B$  will be denoted by  $x'$ . The *duplication of  $D$  over  $B$*  is the arc coloured digraph denoted  $D^B$  and defined as follows:

$$V(D^B) = V(D) \cup V(B'_D)$$

and

$$A(D^B) = A(D) \cup A(B'_D) \cup A_0 \cup A_1$$

in which  $A_0 = \{(x', y) \text{ coloured } i \mid x' \in V(B'_D), y \in V(D) \text{ and } (x, y) \in A(D) \text{ coloured } i\}$ .  $A_1 = \{(y, x') \text{ coloured } i \mid y \in V(D), x' \in V(B'_D) \text{ and } (y, x) \in A(D) \text{ coloured } i\}$ .

We will denote  $B' = V(B'_D)$ . A vertex  $x' \in B'$  (resp., a subset  $S' \subseteq B'$ ) we will call the copy of the vertex  $x \in B$  (resp., the copy of the subset  $S \subseteq B$ ).

The vertex  $x$  (resp., the subset  $S$ ) will be named the original of the vertex  $x'$  (resp., of the subset  $S'$ ).

We will denote by  $\text{Proy}$  the function  $\text{Proy}: V(\sigma(D, \alpha)) \rightarrow V(D)$  such that  $\text{Proy}(x) = v$  if and only if  $x \in V(\alpha_v)$ .

The existence of kernels in the duplication of a digraph  $D$  has been studied in [2]. In this paper, we study the existence of kernels by monochromatic paths in the duplication of an arc coloured digraph  $D$  over a proper subset of vertices of  $V(D)$ .

The composition and the duplication are two operations in digraphs which have been considered several times, see for example [3, 12, 17], and they constitute a powerful tool in the construction of many examples and counterexamples in digraphs.

Also we consider an extension of the concept of kernel perfectness of a digraph and obtain a large variety of monochromatic kernel-perfect digraphs.

## 2. Kernels by Monochromatic Paths in the Composition over $D$ , and in the Duplication of $D$ over $B$

We start this section with a lemma which will be useful in the proof of Theorem 2.1. Its proof is easy and will be omitted.

**Lemma 2.1.** *Let  $D$  be a digraph and  $\alpha = (\alpha_v)_{v \in V(D)}$  a family of pairwise vertex disjoint digraphs. If  $T = (x_0, x_1, \dots, x_n)$  is a directed path in  $\sigma(D, \alpha)$  such that  $\{x_0, x_n\} \subseteq V(\alpha_v)$  for some  $v \in V(D)$ , then  $\text{Proy}(T)$  is a join of directed cycles of  $D$  or a single vertex of  $D$ .*

**Theorem 2.1.** *Let  $D$  be an arc coloured digraph which has no monochromatic directed cycle and  $\alpha = (\alpha_v)_{v \in V(D)}$  a family of arc coloured pairwise vertex disjoint digraphs. A set  $N^* \subseteq V(\sigma(D, \alpha))$  is a kernel by monochromatic paths of  $\sigma(D, \alpha)$  if and only if there exists a kernel by monochromatic paths of  $D$ , say  $N \subseteq V(D)$ , such that  $N^* = \bigcup_{v \in N} N_v$ , in which  $N_v$  is a kernel by monochromatic paths of  $\alpha_v$ .*

**Proof.** Let  $N \subseteq V(D)$  be a kernel by monochromatic paths of  $D$  and  $N_v$  a kernel by monochromatic paths of  $\alpha_v$ ,  $v \in N$ . We will prove that  $N^* = \bigcup_{v \in N} N_v$  is a kernel by monochromatic paths of  $\sigma(D, \alpha)$ .

(a)  $N^*$  is absorbent by monochromatic paths.

Let  $z \in (V(\sigma(D, \alpha)) - N^*)$ . There exists  $v_0 \in V(D)$  such that  $z \in V(\alpha_{v_0})$ . When  $v_0 \in N$ , we have  $N_{v_0} \subseteq N^*$ , in which  $N_{v_0}$  is a kernel by monochromatic paths of  $\alpha_{v_0}$ ; and there exists a  $zN_{v_0}$ -monochromatic directed path (as  $z \in (V(\alpha_0) - N_{v_0})$ ).

When  $v_0 \notin N$ , we have  $v_0 \in (V(D) - N)$  and thus, there exists a monochromatic directed path contained in  $D$ , say  $T = (v_0, v_1, \dots, v_{n-1}, u)$  with  $u \in N$  (because  $N$  is a kernel by monochromatic paths of  $D$ ); since  $z \in V(\alpha_0)$ ; taking  $z_i \in V(\alpha_i)$  and  $z_u \in N_u$ , we have  $T' = (z, z_1, z_2, \dots, z_{n-1}, z_u)$ ; a  $zz_u$ -monochromatic directed path in  $\alpha(D, \sigma)$  with  $z_u \in N_u \subseteq N^*$ .

(b)  $N^*$  is independent by monochromatic paths.

We proceed by contradiction, suppose that there exist  $x_0, x_n \in N^*$  and a  $x_0x_n$ -monochromatic directed path, say  $T = (x_0, x_1, \dots, x_n)$  contained in  $\sigma(D, \alpha)$ . We consider two possible cases:

*Case (b.1).*  $\{x_0, x_n\} \subseteq V(\alpha_v)$ , for some  $v \in V(D)$ .

When  $T \subseteq \alpha_v$ , we have that  $T$  is an  $x_0x_n$ -monochromatic directed path contained in  $\alpha_v$ , with  $\{x_0, x_n\} \subseteq N_v$ , a contradiction.

When  $T \not\subseteq \alpha_v$ , we have from Lemma 2.1 that  $\text{Proy}(T)$  is a join of monochromatic directed cycles contained in  $D$ , contradicting our hypothesis on  $D$ .

*Case (b.2).*  $x_0 \in \alpha_v$  and  $x_n \in \alpha_u$  with  $u \neq v$ .

In this case, it follows from the definition of  $N^*$  that  $x_0 \in N_v$  and  $x_n \in N_u$  with  $\{u, v\} \subseteq N$ . Since  $T$  is monochromatic, We have that  $\text{Proy}(T)$  contains a  $vu$ -monochromatic path, which is contained in  $D$ , contradicting that  $N$  is a kernel by monochromatic paths of  $D$ . We conclude that  $N^*$  is a kernel by monochromatic paths.

Now let  $N^*$  be a kernel by monochromatic paths of  $\sigma(D, \alpha)$ . We will prove that  $N = \{v \in V(D) \mid N^* \cap \alpha_v \neq \emptyset\}$  is a kernel by monochromatic paths of  $D$  and  $N^* \cap V(\alpha_v) = N_v$  is a kernel by monochromatic paths of  $\alpha_v$ , for each  $v \in N$ .

$N$  is absorbent by monochromatic paths.

Let  $v \in (V(D) - N)$  and  $z_0 \in V(\alpha_v)$ ; since  $v \notin N$  we have that  $z_0 \notin N^*$ ; thus there exists a monochromatic directed path  $T = (z_0, \dots, z_n)$  with  $z_n \in N^*$ ; now,  $z_n \in V(\alpha_u)$  for some  $u \in V(D)$ ; moreover, from the definition of  $N$  we have  $u \in N$  and then  $\text{Proy}(T)$  contains a  $vu$ -monochromatic directed path with  $u \in N$ .

$N$  is independent by monochromatic paths.

We proceed by contradiction, suppose that there exist  $v_0, v_n \in N$  and a  $v_0v_n$ -monochromatic directed path  $T = (v_0, v_1, \dots, v_n)$  contained in  $D$ . Since  $v_0, v_n \in N$  there exist  $z_0 \in V(\alpha_{v_0}) \cap N^*$  and  $z_n \in V(\alpha_{v_n}) \cap N^*$ ; now taking any vertex  $z_i \in V(\alpha_{v_i})$  for each  $1 \leq i \leq n - 1$ ; we have from the definition of  $\sigma(D, \alpha)$  that  $(z_0, z_1, \dots, z_n)$  is a  $z_0z_n$ -monochromatic directed path with  $\{z_0, z_n\} \subseteq N^*$ , a contradiction.

Now; let  $v \in V(D)$  be such that  $N^* \cap V(\alpha_v) \neq \emptyset$ . We will prove that  $N_v = N^* \cap V(\alpha_v)$  is a kernel by monochromatic paths of  $\alpha_v$ .

$N_v$  is independent by monochromatic paths.

We proceed by contradiction. Suppose that there exist  $u, x \in N_v$ ,  $u \neq x$ , and a monochromatic directed path  $T$  between them, with  $T \subseteq \alpha_v$ ; clearly,  $T \subseteq \sigma(D, \alpha)$  and  $\{u, x\} \subseteq N^*$ , a contradiction (as  $N^*$  is independent by monochromatic paths in  $\sigma(D, \alpha)$ ).

$N$  is absorbent by monochromatic paths.

Let  $u \in (V(\alpha_v) - N)$  clearly  $u \in (V(\sigma(D, \alpha)) - N^*)$ ; thus there exists  $z \in N^*$  and a  $uz$ -monochromatic directed path  $T \subseteq \sigma(D, \alpha)$ . Let  $T = (u = u_0, u_1, \dots, u_n = z)$ , we will prove that  $T \subseteq \alpha_v$ . When  $u_n = z \in V(\alpha_v)$ ; it follows from Lemma 2.1 that  $\text{Proy}(T)$  is a single vertex i.e.,  $T \subseteq \alpha_v$ ; otherwise  $D$  contains a monochromatic directed cycle, contradicting our hypothesis on  $D$ . When  $u_n \notin V(\alpha_v)$ ; we have  $u_n \in \alpha_w$  for some  $w \in V(D)$  and  $w \in N^*$ . Now take  $x \in N_v^*$  (recall  $N^* \cap \alpha_v \neq \emptyset$ ); it follows from the definition of  $\alpha(D, \alpha)$  that  $(x, u_1, u_2, \dots, u_n = z)$  is a  $xz$ -monochromatic directed path in  $\sigma(D, \alpha)$  with  $x \neq z$ ,  $x, z \in N^*$ , a contradiction. ■

**Lemma 2.2.** *Let  $D$  be an arc coloured digraph,  $B \subset V(D)$ ;  $D^B$  the duplication of  $D$  over  $B$ ; and  $\psi: D[B] \rightarrow B'_D$  the isomorphism defined by the duplication (i.e.,  $\psi(x) = x'$  for any  $x \in V(D[B])$ ); and denote by  $\phi$  the function defined as follows:  $\phi: D \rightarrow D^B - D[B]$*

$$\phi(x) = \begin{cases} x & \text{if } x \notin B, \\ \phi(x) = x' & \text{if } x \in B. \end{cases}$$

*Then  $\phi$  is an isomorphism such that  $(x, y)$  is coloured in  $i$  if and only if  $(\phi(x), \phi(y))$  is coloured in  $i$ ; in particular  $T \subseteq D$  is a monochromatic directed path if and only if  $\phi(T) \subseteq D^B - D[B]$  is a monochromatic directed path.*

This Lemma is a direct consequence of the definition of  $\phi$  and the definition of the duplication of  $D$  over  $B$ .

**Theorem 2.2.** *Let  $D$  be an arc coloured digraph which has no monochromatic directed cycles;  $B \subset V(D)$  and  $D^B$  the duplication of  $D$  over  $B$ .*

*$D$  has a kernel by monochromatic paths if and only if  $D^B$  has a kernel by monochromatic paths.*

**Proof.** Let  $D, B$  and  $D^B$  be as in the hypothesis and suppose that  $D$  has a kernel by monochromatic paths, say  $N$ .

We consider two possible cases:

*Case 1.*  $N \cap B = \emptyset$ .

In this case, we will prove that  $N$  is a kernel by monochromatic paths of  $D^B$ .

$N$  is independent by monochromatic paths in  $D^B$ .

Let  $x, y \in N$ ;  $x \neq y$  and assume for a contradiction that there exists an  $xy$ -monochromatic directed path  $T = (x = x_0, x_1, \dots, x_n = y)$  contained in  $D^B$ .

When  $V(T) \cap B = \emptyset$ , we have  $T \subseteq D^B - D[B]$ , and from Lemma 2.1  $\phi^{-1}(T)$  is an  $xy$ -monochromatic directed path contained in  $D$ , contradicting that  $N$  is independent by monochromatic paths. When  $V(T) \cap B \neq \emptyset$ , we denote  $I = V(T) \cap B$ ; say  $I = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$ ,  $i_1 < i_2 < \dots < i_k$ , we also denote by  $T(I) = (x_0, \dots, x_{i_1-1}, x'_{i_1}, x_{i_1+1}, \dots, x_{i_2-1}, x'_{i_2}, \dots, x_n = y)$  (the succession obtained from  $T$  by substituting  $x_{i_j}$ , for  $x'_{i_j}$  in  $T$ , for each  $j \in \{1, \dots, k\}$ ). It follows from the definition of  $D^B$  that  $T(I)$  is a monochromatic directed path contained in  $D^B - D[B]$ ; and from Lemma 2.2  $\phi^{-1}(T(I))$  is an  $xy$ -monochromatic directed path contained in  $D$ , a contradiction.

$N$  is absorbent by monochromatic paths in  $D^B$ .

Let  $z \in (V(D^B) - N)$ . If  $z \notin B'_D$ , then  $z \in (V(D) - N)$  and there exists a  $zN$ -monochromatic directed path, say  $T$ , with  $T \subseteq D \subseteq D^B$ .

If  $z \in V(B'_D)$ , then there exists  $y \in B$  such that  $z = y' \in V(B'_D)$ ; we have  $y \notin N$  because  $N \cap B = \emptyset$ ; thus there exists a  $yN$ -monochromatic directed path, say  $T = (y, x_1, \dots, x_n)$  and then from definition of  $D^B$  we have that  $T' = (y', x_1, \dots, x_n)$  is a  $zN$ -monochromatic directed path in  $D^B$ .

*Case 2.*  $N \cap B \neq \emptyset$ .

Let  $Z = N \cap B$ , and denote by  $Z' = \{z' \in B'_D \mid z \in Z\}$ .

We will prove that  $N^* = N \cup Z'$  is a kernel by monochromatic paths of  $D^B$ .

$N^*$  is independent by monochromatic paths.

Let  $x, y \in N^*$ ,  $x \neq y$ , and assume for a contradiction that there exists an  $xy$ -monochromatic directed path  $T = (x = x_0, x_1, \dots, x_n = y)$  contained in  $D^B$ . Here we consider several possible cases:

*Case 2.a.*  $x, y \in N$ .

Let  $I' = V(T) \cap V(B'_D) = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$  and denote by  $y_{i_j}$  the original of  $x_{i_j}$  (i.e.,  $y_{i_j} = \psi^{-1}(x_{i_j})$ ). Now let  $T'$  be the succession obtained from  $T$  by substituting each  $x_{i_j}$  for  $y_{i_j}$ . It follows from the definition of  $\psi$  and from the definition of  $D^B$  that  $T'$  contains an  $xy$ -monochromatic directed path contained in  $D$ , with  $x, y \in N$ , a contradiction.

*Case 2.b.*  $x \in N$ ,  $y \in Z'$  and  $x \notin B$ . In this case, we proceed as in Case 2.a to get a contradiction.

*Case 2.c.*  $x \in N \cap B$ ,  $y \in Z'$ .

When  $x$  is the original vertex of  $y$ , taking the succession  $T'$  defined in Case 2.a we have that  $T'$  contains a monochromatic directed cycle, contradicting our hypothesis on  $D$ ; as  $T' \subseteq D$ .

When  $x$  is not the original vertex of  $y$ ; taking again the succession  $T'$  defined in Case 2.a, we have that  $T'$  contains an  $xz$ -monochromatic directed path, in which  $z$  is the original vertex of  $y$  and  $x, z \in N$  with  $x \neq z$ , contradicting that  $N$  is independent by monochromatic paths.

*Case 2.d.*  $x, y \in Z'$ .

Let  $\bar{x}$  (resp.,  $\bar{y}$ ) be the original vertex of  $x$  (resp.,  $y$ ); clearly, in this case  $T'$  (defined in Case 2.a) contains an  $\bar{x}\bar{y}$ -monochromatic directed path which is contained in  $D$ ; with  $\bar{x} \neq \bar{y}$ ,  $\bar{x}, \bar{y} \in N$ , a contradiction. So, we conclude that  $N^*$  is independent by monochromatic paths.

Now we prove that  $N^*$  is absorbent by monochromatic paths.

Let  $z \in (V(D^B) - N^*)$ . When  $z \in B'$ , we have  $z = y'$  in which  $y \in B$  is the original vertex of  $z$ . Since  $N$  is a kernel by monochromatic paths of  $D$ ; there exists a  $yN$ -monochromatic directed path in  $D$ , say,  $T = (y = x_0, x_1, \dots, x_n)$ ; thus  $T' = (y' = z, x_1, \dots, x_n)$  is a  $zN^*$ -monochromatic directed path contained in  $D^B$ . When  $z \notin B'$ , we have  $z \in (V(D) - N)$  and there exists a  $zN$ -monochromatic directed path contained in  $D$ ; say,  $T$ . Clearly,  $T$  is a  $zN^*$ -monochromatic directed path contained in  $D^B$ .

We conclude that  $N^*$  is a kernel by monochromatic paths of  $D^B$ . Now suppose that  $D^B$  has a kernel by monochromatic paths and let  $N^*$  be a



kernel by monochromatic paths of  $D^B$ . We will prove that  $D$  has a kernel by monochromatic paths.

Let  $Z$  be such that  $Z' = N^* \cap V(B'_D)$  in which  $Z'$  is defined by the process introduced in the construction of  $B'_D$ , when  $Z' = \emptyset$  we define  $Z = \emptyset$ . Denote by  $N = (N^* - Z') \cup Z$ . We will show that  $N$  is a kernel by monochromatic paths of  $D$ .

$N$  is independent by monochromatic paths in  $D$ .

Assume by contradiction that there exist  $x, y \in N$ ;  $x \neq y$ ; and an  $xy$ -monochromatic directed path  $T = (x = x_0, x_1, \dots, x_n = y)$  contained in  $D$ .

Let  $\bar{x}$  and  $\bar{y}$  be defined as follows:  $\bar{x} = x$  if  $x \in (N^* - Z')$  and  $\bar{x}$  is the copy of  $x$  if  $x \in Z$ ,  $\bar{y} = y$  if  $y \in (N^* - Z')$  and  $\bar{y}$  is the copy of  $y$  if  $y \in Z$ . Clearly,  $T' = (\bar{x}, x_1, \dots, x_{n-1}, \bar{y})$  is a monochromatic directed path in  $D^B$  with  $\bar{x} \neq \bar{y}$  and  $\bar{x}, \bar{y} \in N^*$ , a contradiction.

$N$  is absorbent by monochromatic paths in  $D$ .

Let  $z \in (V(D) - N)$ , then from the definition of  $N$ , we have  $z \in (V(D^B) - N)$ , thus there exists a  $zN^*$ -monochromatic directed path, say  $T = (z = x_0, x_1, \dots, x_n)$  contained in  $D^B$ . Let  $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} = V(T) \cap V(B'_D)$ ;  $y_{i_j}$  the original vertex of  $x_{i_j}$  and  $T'$  the succession obtained from  $T$  by substituting  $x_{i_j}$  for  $y_{i_j}$  for each  $1 \leq j \leq k$  in  $T$ . Clearly,  $T'$  contains a  $zN$ -monochromatic directed path, and  $T' \subseteq D$ .

We conclude that  $N$  is absorbent by monochromatic paths. ■

### 3. Monochromatic Kernel Perfectness of Composition and Duplication

The following definition is a generalization of the concept of kernel perfectness of a digraph.

**Definition 3.1.** Let  $D$  be an arc coloured digraph,  $D$  is said to be a monochromatic kernel perfect digraph whenever for every nonempty subset  $B$  of vertices of  $D$ , the digraph  $D[B]$  has a kernel by monochromatic paths.

**Theorem 3.1.** Let  $D$  be an arc coloured digraph which has no monochromatic directed cycle and  $\alpha = (\alpha_v)_{v \in V(D)}$  a family in which the  $\alpha_v$  are mutually disjoint arc coloured digraphs.

$D$  and each  $\alpha_v, v \in V(D)$  are monochromatic kernel perfect digraphs if and only if  $\sigma(D, \alpha)$  is a monochromatic kernel perfect digraph.

**Proof.** Theorem 3.1 follows directly from Theorem 2.1 and the two following assertions: (1) The disjoint union of monochromatic kernel perfect digraphs is also a monochromatic kernel perfect digraph. (2) Every connected induced subdigraph of  $\sigma(D, \alpha)$  has the form  $\sigma(D', \alpha')$  for a suitable  $D'$  and  $\alpha' = (\alpha'_v)_{v \in V(D')}$  (actually  $D'$  is an induced subdigraph of  $D$  and  $\alpha'_v$  is an induced subdigraph of  $\alpha_v$  for each  $v \in V(D')$ ). ■

**Theorem 3.2.** *Let  $D$  be an arc coloured digraph which has no monochromatic directed cycle,  $B \subset V(D)$  and  $D^B$  the duplication of  $D$  over  $B$ . Then  $D$  is a monochromatic kernel perfect digraph if and only if  $D^B$  is a monochromatic kernel perfect digraph.*

**Proof.** Clearly, an arc coloured digraph  $D$  is a monochromatic kernel perfect digraph if and only if each induced subdigraph of  $D$  is a monochromatic kernel perfect digraph. Thus if  $D^B$  is a monochromatic kernel perfect digraph, then  $D$  is a monochromatic kernel perfect digraph.

Now suppose that  $D$  is a monochromatic kernel perfect digraph and let  $A \subseteq V(D^B)$ . We will prove that  $D^B[A]$  has a kernel by monochromatic paths. Here we consider two possible cases:

*Case 1.*  $A \cap V(B'_D) = \emptyset$ .

In this case,  $A \subseteq V(D^B - V(B'_D))$  and  $D^B[A] \cong D[A]$  and since  $D[A]$  has a kernel by monochromatic paths; it follows that  $D^B[A]$  has a kernel by monochromatic paths.

*Case 2.*  $A \cap V(B'_D) \neq \emptyset$ .

Let  $C' = \{x' \in V(D^B) \mid x' \in A \cap V(B'_D)\}$  and  $E = A - C'$  be, thus  $A = C' \cup E$ .

*Case 2.1.*  $E \cap C = \emptyset$ . (In Which  $C = \psi^{-1}(C')$ ).

In this case, we have  $D^B[E \cup C'] \cong D^B[E \cup C] \cong D[E \cup C]$  and then  $D^B[A] \cong D[E \cup C]$  has a kernel by monochromatic paths.

*Case 2.2.*  $E \cap C \neq \emptyset$ .

It follows from the hypothesis that  $D[E \cup C]$  has a kernel by monochromatic paths, say  $N$ .

When  $N \cap B = \emptyset$  it follows as in Case 1 of the proof of Theorem 2.2 that  $N$  is a kernel by monochromatic paths of  $D^C[E \cup C]$  (the duplication of  $D[E \cup C]$  over  $C$ ); therefore  $N$  is independent by monochromatic paths in  $D^B[E \cup C']$ . Since  $N$  is absorbent by monochromatic paths in  $D[E \cup C]$  it follows that  $N$  is absorbent by monochromatic paths in  $D^B[E \cup C']$ .

(Clearly, to each monochromatic directed path in  $D[E \cup C]$ , say  $T$  there corresponds an unique monochromatic directed path in  $D^B[E \cup C']$ ,  $T'$  obtained from  $T$  by substituting each vertex  $x$  in  $V(T) \cap (C - E)$  for its copy  $x'$  in  $C'$ ).

When  $N \cap B \neq \emptyset$ , we denote by  $Z = N \cap B$ ; we have proved in Case 2 of the proof of Theorem 2.2 that  $N \cup Z'$  is a kernel by monochromatic paths of  $D^C[E \cup C]$  (the duplication of  $D[E \cup C]$  over  $C$ ). So  $N \cup Z'$  is independent by monochromatic paths in  $D^B[E \cup C']$ . Now, let  $z \in (V(D^B[E \cup C']) - (N \cup Z'))$ ; clearly,  $z \in (V(D^C[E \cup C]) - (N \cup Z'))$  and then there exists a  $z \in (N \cup Z')$ -monochromatic directed path, say  $T = (z = x_0, x_1, \dots, x_n)$ ; if  $T \cap (C - E) = \{x_{i_1}, \dots, x_{i_k}\}$  then let  $T'$  be the succession obtained from  $T$  by substituting each  $x_{i_j}$ ,  $1 \leq j \leq k$  for its copy  $x'_{i_j}$  in  $C'$ . Since  $D$  has no monochromatic directed cycles we have  $x'_{i_j} \notin V(T)$  for each  $1 \leq j \leq k$ . Therefore from the definition of  $D^B$  we have that  $T'$  is a monochromatic directed path contained in  $D^B[E \cup C']$  from  $z$  to  $(N \cup Z')$ . We conclude that  $N \cup Z'$  is a kernel by monochromatic paths of  $D^B[E \cup C']$ . ■

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