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HISTORIES IN PATH GRAPHS

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Abstract

For a given graph G and a positive integer r the r-path graph, $P_r(G)$, has for vertices the set of all paths of length r in G. Two vertices are adjacent when the intersection of the corresponding paths forms a path of length $r-1$, and their union forms either a cycle or a path of length $k+1$ in G. Let $P_r^k(G)$ be the k-iteration of r-path graph operator on a connected graph G. Let H be a subgraph of $P_r^k(G)$. The k-history $P_r^{-k}(H)$ is a subgraph of G that is induced by all edges that take part in the recursive definition of H. We present some general properties of k-histories and give a complete characterization of graphs that are k-histories of vertices of 2-path graph operator.

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1. Introduction

Path graphs were introduced by Broersma and Hoede in [4]. Let G be a graph. The vertex set of *path graph* $P_r(G)$ is the set of all paths of length r in G, $r \geq 1$. Two vertices of $P_r(G)$ are adjacent if and only if the intersection of corresponding paths is a path of length $r-1$ and the union is a path or a cycle of length $r+1$. The most frequently studied path graphs are 2-path graphs. Characterization of 2-path graphs is given in [14] and [10]. Traversability of 2-path graphs is studied in [16]. Distance properties of

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2-path graphs are studied in [7, 8] and [9] and the connectivity of path graphs is studied in [2, 5, 6] and [3]. Papers [1] and [11] are devoted to the problem of isomorphism of path graphs. Dynamics of iterated path graphs is discussed in [13] and [15].

The history of a vertex with respect to the line operator was used in [12] to prove the asymptotical behavior of diameter and radius of iterated line graphs. Line graphs could be understood as a special case of r-path graphs with $r = 1$. Histories of vertices of path graphs were used in [7] for the study of diameters in iterated path graphs and in [9] to find an estimation for cardinalities of maximal independent sets in path graphs. The structure of the paper is following. In Section 2 we formulate a definition of k-history of a graph and prove some properties of k-histories with respect to r-path operator for any $r \geq 2$. In Section 3 we completely characterize graphs that are k-histories of vertices in k-iterated 2-paths graphs.

2. Histories in Iterated r-path Graphs

Let G be a graph and v be a vertex of $P_r(G)$. Then the *history* of v, $P_r^{-1}(v)$ is the path of length r in G that corresponds to v. The history of a subgraph H of $P_r(G)$, $P_r^{-1}(H)$ is the graph $\bigcup_{v \in H} P_r^{-1}(v)$. The k-history of $H \subset P_r^k(G)$ is defined recursively as $P_r^{-k}(H) = P_r^{-1}(P_r^{-(k-1)}(H))$. We set $P_r^0(H) = H$.

In other words, the j-history of H is the subgraph of $P_r^{k-j}(G)$, formed by all edges that take part in the recursive definition of H. Hence, while the path operator can be applied to any graph, the history operator is defined on path graphs only.

Example 1. Let p be a path of length rk, then $P_r^k(p)$ is a singleton, it means a graph with a single vertex. Let us denote this vertex by v. Then $P_r^{-j}(v)$ is a graph isomorphic to a path of length jr for any $0 \leq j \leq k$.

Let c be a cycle of length $k \geq r+1$ then $P_r^j(c)$ is isomorphic to a cycle of length k for any $j \geq 0$. Let $v \in P_r^j(c)$ then $P_r^{-i}(v)$ is defined for any $0 \leq i \leq j$. $P_r^{-i}(v)$ is a path of length ir when $ir < k$ otherwise it is a cycle of length k.

Observation 2. *If H', H'' are two subgraphs of $P_r(G)$ with $V(H') = V(H'')$ then $P_r^{-1}(H') = P_r^{-1}(H'')$.*

For simplicity we shall omit the subscript r in the notation of path operator when it is clear from the context. The j-histories satisfy the usual property of powers of operators in the following form.

Lemma 3. *Let G be a graph, $k \geq 1$, and let H be a subgraph of $P^k(G)$.*

1. *Let $1 \leq j \leq k$, then $P^{-1}(P^{-(j-1)}(H)) = P^{-(j-1)}(P^{-1}(H)) = P^{-(j)}(H)$*
2. *Let m, n be integers such that $0 \leq m + n \leq k$. Then $P^{-m}(P^{-n}(H)) = P^{-(m+n)}(H)$.*
3. *Let $1 \leq j \leq k$, then $P^{(-j)}(H) = \bigcup_{v \in H} P_r^{-j}(v)$.*
4. *Let $0 \leq n \leq k$, $0 \leq m$. Then $P^{(m-n)}(H)$ is a subgraph of $P^m(P^{-n}(H))$.*

Proof. Statements 1–4 are direct consequences of definition of path and history operators. We should mention that it is not possible to change the inclusion in property 4 to equality. It is enough to consider the history of a vertex in a cycle (example 1). ■

In [12] it was proved that that k-history of a vertex v in an iterated line graph $L^k(G)$ is a connected graph with at most k edges. We prove analogous results for arbitrary path graphs.

Lemma 4. *Let G be a graph and $r \geq 2, k \geq 1$. If uv is an edge in $P_r^k(G)$ then $P_r^{-k}(u)$ and $P_r^{-k}(v)$ have at least $r-1$ common edges.*

Proof. Induction on k. If $k = 1$ then, since u and v are adjacent, $P_r^{-1}(u)$ and $P_r^{-1}(v)$ are paths of length r with $r - 1 \geq 1$ common edges. Let now the assertion be true for some $k - 1 \geq 1$. Let $u'v'$ be the common edge of $P_r^{-(k-1)}(u)$ and $P_r^{-(k-1)}(v)$. Again, $P_r^{-1}(u')$ and $P_r^{-1}(v')$ are paths of length r with $r - 1 \geq 1$ common edges belonging to both $P_r^{-k}(u)$ and $P_r^{-k}(v)$. ■

Lemma 5. *Let G be a graph and H a connected subgraph of $P_r(G)$ with m vertices. Then $P_r^{-1}(H)$ contains at most $m + r - 1$ edges.*

Proof. We prove the assertion by induction on m. If H contains just one vertex then $P_r^{-1}(H)$ is a path of length r, so the hypothesis is true. Suppose now that the statement holds for any graph consisting of less than m vertices. Let v be a vertex in H, such that $H - v$ is connected. Then the number of edges in $P_r^{-1}((H) - v)$ is at most $(m - 1) + (r - 1)$. The history of v has at most one edge different from edges in the history of any vertex adjacent to v. Therefore, the number of edges in $P_r^{-1}(H)$ is at most $(m + r - 1)$. ■

Lemma 6. *Let G be a graph and v a vertex in $P_r^k(G)$, $k \geq 0, r \geq 2$. Then $P_r^{-k}(v)$ is a connected graph with at most rk edges.*

Proof. First we prove that $P^{-k}(v)$ is connected. We will use induction on k. It is clear that $P_r^{-1}(v)$ is connected. Now let us suppose that the assertion is true for some $k-1 > 1$. Let $P_r^{-1}(v) = a_1a_2 \dots a_r$. Now, following Lemma 3, $P_r^{-k}(v) = P_r^{-1}(P_r^{-(k-1)}(v)) = P_r^{-(k-1)}(P_r^{-1}(v)) = P_r^{-(k-1)}(a_1a_2 \dots a_r)$. Using property 3 we obtain $P_r^{-k}(v) = \bigcup_{i=1}^{r} P_r^{-(k-1)}(a_i)$. By the inductive hypothesis, for $1 \leq i \leq r$, $P_r^{-(k-1)}(a_i)$ is a connected graph. Lemma 4 implies that each pair $P_r^{-k}(a_i), P_r^{-k}(a_{i+1})$where $1 \leq i \leq r-1$, has a common edge. Hence the graph $P_r^{-k}(v)$ is connected.

Now, using induction again, we will prove that $P_r^{-k}(v)$ contains at most rk edges. The assertion is trivial for $k = 0$. Let it be true for $k-1 \geq 0$. Then $P_r^{-(k-1)}(v)$ contains at most $r(k-1)$ edges. Since $P_r^{-(k-1)}(v)$ is connected, it has a spanning tree, which cannot contain more edges. Therefore $P_r^{-(k-1)}(v)$ consists of $r(k-1)+1$ vertices at most. Then, following Lemma 5, $P_r^{-k}(v) = P_r^{-1}(P_r^{-(k-1)}(v))$ contains at most $(r(k-1)+1)+(r-1) = rk$ edges. ∎

Now we can formulate a necessary and sufficient condition for a path to be a k-history of some vertex. From Example 1 and Lemma 6 it follows

Proposition 7. *Let p be a path in graph G such that $P_r^k(G)$ is not empty. Then p is the k-history of some vertex v in $P_r^k(G)$ if and only if the length of p is rk.*

A sequence of vertices $(v_1, v_2, \dots, v_m)$ in graph G is a walk when (v_i, v_{i+1}) is an edge in G for any $0 \leq i \leq m-1$. We call the walk *r-regular* if any $r+1$ consecutive vertices are distinct. In other words any sequence of $r+1$ consecutive vertices in r-regular walk is a path of length r. We say that a walk W covers subgraph H of G if $E(H)$ is equal to the set of all edges in W.

Lemma 8. *Let W be a r-regular walk in graph G of length $k \geq r$. Then there exists a subgraph H of $P_r(G)$ covered by a r-regular walk of length $k-r$ such that $P_r^{-1}(H)$ is formed by all vertices and edges of walk W.*

Proof. If $k = r$ then W is a path of length r and $H = P_r(W)$ is a singleton which is a path of length 0. Let $k > r$. Denote by u_i the vertex in $P_r(G)$ corresponding to the path $(v_i, v_{i+1}, \dots, v_{i+r}), 1 \leq i \leq k-r+1$. Vertices u_i,

u_{i+1} are adjacent in $P_r(G)$ because $(v_i, v_{i+1}, \ldots, v_{i+r+1})$ is a path of length $r+1$ when $v_i \neq v_{i+r+1}$ or a cycle when $v_i = v_{i+r+1}$, respectively. From the r-regularity of W it follows that $v_{i+r+1} \neq v_{i+j}$ for any $1 \leq j \leq r$. Hence vertices $(u_1, u_2, \ldots, u_{k+1-r})$ form a walk W' of length $k-r$. Let H be formed by vertices and edges of W'. Clearly , $P_r^{-1}(H) = W$.

Now we show that W' is r-regular. Suppose that it is not true. Then there exists a vertex $u_i \in W'$ such that $u_i = u_{i+j}$ and $j \leq r$. In this case $(v_i, v_{i+1}, \ldots, v_{i+r}) = (v_{i+j}, v_{i+j+1}, \ldots, v_{i+j+r})$ or $(v_i, v_{i+1}, \ldots, v_{i+r}) = (v_{i+j+r}, v_{i+j+r-1}, \ldots, v_{i+j})$. Both cases contradict to the r-regularity of W, hence W' is r-regular. ■

Corollary 9. *Let H be a subgraph of G such that there exists a r-regular walk W of length kr that covers H. Then there exists a vertex $v \in P_r^k(G)$ such that $H = P_r^{-k}(v)$.*

The minimal degree of a vertex in a graph G is denoted by $\delta(G)$ the size of a shortest cycle in G is called *girth* and we denote it *girth*(G).

Theorem 10. *Let G be a connected graph with $\delta(G) \geq 2$ and girth$(G) > r$ then there exist $k \geq 1$ and $v \in P_r^k(G)$ such that $G = P_r^{-k}(v)$.*

Proof. To prove the statement of the theorem it is enough to construct a r-regular walk covering graph G. We can use a depth-first search strategy to construct a walk. Suppose that each vertex is labelled by an integer that is its order in the process of search. The starting vertex has label 1 and the last-found vertex has the label n where n is the number of vertices in G. We also suppose that the edges of G have orientation. Each edge of the depth-first search tree is oriented from the vertex with the smaller label to the larger one and is called *direct*. All other edges are oriented from the larger value of label to the smaller one and are called *back* edges. The walk W is created by traversing all edges of G. When all edges incident with a vertex are traversed we call it *completed.* We start with vertex 1 and use following rules:

(1) From the current vertex with label i we traverse by a direct edge to a vertex j if the subtree with the root j contains at least one non-completed vertex. The vertex j becomes the current vertex.

(2) If the the current vertex i is a leaf of the depth-first search tree or the subtree with root i has all vertices completed, we traverse a back edge (i, j)

that was not yet used in the previous traversal and j becomes the current vertex.

(3) If in condition (2) the current vertex is already completed and there are still non-completed vertices in subtrees of vertices with the smaller label then i, we traverse the direct edge (j, i) in the opposite direction.

(4) When all vertices are completed we stop the traversal.

When we traverse the direct edge we move always towards non-completed vertices. Because in G are not vertices of degree 1 the last edge of the traversal is a back one.

In the traversal procedure each back edge is traversed exactly once and some direct edges can be repeated in the resulting walk. Because the girth of G is at least $r+1$ the subsequence of W between any two repeated vertices has length at least $r+1$ and the walk is r-regular. When the length of the walk is not divisible by r it is possible to prolong it repeating a part of W in the direct direction (the last edge was a back one). ∎

In the the proof of Theorem 10 we have constructed a r-regular walk that can be arbitrary prolonged. It is enough to repeat a subsequence corresponding to any cycle. It means that if G fulfils the conditions of theorem, then for any $K \geq k$ G is a k-history of some vertex $v \in P_r^K(G)$.

3. Case $r = 2$

In this part we consider histories of path graphs where vertices correspond to paths of length 2.

The path graph of a connected graph G is either connected or consists of one connected component and a set of isolated vertices. The path graph of an isolated vertex is empty, so for construction of iterations $P^i(G)$ we consider main connected components only. Graphs with the infinite sequence of iterations where $P^i(G)$ is not isomorphic to $P^{i+k}(G)$ for any $i, k \geq 1$ are *P-divergent*. Graphs that are not P-divergent are *P-convergent*. From Theorem 10 it follows that any connected graph without pendant vertices is a k-history of a vertex if $k \geq k_0$ for some $k_0 \geq 1$. So it is enough to study graphs that contain pendant vertices. In [15] it was proved that the sequence $G, P^1(G), P^2(G), \ldots$ is finite only if G is a tree and does not contain any of the graphs G_0 or G_j from Figure 1 as subgraphs. The parameter j of G_j is the distance between vertices u and v.

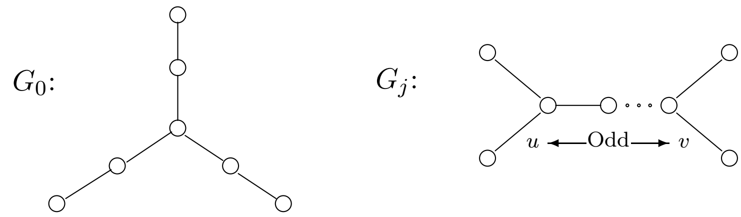

Figure 1. Convergent graphs

A tree T is called a *caterpillar* if it consists of a diametric path of length d and some pendant vertices that are adjacent to vertices of this path. Vertices of degree at least 3 we shall call *root* vertices or simply *roots*. If the distance between any two roots is even, we call the caterpillar *even*. Trees that contain neither G_0 nor G_j are either paths or even caterpillars. The star $K_{1,n}$ is a special case of even caterpillar with diametric path of length $d = 2$. Path graph of a star is a set of isolated vertices so the star can not be a k-history of a single vertex.

For all other connected graphs the sequence of iterated path-graphs is infinite. The only non-trivial P-convergent graphs are cycles, G_0 and G_j, where j is odd [15]. From Example 1 it is clear that the cycle C_d is k-history of a vertex for any $k \geq d/2$.

When the sequence of iterated path graphs converges to the empty graph, the possible value of the parameter k is bounded by the index of the last non-empty iteration. So first we concentrate on even caterpillars. The path joining any pendant vertex with a closest root vertex will be called an *end*. The parity of an end is the parity of its length. We prove that an even caterpillar with at most 2 odd ends is a k-history of some vertex. First we prove the next lemma.

Lemma 11. *Let T be an even caterpillar with e odd ends where $e = 1$ or $e = 2$, and k edges. Then there exists a graph H such that $T = P^{-1}(H)$,H has $k - e$ edges, and H is a path or a caterpillar with two odd ends.*

Proof. Let d be the length of the diametric path $(u_0, u_1, \ldots, u_d)$ in T. We consider two cases.

(a) Let T have two even ends of lengths $d_1 \geq d_2$. In this case all other ends are of length 1 and these ends are pendant vertices on the diametrical path. When $d_1 = d_2 = 2$, pendant vertices u and v are adjacent to u_2

and u_{j+2} where j is the distance between root vertices. In this case graph H is induced by vertices $(u, u_2, u_1), (v, u_{j+2}, u_{j+3})$ and (u_i, u_{i+1}, u_{i+2}) where $0 \leq i \leq d-2$. Graph H is a path of length d. When T has only one odd end then $j = 0$ and we set $u = v$. The constructed graph H is then also a path of length d.

Now let $d_1 > 2$ and the numbering of the diametrical path starts from the end of length d_1. Let the odd ends be $(u, u_{d_1}), (v, u_{d_1+j})$ then H is induced by vertices (u, u_{d_1}, u_{d_1-1}) and $(v, u_{d_1+j}, u_{d_1+j-1})$ and (u_i, u_{i+1}, u_{i+2}) where $0 \leq i \leq d-2$. The resulting graph is a caterpillar with two pendant vertices and diametric path with $d-2$ edges. When T has just one odd end we again set $u = v$ and the resulting graph is again a caterpillar with two pendant vertices and diametric path with $d-2$ edges.

(b) Let T have one even end and two odd ends. Let d_1 be the length of the even end and d_2 be the length of the odd end of the diametric path. Let (u, u_{d_1}) be the pendant edge. We suppose that vertices of the diametric path are numbered starting from the even end. Graph H is induced by vertices (u, u_{d_1}, u_{d_1-1}) and (u_i, u_{i+1}, u_{i+2}) where $0 \leq i \leq d-2$. When $d_1 > 2$ then graph H is a caterpillar with the diametric path of length $d-2$ and one pendant edge, otherwise it is a path of length $d-1$. Because there are no other possible cases, the proof is complete. ■

Now we prove that each caterpillar with at most 2 odd ends is a k-history of a vertex for some value $k \geq 1$. Let T be an even caterpillar with $2k$ or $2k-1$ edges and 2 or 1 odd ends, respectively. Using the construction from Lemma 11 we create the sequence $(T_0, T_1, \ldots, T_k)$ where $T = T_0$, $P^{-1}(T_{i+1}) = T_i$ and T_k is a single vertex. When T_i is a caterpillar then we use the construction from the lemma. When T_i is a path of length $2d$ and $d > 1$ then T_{i+1} is a path of length $2d-2$ otherwise $d = 1$ and T_{i+1} is a vertex. T_i is a subgraph of $P^i(T)$, hence $P^{-i}(T_i) = T$ and vertex T_k is the k-history of T.

Proposition 12. *Let T be an even caterpillar with $2k$ edges and two odd ends or with $2k-1$ edges and one odd end. Then there exists a vertex $v \in P^k(T)$ such that v is the k-history of T.*

Now we show that an even caterpillar with at least three odd ends is not a k-history of any vertex for any value of k. For this purpose we define a special class of graphs that contains even caterpillars as a subclass. Let G be a connected bipartite graph with partitions A and B where all vertices in

A have degrees 1 or 2, A has at least 1 vertex of degree 1, and B has at least 2 vertices. We call this graph 1-2-*bipartite*. When T is an even caterpillar, then pendant vertices of odd ends are placed to set A. Because the distance from a pendant vertex of an odd end to the closest root is odd and distance between any to roots is even, all roots are in the set B.

Lemma 13. *Let G be a 1-2-bipartite graph. Then the main connected component of $P(G)$ is a 1-2-bipartite graph or a star $K_{1,n}$.*

Proof. Let $A = a_1, a_2 \dots a_p$ and $B = b_1, b_2 \dots b_r$ be partitions of 1-2-bipartite graph. Let the set A' be the subset of vertices in $P(G)$ that contains all vertices (a_i, b_j, a_k) and let the set B' contains all vertices (b_i, a_j, b_k). No two vertices in B' are adjacent and no two vertices in A' are adjacent. Hence $P(G)$ is bipartite. The degree of vertex (a_i, b_j, a_k) in $P(G)$ is $deg(a_i) + deg(a_k) - 2$, hence all vertices in A' are of the degree at most 2. When B' has just one vertex, the main component of $P(G)$ is a star otherwise it is a 1-2-bipartite graph. ■

Lemma 14. *Let G be a 1-2-bipartite graph with partition (A, B) and a be a pendant vertex in A, then each vertex u from the main component of $P(G)$ such that $a \in P^{-1}(u)$ has degree 1 and its history does not contain other pendant vertex.*

Proof. The path corresponding to vertex u is (a, b_i, a_j) and $deg(u) = deg(a) + deg(a_j) - 2$. The degree of vertex a_j is 2 because u is not isolated. ■

Proposition 15. *Let G be a 1-2-bipartite graph with at least 3 pendant vertices in the set A, then G is not a k-history of any vertex $v \in P^k(G)$. For any $k \geq 1$.*

Proof. Suppose that there exists $v \in P^k(G)$ such that $P^{-k}(v) = G$. From Lemmas 13 and 14 it follows that each H such that $P^{-j}(H) = G$ contains at least 3 pendant vertices. This is a contradiction with the number of pendant vertices of $P^{-1}(v)$. ■

The characterization of convergent graphs that are k-histories of vertices follows from the Propositions 7, 12 and 15. We shall prove that the only divergent graphs, that are not k-histories of some vertices, are 1-2-bipartite with at least 3 odd ends. First we formulate some technical lemmas.

Lemma 16. *Let G be a union of an induced path W of length $d \geq 2$ and a cycle C such that W is rooted at a vertex of C. If d is even, then there exists a cycle $C' \subseteq P^{d/2}(G)$ such that $P^{-d/2}(C') = G$. If d is odd, then there exists a cycle C' with a pendant edge e such that $P^{-(d-1)/2}(e \cup C') = G$.*

Proof. Let $W = (a_0, a_1, \ldots, a_d)$ and $C = (c_0, c_1, \ldots, c_{k-1})$ where $a_0 = c_0 = c_k$. Let G' be the subgraph of $P(G)$ induced by vertices (a_1, c_0, c_1), (a_1, c_k, c_{k-1}), (c_i, c_{i+1}, c_{i+2}), $0 \leq i \leq k-2$, and all vertices of $P(W)$. G' is the union of a cycle and a path of length $d-2$, and $P^{-1}(G') = G$. Repeating the above construction we get the statement of the lemma. ∎

Lemma 17. *Let G be a cycle of length $2d-1$ with one pendant edge. Then in $P^d(G)$ there exists a cycle C of length $2d+1$ such that $P^{-d}(C) = G$.*

Proof. It is easy to see that it is possible to construct a sequence of graphs $G = H_0, H_1, H_2, \ldots, H_{d-1}$, such that H_i is a subgraph of $P(H_{i-1})$, H_i is a cycle with two pendant edges such that the roots of pendant edges divide the cycle into paths of lengths $2i$ and $(2d-1-2i)$, and $P^{-1}(Hi) = H_{i-1}$. So H_{d-1} has two pendant edges rooted in adjacent vertices of a cycle of length $2d-1$. $P(H_{d-1})$ contains exactly one cycle C of length $2d+1$ and all edges of H_{d-1} are in its history. From the above construction it follows that $P^{-d}(C) = G$. ∎

Lemma 18. *Let G be a caterpillar with two root vertices u and v with odd distance j and four ends, three of length 1 and one of length d. Let us denote this caterpillar $G_{j,d}$. Then there exists $m \geq 1$ and a cycle C in $P^m(G)$, such that $P^{-m}(C) = G$.*

Proof. If $j = 1, d = 1$ then the main component of $P(G_{1,1})$ is a cycle C and $P^{-1}(C) = G$.

Let $d > 1$ and $j > 1$. We construct a sequence of graphs $H_1, H_2, \ldots, H_{(j-1)/2}$. The main component of $P(G)$ is a caterpillar $G_{j-2,d}$ with one more pendant vertex. We set graph H_1 to be equal to caterpillar $G_{j-2,d}$. It is clear that $P^{-1}(H_1) = G$. The construction of H_i from H_{i-1} is the same as described above. We should note, that $P^{-1}(H_{i+1}) = H_i$, $P^{-(j-1)/2}(H_{(j-1)/2}) = G$ and $H_{(j-1)/2} = G_{1,d}$. When $d = 1$ assertion of the lemma follows. When $d = 2$ then the main component of $P(H_{(j-1)/2})$ is a cycle with pendant path of length 2 and assertion follows from Lemma 16.

Suppose that $d \geq 3$, then the main component of $P(H_{(j-1)/2})$ is a union of a cycle $C = (c_0, c_1, c_2, c_3)$ and a caterpillar with the diametric path

$W = (c_0, a_1, \ldots, a_{d-1})$ and pendant edge (b, a_1). Deleting the vertex c_2 from the main component we obtain graph $H' = G_{1,d-1}$. As $P^{-1}(H') = H_{(j-1)/2}$, repeating this construction we finish with a caterpillar $G_{1,2}$, hence the proof is complete. ■

Proposition 19. *Let G be a connected P-divergent graph that is not* 1-2-*bipartite, then there exists $k \geq 1$ and $v \in P^k(G)$ such that $P^{-k}(v) = G$.*

Proof. Let G be a P-divergent graph different from 1-2-bipartite. When G is not bipartite, then it contains an odd cycle C_{odd}. Let v be a pendant vertex of G. There is a path joining v to C_{odd} and by Lemma 16 and 17 there exists $k_v \geq 1$ and a cycle C'_{odd} in $P^{k_v}(G)$ such that the k_v-history of C'_{odd} contains the pendant path of vertex v. We apply this procedure to all pendant paths and obtain a graph G' without pendant vertices such that $P^{-k}(G') = G$. By Theorem 10 the assertion follows.

When G is P-divergent bipartite but not 1-2-bipartite, then it contains at least two vertices of degree ≥ 3 from different partitions. It means that G contains a caterpillar G_j with j odd. Let v be a pendant vertex in G, then there exists a path from v to a vertex u of G_j. Let the length of this path be d. When u is a pendant vertex of G_j then v is a pendant vertex of caterpillar $G_{d+1,j}$. By Lemma 18 there exists k_v such that this caterpillar is included in the k_v-history of some cycle.

Let now u be a vertex on the path between the root vertices x and y of G_j. One of the paths $(u - x)$ or $(u - y)$ has odd length i so v is a pendant vertex of a caterpillar $G_{i,d}$ and by Lemma 18 there is a cycle such that the pendant path is included in its k_v-history. When we apply this procedure to all pendant paths, we create G' without pendant vertices and by Theorem 10 the assertion follows. ■

The last remaining type of graphs are 1-2-bipartite P-divergent graphs with at most 2 odd ends. A connected 1-2-bipartite graph is divergent only if it contains a copy of graph G_0 or a cycle (of even length). In both cases $P(G)$ contains at least one cycle. From Lemma 16 it follows that there exists a $k_0 \geq 0$ and a subgraph H_0 of $P^{k_0}(G)$ such that H_0 has no even ends, $P^{-k_0}(H_0) = G$ and H_0 has the same number of odd ends as G. So H_0 has at most two pendant vertices. It is easy to see that there exist a 2-regular walk that starts in the first pendant vertex, traverse all other vertices of H_0 and ends in the second pendant vertex. When G has only one odd end, the first and the last vertex of the walk is the same. From Corollary 9 it follows that G is a k-history of some vertex.

Now we can formulate a characterization of k-histories for 2-path graphs.

Theorem 20. *A graph G is a k-history of some vertex in $P_2^k(G)$ for some $k \geq 0$ if and only if G is a connected graph different from a path of odd length, from a star $K_{1,r}$ where $r \geq 3$ and from a 1-2-bipartite graph with at least 3 odd ends.*

References

[1] R.E.L. Aldred, M.N. Ellingham, R. Hemminger and P. Jipsen, *P_3-isomorphism for graphs*, J. Graph Theory **26** (1997) 35–51.

[2] C. Balbuena and D. Ferrero, *Edge-connectivity and super edge-connectivity of P_2-path graphs*, Discrete Math. 269 (2003) 13–20.

[3] C. Balbuena and P. García-Vázquez, *Edge-connectivity in P_k-path graphs*, Discrete Math. **286** (2004) 213–218.

[4] H.J. Broersma and C. Hoede, *Path graphs*, J. Graph Theory **13** (1989) 427–444.

[5] D. Ferrero, *Edge connectivity of iterated P_3-path graphs*, Congr. Numer. **155** (2002) 33–47.

[6] D. Ferrero, *Connectivity of path graphs*, Acta Mathematica Universitatis Comenianae **LXXII** (1) (2003) 59–66.

[7] M. Knor and Ľ. Niepel, *Diameter in iterated path graphs*, Discrete Math. **233** (2001) 151–161.

[8] M. Knor and Ľ. Niepel, *Centers in path graphs*, JCISS **24** (1999) 79–86.

[9] M. Knor and Ľ. Niepel, *Independence number in path graphs*, Computing and Informatics **23** (2004) 179–187.

[10] H. Li and Y. Lin, *On the characterization of path graphs*, J. Graph Theory **17** (1993) 463–466.

[11] X. Li, *Isomorphism of P_3-graphs*, J. Graph Theory **21** (1996) 81–85.

[12] Ľ. Niepel, M. Knor and Ľ. Šoltés, *Distances in iterated line graphs*, Ars Combin. **43** (1996) 193–202.

[13] E. Prisner, Graph dynamics, Pitman Research Notes in Mathematics Series Vol 338 (Longman, Essex, 1995).

[14] E. Prisner, *Recognizing k-path graphs*, Discrete Appl. Math. **99** (2000) 169–181.

[15] E. Prisner, *The dynamics of the line and path graph operators*, Discrete Math. **103** (1992) 199–207.

[16] X. Yu, *Trees and unicyclic graphs with hamiltonian path graphs*, J. Graph Theory **14** (1990) 705–708.

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