

## ON DISTANCE LOCAL CONNECTIVITY AND VERTEX DISTANCE COLOURING

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### Abstract

In this paper, we give some sufficient conditions for distance local connectivity of a graph, and a degree condition for local connectivity of a  $k$ -connected graph with large diameter. We study some relationships between  $t$ -distance chromatic number and distance local connectivity of a graph and give an upper bound on the  $t$ -distance chromatic number of a  $k$ -connected graph with diameter  $d$ .

**Keywords:** degree condition, distance local connectivity, distance chromatic number.

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### 1. INTRODUCTION

By a graph we mean a simple undirected graph. We use [2] for terminology and notation not defined here. Let  $\text{dist}_G(x, y)$  denote the distance between vertices  $x$  and  $y$  in  $G$ . An  $x, y$ -path is a path between vertices  $x$  and  $y$  in  $G$ . Let  $d = \max \text{dist}_G(xy) : x, y \in V(G)$  denote the diameter of  $G$ . An  $x, y$ -path  $P$  is called *diameter-path*, if  $\text{dist}_G(x, y) = d$  and  $|E(P)| = d$ . Let  $d_G(x)$  denote the degree of a vertex  $x$  in  $G$ ,  $\delta(G)$  the minimum degree of  $G$  and  $\Delta(G)$  the maximum degree of  $G$ . For a nonempty set  $U \subseteq V(G)$ , the induced subgraph on  $U$  is denoted by  $\langle U \rangle$ . For a nonempty

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set  $A \subset V(G)$ ,  $G - A$  denotes the subgraph of  $G$  that we obtain by deleting all vertices of  $A$  and all edges adjacent to at least one vertex of  $A$ . Let  $\sigma_k(G) = \min\{\sum_{i=1}^k d_G(x_i) \mid \{x_1, \dots, x_k\} \subset V(G), \text{ independent}\}$ . The *square* of a graph  $G$ , denoted by  $G^2$ , is the graph in which  $V(G^2) = V(G)$  and  $E(G^2) = E(G) \cup \{\{u, v\} \mid \text{dist}_G(u, v) = 2\}$ .

Let  $N_G(x) = \{y \in V(G), xy \in E(G)\}$ , let  $N_G[x] = N_G(x) \cup \{x\}$ . The set  $N_G(x)$  is called the *neighbourhood* of the first type of  $x$  in  $G$ . We say that  $x$  is a *locally connected vertex* of  $G$ , if  $\langle N_G(x) \rangle$  is connected. We say that  $G$  is a *locally connected graph*, if every vertex of  $G$  is locally connected. Chartrand and Pippert [3] proved the following Ore-type condition for local connectivity of graphs:

**Theorem A** [3]. *Let  $G$  be a connected graph of order  $n$ . If*

$$d_G(u) + d_G(v) > \frac{4}{3}(n - 1)$$

*for every pair of vertices  $u, v \in V(G)$ , then  $G$  is locally connected.*

Let  $N_2(x)$  be a subgraph induced by the set of edges  $uv$ , such that

$$\min\{\text{dist}_G(x, u), \text{dist}_G(x, v)\} = 1.$$

The subgraph  $N_2(x)$  is called the *neighbourhood of the second type* of  $x$  in  $G$ . We say that  $x$  is an  *$N_2$ -locally connected vertex* of  $G$ , if  $N_2(x)$  is connected. We say that  $G$  is  *$N_2$ -locally connected*, if every vertex of  $G$  is  $N_2$ -locally connected.

Now define the distance neighbourhood of the first type of a vertex of  $G$  as in [5]. Let  $m$  be a positive integer and let  $x$  be an arbitrary vertex of a graph  $G$ . The  $N_1^m$ -neighbourhood of  $x$  in  $G$ , denoted by  $N_1^m(x)$ , is the set of all vertices  $y \in V(G)$ ,  $y \neq x$ , such that  $\text{dist}_G(x, y) \leq m$ . Let  $N_1^m[x] = N_1^m(x) \cup \{x\}$ . A vertex  $x$  is called  $N_1^m$ -locally connected if  $\langle N_1^m(x) \rangle$  is connected. A graph  $G$  is said to be  $N_1^m$ -locally connected if every vertex of  $G$  is  $N_1^m$ -locally connected.

The distance local connectivity of the second type is analogously defined as the neighbourhood of the second type. Let  $m$  be a positive integer and let  $x$  be an arbitrary vertex of a graph  $G$ . The  $N_2^m$ -neighbourhood of  $x$ , denoted by  $N_2^m(x)$ , is the subgraph induced by all edges  $\{u, v\}$  of  $G$ ,  $u \neq x$ ,  $v \neq x$ , with  $\min\{\text{dist}_G(x, u), \text{dist}_G(x, v)\} \leq m$ . We say that  $x$  is  $N_2^m$ -locally connected in  $G$  if  $N_2^m(x)$  is connected. A graph  $G$  is said to be  $N_2^m$ -locally connected if every vertex of  $G$  is  $N_2^m$ -locally connected in  $G$ .

Let  $t$  be a positive integer. The  $t$ -distance chromatic number of a graph  $G$ , denoted  $\chi^{(t)}(G)$ , is the minimum number of colours required to colour all vertices of  $G$  in such a way that any two vertices  $x, y$  with  $\text{dist}_G(x, y) \leq t$  have distinct colours. Let  $\chi(x)$  denote the colour of a vertex  $x$  in  $G$ . Recall that the vertex distance colouring was introduced by Kramer and Kramer in [7] and [8]. In the 90's, several results on vertex distance colourings were presented, cf. Baldi in [1], Skupień in [11], Chen *et al.* in [4].

The following result was proved by Jendrol' and Skupień in [6].

**Theorem B** [6]. *Given a planar graph  $G$ , let  $D = \max\{8, \Delta(G)\}$ . Then the  $t$ -distance chromatic number of  $G$  is*

$$\chi^{(t)}(G) \leq 6 + \frac{3D + 3}{D - 2}((D - 1)^{t-1} - 1).$$

Madaras and Marcinová strengthened this condition in [9].

**Theorem C** [9]. *Let  $G$  be a planar graph, let  $D = \max\{8, \Delta(G)\}$ . Then*

$$\chi^{(t)}(G) \leq 6 + \frac{2D + 12}{D - 2}((D - 1)^{t-1} - 1).$$

## 2. DISTANCE LOCAL CONNECTIVITY OF A GRAPH IN $k$ -CONNECTED GRAPHS

The concept of the local connectivity of a graph was introduced in 1970's. Ryjáček used the concept of the local connectivity of a vertex in [10] for local completing in his closure concept for claw-free graphs. This closure concept gave a solution for several hamiltonian problems. A degree condition is one of the easily verified conditions. Chartrand and Pippert in [3] proved a degree condition for the local connectivity of connected graphs (see Theorem A). The same degree condition can guarantee the local connectivity of any vertex of a connected locally connected graph. In this chapter, degree conditions for the local connectivity of a  $k$ -connected graph with a large diameter will be presented as a strengthening of the result of Chartrand and Pippert. Holub and Xiong in [5] proved degree conditions for distance local connectivity of 2-connected graphs. As a strengthening of this condition, degree conditions for distance local connectivity of a  $k$ -connected graph with a large diameter will be shown, too.

**Theorem 1.** *Let  $k \geq 2$  be an integer,  $G$  be a  $k$ -connected graph of order  $n$ . Let  $d$  be the diameter of  $G$ , let  $d \geq 5$ . If*

$$d_G(u) + d_G(v) > \frac{4}{3}(n - kd + 5k - 3)$$

*for every pair of vertices  $u, v \in V(G)$ , then  $G$  is locally connected.*

**Theorem 2.** *Let  $k \geq 2$  be an integer,  $G$  be a  $k$ -connected graph of order  $n$ . Let  $d$  be the diameter of  $G$ ,  $m$  be an integer such that  $2 \leq m \leq \frac{1}{2}(d - 7)$ . If*

$$(1) \sigma_t \geq n - kd + 2mk + 6k - t, \quad \text{where } t = \frac{2}{3}m + 1 \quad \text{if } m \equiv 0 \pmod{3},$$

$$(2) \sigma_t \geq n - kd + 2mk + 6k - 2 - t, \quad \text{where } t = \frac{2}{3}(m - 1) + 3 \quad \text{if } m \equiv 1 \pmod{3},$$

$$(3) \sigma_t \geq n - kd + 2mk + 4k - 1 - t, \quad \text{where } t = \frac{2}{3}(m - 2) + 3 \quad \text{if } m \equiv 2 \pmod{3}.$$

*then  $G$  is  $N_1^m$ -locally connected.*

Before proofs of these two theorems, some auxiliary statements will be shown.

**Lemma 1.** *Let  $k \geq 2$  be an integer,  $G$  be a  $k$ -connected graph and  $x$  be an arbitrary vertex of  $G$ . Let  $d$  be the diameter of  $G$ , let  $d \geq 5$ . If  $x$  does not belong to any diameter-path in  $G$ , then there are at least  $kd - 5k + 2$  vertices  $y$  such that  $\text{dist}_G(x, y) > 2$ .*

**Proof.** Let  $P$  denote a diameter-path in  $G$ , let  $u, v$  be the end vertices of  $P$ . Since  $G$  is  $k$ -connected, there are at least  $k$  vertex-disjoint  $u, v$ -paths in  $G$  by Menger's theorem. Choose  $P_1, \dots, P_k$  with minimum sum of their lengths. Note that  $|E(P_i)| \geq d$ ,  $i = 1, \dots, k$ . Now it will be shown that there are at least  $d - 3$  vertices at the required distance from  $x$  on each of  $P_i$ ,  $i = 1, \dots, k$ . Let  $M_j = \{y \in P_j \mid \text{dist}_G(x, y) \leq 2\}$ ,  $j = 1, \dots, k$ . For each path of  $P_i$ ,  $i = 1, \dots, k$ , there are two following cases:

*Case 1.* If  $M_j = \emptyset$ , then there are at least  $d + 1$  vertices at the required distance from  $x$  on  $P_j$ .

*Case 2.* If  $M_j \neq \emptyset$ , then let  $a_j \in M_j$  such that  $\text{dist}_G(a_j, u) = \min_{m \in M_j} \text{dist}_G(m, u)$  and let  $b_j \in M_j$  such that  $\text{dist}_G(b_j, v) = \min_{m \in M_j} \text{dist}_G(m, v)$ . Since  $x$  does not belong to any diameter path, we have

$$\text{dist}_G(u, a_j) + \text{dist}_G(a_j, x) + \text{dist}_G(x, b_j) + \text{dist}_G(b_j, v) \geq d + 1.$$

Since  $\text{dist}_G(a_j, x) \leq 2$  and  $\text{dist}_G(b_j, x) \leq 2$ , we obtain

$$\text{dist}_G(u, a_j) + \text{dist}_G(b_j, v) \geq d - 3.$$

Hence there are at least  $d - 3$  vertices at the required distance from  $x$  on  $P_j$ .

On the paths  $P_i$ ,  $i = 1, \dots, k$ , there are at least  $k(d - 3)$  vertices at the required distance from  $x$  in  $G$ . Since  $u$  and  $v$  can be counted only once, there are at least  $k(d - 5) + 2$  different vertices at the required distance from  $x$  in  $G$ . ■

**Proof of Theorem 1.** Suppose  $G$  is not locally connected. Then there is a vertex  $x$  such that  $x$  is not locally connected in  $G$ . There are at least two components of  $\langle N_G(x) \rangle$ . Let  $G_1$  denote a smallest component of  $\langle N_G(x) \rangle$  and let  $G_2$  be the union of all the other components of  $\langle N_G(x) \rangle$ . Let  $g_1 = |V(G_1)|$ , let  $g_2 = |V(G_2)|$ . Let  $Z = \{y \in V(G); \text{dist}_G(x, y) = 2\}$ , let  $z = |Z|$ . Let  $p = |\{y \in V(G); \text{dist}_G(x, y) > 2\}|$ .

*Case 1.* Suppose that  $x$  does not belong to any diameter-path in  $G$ . By Lemma 1, the number  $p \geq kd - 5k + 2$ . Clearly  $n = g_1 + g_2 + z + p + 1$ . Choose arbitrary vertices  $u$  and  $v$  such that  $u \in V(G_1)$  and  $v \in V(G_2)$ . By the assumptions of Theorem 1

$$d_G(x) + d_G(u) > \frac{4}{3}(n - kd + 5k - 3).$$

Since  $d_G(x) = g_1 + g_2$  and  $d_G(u) \leq g_1 + z = n - 1 - p - g_2 \leq n - 1 - g_2 - kd + 5k - 2$ , we obtain

$$g_1 + g_2 + n - g_2 - 1 - kd + 5k - 2 > \frac{4}{3}(n - kd + 5k - 3).$$

Clearly  $g_1 > \frac{1}{3}(n - kd + 5k - 3)$  and  $g_2 > \frac{1}{3}(n - kd + 5k - 3)$  since  $g_2 \geq g_1$ . Therefore

$$z < \frac{1}{3}(n - kd + 5k - 3).$$

For vertices  $u$  and  $v$

$$d_G(u) + d_G(v) \leq g_1 + z + g_2 + z < \frac{4}{3}(n - kd + 5k - 3),$$

a contradiction.

*Case 2.* Suppose that  $x$  belongs to a diameter-path  $P$ . Let  $e, f$  be the end vertices of  $P$ . Since  $G$  is  $k$ -connected, there are at least  $k$  vertex-disjoint  $e, f$ -paths in  $G$ . Choose  $P_1, \dots, P_k$  with a minimum sum of their lengths. For each of  $P_i, i = 1, \dots, k$  the following cases can happen.

*Subcase 2.1.*  $V(P_i) \cap Z = \emptyset$ . Then there are at least  $d + 1$  vertices on  $P_i$  at distance at least 3 from  $x$  in  $G$ .

*Subcase 2.2.*  $V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset$ , but  $V(P_i) \cap Z \neq \emptyset$ . Let  $d_i = |V(P_i) \cap Z|$ . If  $d_i \leq 4$ , then there are at least  $d - 3$  vertices on  $P_i$  at distance at least 3 from  $x$  in  $G$ .

Now suppose that  $d_i \geq 5$ . If there is a vertex  $w \in V(G_1) \cup V(G_2)$  such that  $w$  is adjacent to every vertex of  $V(P_i) \cap Z$ , then there are at least  $d - 2$  vertices at distance at least 3 from  $x$  in  $G$  since  $\text{dist}_G(e, f) \geq d$ . If none of the vertices of  $V(G_1) \cup V(G_2)$  is adjacent to every vertex of  $V(P_i) \cap Z$ , then

$$\begin{aligned} d_G(u) &\leq g_1 + z - (d_i - 3) \leq g_1 + z - 1, \quad \forall u \in V(G_1), \\ d_G(v) &\leq g_2 + z - (d_i - 3) \leq g_2 + z - 1, \quad \forall v \in V(G_2). \end{aligned}$$

*Subcase 2.3.*  $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset$ . Let  $d_i^1 = |V(P_i) \cap V(G_1)|$ ,  $d_i^2 = |V(P_i) \cap V(G_2)|$  and  $d_i = |V(P_i) \cap Z|$ . Note that  $d_i \geq 2$ . The following two possibilities have to be considered.

(i)  $d_i^1 = 0$  or  $d_i^2 = 0$ . Up to symmetry, suppose that  $d_i^2 = 0$ . If  $d_i^1 = 1$  and  $d_i = 2$ , then there are at least  $d - 3$  vertices on  $P_i$  at distance at least 3 from  $x$  in  $G$ .

Now suppose that  $d_i^1 = 1$  and  $d_i > 2$ . If there is a vertex  $w \in V(G_1)$  such that  $w$  is adjacent to every vertex of  $V(P_i) \cap Z$ , then there are at least  $d - 2$  vertices at distance at least 3 from  $x$  in  $G$  since  $\text{dist}_G(e, f) \geq d$ . If there is no vertex  $w \in V(G_1)$  adjacent to every vertex of  $V(P_i) \cap Z$ , then

$$d_G(u) \leq g_1 + z - (d_i - 2) \leq g_1 + z - 1, \quad \forall u \in V(G_1).$$

Now suppose that  $d_i^1 > 1$ . If there is a vertex  $w \in V(G_1)$  such that  $w$  is adjacent to every vertex of  $V(P_i) \cap Z$ , then there are at least  $d - 2$  vertices at distance at least 3 from  $x$  in  $G$  since  $\text{dist}_G(e, f) \geq d$ . If there is no vertex  $w \in V(G_1)$  adjacent to every vertex of  $V(P_i) \cap Z$ , then

$$d_G(u) \leq g_1 + z - (d_i^1 - 2) - (d_i - 1) \leq g_1 + z - 1, \quad \forall u \in V(G_1).$$

(ii)  $d_i^1 > 0$  and  $d_i^2 > 0$ . If  $P_i$  is a diameter-path containing  $x$ , then there are at least  $d-4$  vertices on  $P_i$  at distance at least 3 from  $x$  in  $G$ . If  $P_i$  does not contain  $x$ , then  $d_i \geq 3$ . If there is a vertex  $w \in V(G_1) \cup V(G_2)$  such that  $w$  is adjacent to every vertex of  $V(P_i) \cap Z$ , then there are at least  $d-2$  vertices at distance at least 3 from  $x$  in  $G$  since  $\text{dist}_G(e, f) \geq d$ . If there is no vertex  $w \in V(G_1) \cup V(G_2)$  adjacent to every vertex of  $V(P_i) \cap Z$ , then

$$d_G(u) \leq g_1 + z - (d_i - 2) \leq g_1 + z - 1, \quad \forall u \in V(G_1),$$

$$d_G(v) \leq g_2 + z - (d_i - 2) \leq g_2 + z - 1, \quad \forall v \in V(G_2).$$

Let  $l_1$  denote the number of such the paths  $P_1, \dots, P_k$ , for which one of the following conditions is satisfied

- $V(P_i) \cap V(Z) \neq \emptyset, V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset, d_i \geq 5$  and there is no vertex  $w \in V(G_1) \cup V(G_2)$  adjacent to every vertex of  $V(P_i) \cap Z$ ,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 = 1, d_i^2 = 0, d_i > 2$  and there is no vertex  $w \in V(G_1)$  adjacent to every vertex of  $V(P_i) \cap Z$ ,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 > 1, d_i^2 = 0$  and there is no vertex  $w \in V(G_1)$  adjacent to every vertex of  $V(P_i) \cap Z$ ,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 d_i^2 \neq 0, x \notin V(P_i)$  and there is no vertex  $w \in V(G_1) \cup V(G_2)$  adjacent to every vertex of  $V(P_i) \cap Z$ .

Let  $l_2$  denote the number of such the paths  $P_1, \dots, P_k$ , for which one of the following conditions is satisfied

- $V(P_i) \cap V(Z) \neq \emptyset, V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset, d_i \geq 5$  and there is no vertex  $w \in V(G_1) \cup V(G_2)$  adjacent to every vertex of  $V(P_i) \cap Z$ ,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 = 0, d_i^2 = 1, d_i > 2$  and there is no vertex  $w \in V(G_2)$  adjacent to every vertex of  $V(P_i) \cap Z$ ,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^2 > 1, d_i^1 = 0$  and there is no vertex  $w \in V(G_2)$  adjacent to every vertex of  $V(P_i) \cap Z$ ,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 d_i^2 \neq 0, x \notin V(P_i)$  and there is no vertex  $w \in V(G_1) \cup V(G_2)$  adjacent to every vertex of  $V(P_i) \cap Z$ .

Let  $l = l_1 + l_2$ . Then there are at least  $kd - 5k + 2 - l - 1$  vertices at distance at least 3 from  $x$  in  $G$  and

$$d_G(u) \leq g_1 + z - l_1, \quad \forall u \in V(G_1),$$

$$d_G(v) \leq g_2 + z - l_2, \quad \forall v \in V(G_2).$$

Suppose that  $l_2 \geq l_1$ . By the assumptions, for every  $u \in V(G_1)$

$$d_G(x) + d_G(u) > \frac{4}{3}(n - kd + 5k - 3).$$

Since  $d_G(x) = g_1 + g_2$  and  $d_G(u) \leq g_1 + z - l_1 \leq n - 1 - g_2 - l_1 - kd + 5k - 2 + l$ , we have

$$g_1 + g_2 + n - g_2 - l_1 - kd + 5k - 3 + l > \frac{4}{3}(n - kd + 5k - 3).$$

Clearly

$$g_1 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l \text{ and}$$

$$g_2 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l,$$

since  $g_2 \geq g_1$ . Thus

$$z < \frac{1}{3}(n - kd + 5k - 3) + 2l_1 - l.$$

For vertices  $u$  and  $v$ , it holds that

$$d_G(u) + d_G(v) \leq g_1 + g_2 + 2z - l_1 - l_2 < \frac{4}{3}(n - kd + 5k - 3) + l_1 - l_2,$$

a contradiction, since  $l_2 \geq l_1$ . Hence suppose that  $l_1 > l_2$ . Then we get

$$g_1 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l \geq \frac{1}{3}(n - kd + 5k - 3) - l_1.$$

Thus

$$g_2 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l,$$

$$z < \frac{1}{3}(n - kd + 5k - 3) + l - l_1 + l_1 - l = \frac{1}{3}(n - kd + 5k - 3).$$

Then

$$d_G(u) + d_G(v) \leq g_1 + g_2 + 2z - l_1 - l_2 < \frac{4}{3}(n - kd + 5k - 3) + 1.$$



Hence

$$d_G(u) + d_G(v) \leq \frac{4}{3}(n - kd + 5k - 3),$$

a contradiction. ■

The following example shows that the conditions of Theorem 1 are sharp.

**Example.** Let  $K_1, \dots, K_{k_1}$  be  $k_1$  cliques of order  $k$ , let  $L_1, \dots, L_{k_2}$  be  $k_2$  cliques of order  $k$ . Let  $K_0, L_0$  be two cliques of order  $l_1 > 2k - 1$ , let  $M$  be a clique of order  $l_1 - k$ . All considered cliques  $K_i, L_i$  are vertex-disjoint. Construct a graph  $G$  by joining a new vertex  $x$  with each vertex of  $K_0 \cup L_0$ , a new vertex  $u$  with each vertex of  $K_{k_1}$  and a new vertex  $v$  with each vertex of  $L_{k_2}$ . Now join each vertex of  $K_i$  with each vertex of  $K_{i-1}$  for  $i = 1, \dots, k_1$ , each vertex of  $L_i$  with each vertex of  $L_{i-1}$  for  $i = 1, \dots, k_2$  and each vertex of  $K_0 \cup L_0$  with each vertex of  $M$ . Clearly the prescribed graph  $G$  is  $k$ -connected and the vertex  $x$  is not locally connected. The diameter of  $G$  is  $d = k_1 + k_2 + 4$ . It holds that

$$n = 1 + 2l_1 + l_1 - k + (k_1 + k_2)k + 2 = 3l_1 + (d - 5)k + 3.$$

Thus

$$3l_1 = n - kd + 5k - 3.$$

Furthermore

$$\begin{aligned} d_G(x) &= 2l_1, \\ d_G(y) &= 2l_1, \quad \forall y \in K_0, \\ d_G(z) &= 2l_1, \quad \forall z \in L_0. \end{aligned}$$

Hence for every pair  $a, b$  of vertices of  $N_G[x]$  holds that

$$d_G(a) + d_G(b) = 4l_1 = \frac{4}{3}(n - kd + 5k - 3).$$

and  $x$  is not locally connected.

The following lemma is a proposition analogous to Lemma 1 for the  $N_1^m$ -local connectivity of a vertex of a graph.

**Lemma 2.** *Let  $k \geq 2$  be an integer,  $G$  be a  $k$ -connected graph. Let  $d$  be the diameter of  $G$  and  $m \leq \frac{1}{2}(d - 1)$  be an integer. Then, for each vertex  $x$  of  $G$ , there are at least  $kd - 2km + 2$  vertices at distance at least  $m$  from  $x$  in  $G$ .*

**Proof.** Let  $P$  denote a diameter-path in  $G$ , let  $u, v$  be the end vertices of  $P$ . Since  $G$  is  $k$ -connected, there are at least  $k$  vertex-disjoint  $u, v$ -paths in  $G$  by Menger's theorem. Choose  $P_1, \dots, P_k$  with minimum sum of their lengths. Note that  $|E(P_i)| \geq d$ ,  $i = 1, \dots, k$ . Now it will be shown that there are at least  $d - 2m + 2$  vertices at the required distance from  $x$  on each of  $P_i$ ,  $i = 1, \dots, k$ . Let  $M_j = \{y \in P_j \mid \text{dist}_G(x, y) \leq m - 1\}$ ,  $j = 1, \dots, k$ . For each path of  $P_i$ ,  $i = 1, \dots, k$  there are two following cases:

*Case 1.* If  $M_i = \emptyset$ , then there are at least  $d + 1$  vertices at the required distance from  $x$  on  $P_i$ .

*Case 2.* If  $M_i \neq \emptyset$ , then let  $a_i \in M_i$  such that  $\text{dist}_G(a_i, u) = \min_{m \in M_i} \text{dist}_G(m, u)$  and let  $b_i \in M_i$  such that  $\text{dist}_G(b_i, v) = \min_{m \in M_i} \text{dist}_G(m, v)$ . Clearly

$$\text{dist}_G(u, a_i) + \text{dist}_G(a_i, x) + \text{dist}_G(x, b_i) + \text{dist}_G(b_i, v) \geq d.$$

Since  $\text{dist}_G(a_i, x) \leq m - 1$  and  $\text{dist}_G(b_i, x) \leq m - 1$ , we have

$$\text{dist}_G(u, a_i) + \text{dist}_G(b_i, v) \geq d - 2m + 2.$$

Hence there are at least  $d - 2m + 2$  vertices at the required distance from  $x$  on  $P_i$ .

On the paths  $P_i$ ,  $i = 1, \dots, k$  there are at least  $k(d - 2m + 2)$  vertices at the required distance from  $x$  in  $G$ . Since  $u$  and  $v$  can be counted only once, there are at least  $kd - 2km + 2$  different vertices at the required distance from  $x$  in  $G$ . ■

Let  $C$  be a cycle,  $x \in V(C)$  and  $\vec{C}$  be an orientation of  $C$ . Let  $x^{-(i)}$  denote the  $i$ -th predecessor of  $x$  on  $C$  and  $x^{+(i)}$  denote the  $i$ -th successor of  $x$  on  $C$  in the orientation  $\vec{C}$ .

**Lemma 3** [5]. *Let  $G$  be a 2-connected graph,  $x \in V(G)$ , and  $m$  be a positive integer. If  $x$  is not  $N_1^m$ -locally connected, then there is an induced cycle  $C$  of length at least  $2m + 2$  such that, in an orientation of  $C$ ,*

- $\text{dist}_G(x^{-(i)}, x) = i$  and  $\text{dist}_G(x^{+(i)}, x) = i$ ,  $i = 1, \dots, m$ ,
- $\text{dist}_G(y, x) > m$ , for every  $y \in V(C) \setminus \{x, x^{-(1)}, \dots, x^{-(m)}, x^{+(1)}, \dots, x^{+(m)}\}$ .

The following consequence proved by Holub and Xiong we use in the proof of Theorem 2.

**Corollary 1** [5]. *Let  $m \geq 2$  be an integer,  $G$  a 2-connected graph. If  $x \in V(G)$  is not  $N_1^m$ -locally connected, then there is a set  $M \subset V(G)$  such that*

- (1)  $M$  is independent in  $(G - x)^2$ ,  $M \subset N_1^{m+1}(x)$  and  $|M| \geq \frac{2}{3}m + 1$ , if  $m \equiv 0 \pmod{3}$ ,
- (2)  $M$  is independent in  $(G - N_G[x])^2$ ,  $M \subset (N_1^{m+1}(x) \setminus N_1^1(x))$  and  $|M| \geq \frac{2}{3}(m - 1) + 1$ , if  $m \equiv 1 \pmod{3}$ ,
- (3)  $M$  is independent in  $G^2$ ,  $M \subset N_1^m[x]$  and  $|M| \geq \frac{2}{3}(m - 2) + 2$ , if  $m \equiv 2 \pmod{3}$ .

**Proof of Theorem 2.** Suppose that  $G$  is not  $N_1^m$ -locally connected. Then there is a vertex  $x \in V(G)$  such that  $x$  is not  $N_1^m$ -locally connected in  $G$ . Hence  $\langle N_1^m(x) \rangle$  consists of at least two components. Let  $G_1$  denote arbitrary component of  $\langle N_1^m(x) \rangle$ , let  $G_2$  denote the union of all the other components of  $\langle N_1^m(x) \rangle$ .

*Case 1.*  $m \equiv 0 \pmod{3}$ . By Corollary 1 case (1), there is a set  $M \subset N_1^{m+1}(x)$  such that  $|M| = \frac{2}{3}m + 1$  and  $M$  is independent in  $(G - x)^2$ . Let  $t = |M|$ . Using Lemma 3, the set  $M$  can be chosen in the following way:  $M = \{x_1, x_2, \dots, x_t\}$ , where  $x_{2j-1} = x^{-(3j-2)}$ ,  $x_{2j} = x^{+(3j-2)}$ ,  $j = 1, \dots, \frac{m}{3}$ ,  $x_t = x^{+(m+1)}$ . Let  $A = \{y \in V(G) \mid \text{dist}_G(x, y) > m + 2\}$ , let  $a = |A|$ . By Lemma 2, the number  $a \geq kd - 2(m + 3)k + 2$ . Since  $M$  is independent in  $(G - x)^2$ , we have, for every pair  $u, v \in M \setminus \{x\}$ ,

$$N_{G-x}(u) \cap N_{G-x}(v) = \emptyset.$$

Since  $x$  is adjacent to at most two vertices of  $M$ , we obtain

$$\sum_{x_i \in M} d_G(x_i) \leq (n - 1) - t - a + 2 = n - t - a + 1.$$

Since  $a \geq kd - 2(m + 3)k + 2$ , we have

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 6k - 1,$$

a contradiction.

*Case 2.*  $m \equiv 1 \pmod{3}$ . By Corollary 1 case (2), there is a set  $M \subset N_1^{m+1}(x)$  such that  $|M| = \frac{2}{3}(m-1)+1$  and  $M$  is independent in  $(G - N_G[x])^2$ . Let  $t = |M|$ . Using Lemma 3, the set  $M$  can be chosen in the following way:  $M = \{x_1, x_2, \dots, x_t\}$ , where  $x_{2j-1} = x^{-(3j-1)}$ ,  $x_{2j} = x^{+(3j-1)}$ ,  $j = 1, \dots, \frac{m-1}{3}$ ,  $x_t = x^{+(m+1)}$ . Let  $A = \{y \in V(G) \mid \text{dist}_G(x, y) > m+2\}$ , let  $a = |A|$ . By Lemma 2, the number  $a \geq kd - 2(m+3)k + 2$ . Since  $M$  is independent in  $(G - N_G[x])^2$ , we have, for every pair  $u, v \in M$ ,

$$N_G(u) \cap N_G(v) = \emptyset.$$

Since each vertex of  $N_G(x)$  is adjacent to at most one vertex of  $M$ , we obtain

$$\sum_{x_i \in M} d_G(x_i) \leq (n-1) - t - a.$$

Since  $a \geq kd - 2(m+3)k + 2$ , we have

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 6k - 3,$$

a contradiction.

*Case 3.*  $m \equiv 2 \pmod{3}$ . By Corollary 1 case (3), there is a set  $M \subset N_1^m[x]$  such that  $|M| = \frac{2}{3}(m-2) + 2$  and  $M$  is independent in  $G^2$ . Let  $A = \{y \in V(G) \mid \text{dist}_G(x, y) > m+1\}$ , let  $a = |A|$ . By Lemma 2, the number  $a \geq kd - 2(m+2)k + 2$ . Since  $M$  is independent in  $G^2$ , we have, for every pair  $u, v \in M$ ,

$$N_G(u) \cap N_G(v) = \emptyset.$$

Let  $t = |M|$ . Hence

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - a.$$

Since  $a \geq kd - 2(m+2)k + 2$ , we obtain

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 4k - 2,$$

a contradiction. ■

## 3. VERTEX DISTANCE COLOURING

There are several results on  $t$ -distance chromatic number for planar graphs. In this paragraph, results on  $t$ -distance chromatic number in  $k$ -connected, not necessary planar, graphs are presented. Moreover, the relations between distance local connectivity and  $t$ -distance chromatic number in 2-connected graphs are given. Main results of this section are the following theorems.

**Theorem 3.** *Let  $G$  be a  $k$ -connected graph of order  $n$ ,  $d$  be the diameter of  $G$ . Let  $t < d$  be a positive integer. Then the distance-chromatic number*

$$\chi^{(t)}(G) \leq \begin{cases} n - 1 & \text{if } t = d - 1, \\ n - (d - t - 2)k - 2 & \text{if } t < d - 1. \end{cases}$$

**Theorem 4.** *Let  $G$  be a 2-connected graph of order  $n$ , let  $t, k$  be positive integers. If*

$$\chi^{(t)}(G) > n - (2k - 1)(t + 1),$$

*then  $G$  is  $N_1^m$ -locally connected, where  $m = k(t + 1) - 1$ .*

**Theorem 5.** *Let  $G$  be a 2-connected graph of order  $n$ ,  $k$  be a positive integer and  $t$  be an even positive integer. If*

$$\chi^{(t)}(G) > n - 2k(t + 1),$$

*then  $G$  is  $N_2^m$ -locally connected, where  $m = k(t + 1) + \frac{t}{2} - 1$ .*

The distance local connectivity number of a 2-connected graph  $G$ , denoted  $dlc(G)$ , is the smallest positive integer  $m$  for which  $G$  is  $N_1^m$ -locally connected. Since  $G$  is 2-connected, the number  $dlc(G)$  is well-defined. Note that local connectivity of a graph is the  $N_1^1$ -local connectivity. The following statement is a straightforward consequence of Theorem 4.

**Corollary 2.** *Let  $G$  be a 2-connected graph, let  $t$  be a positive integer. If  $dlc(G) = m$ , then  $\chi^{(t)}(G) \leq n - (k - 1)(t + 1)$ , where  $k = \lfloor \frac{2m}{t+1} \rfloor$ .*

**Proof of Theorem 3.** Let  $u, v$  denote the end vertices of a diameter path in  $G$ . Since  $G$  is  $k$ -connected, there are at least  $k$  vertex-disjoint  $u, v$ -paths  $P_1, \dots, P_k$  in  $G$  by Menger's theorem. Since  $\text{dist}_G(u, v) = d$ , each of  $P_i$ ,  $i = 1, \dots, k$ , has length at least  $d$ . Let  $u_{i,j}$  denote a vertex on  $P_i$  such

that  $\text{dist}_G(u, u_{i,j}) = j$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, d$ . Since  $d > t$ , there is at least one vertex  $u_{i,j}$  on  $P_i$ ,  $i = 1, \dots, k$ , including the end-vertex  $v$ , such that  $j > t$ ,  $j = t + 1, \dots, d$ . If  $d - t = 1$ , then  $u_{i,t+1} = v$  for every  $i \in \{1, \dots, k\}$ . We define colouring  $\chi$  of vertices of  $G$  in such a way that  $\chi(v) = \chi(u)$  and  $\chi(x) \neq \chi(y)$  for all pairs  $x, y \in V(G) \setminus \{u, v\}$ . Clearly  $\chi$  is a  $t$ -distance colouring of  $G$  and

$$\chi^{(t)}(G) \leq n - 1.$$

Suppose that  $d - t > 1$ . We define a colouring  $\chi$  of vertices of  $G$  in such a way that the vertices of  $N_1^{t+1}(u)$  have distinct colours in  $G$ ,  $\chi(u) = \chi(u_{i,t+1})$  and  $\chi(v) = \chi(u_{i,d-t-1})$  for some  $i \in \{1, \dots, k\}$ . Moreover, if  $d - t > 2$ , then, for every  $i \in \{1, \dots, k\}$ ,  $\chi(u_{i,j+t+1}) = \chi(u_{i,j})$ , since  $\text{dist}_G(u_{i,j}, u_{i,j+t+1}) = t + 1$ ,  $j = 1, \dots, d - t - 2$ . Clearly  $\chi$  is a  $t$ -distance colouring of  $G$ . Hence there are at least  $k(d - t - 2) + 2$  vertices with previously used colours, implying that

$$\chi^{(t)}(G) \leq n - k(d - t - 2) - 2. \quad \blacksquare$$

For the proofs of Theorem 4 and Theorem 5 we need some auxiliary statements. The following lemma is the analogue of Lemma 3.

**Lemma 4.** *Let  $G$  be a 2-connected graph,  $x \in V(G)$  and  $m$  be a positive integer. If  $x$  is not  $N_2^m$ -locally connected, then there is an induced cycle  $C$  containing  $x$  of length at least  $2m + 3$  such that, in an orientation of  $C$ ,*

$$\text{dist}_G(x^{-i}, x) = i \quad \text{and} \quad \text{dist}_G(x^{+i}, x) = i, \quad i = 1, \dots, m + 1,$$

**Proof.** The vertex  $x$  is not  $N_2^m$ -locally connected. The  $N_2^m$ -neighbourhood of a vertex  $x$  consists of at least two components  $G_1, G_2$ . Since  $G$  is 2-connected, there is a cycle  $C$  containing  $x$ , such that  $x^{-1} \in G_1$  and  $x^{+1} \in G_2$  in an orientation of  $C$ . Choose  $C$  shortest possible with this property. Since  $x$  is not  $N_2^m$ -locally connected,  $|V(C)| \geq 2m + 3$ . It is easy to see that  $C$  has the required property since otherwise there is a shorter cycle.  $\blacksquare$

From the definition of a  $t$ -distance colouring we obtain the following clear observation.

**Proposition 1.** *Let  $G$  be a 2-connected graph of order  $n$ , let  $t$  be a positive integer, let  $d$  denote the diameter of  $G$ . Then  $\chi^{(t)}(G) = n$  if and only if  $d \leq t$ .*

**Corollary 3.** *Let  $G$  be a 2-connected graph of order  $n$ , let  $t$  be a positive integer. If  $\chi^{(t)}(G) = n$ , then  $G$  is  $N_1^t$ -locally connected.*

**Proof.** Suppose that  $G$  is not  $N_1^t$ -locally connected, i.e., there is a vertex  $x \in V(G)$  such that  $x$  is not  $N_1^t$ -locally connected in  $G$ . By Proposition 1,  $d \leq t$ . By Lemma 3, there is an induced cycle  $C$  in  $G$  of length at least  $2t + 2$ , which contradicts the fact that  $d \leq t$ . ■

**Proof of Theorem 4.** Suppose that  $G$  is not  $N_1^m$ -locally connected, i.e., there is a vertex  $x$  which is not  $N_1^m$ -locally connected. By Lemma 3 there is an induced cycle  $C$  containing  $x$ , such that  $|V(C)| \geq 2m + 2$ . Moreover  $\text{dist}_G(x, x^{-(i)}) = \text{dist}_G(x, x^{+(i)}) = i$  for  $i = 1, \dots, m$ . Since  $x$  is not  $N_1^m$ -locally connected, the cycle  $C$  can be chosen such that  $x^{-(1)}$  and  $x^{+(1)}$  belong to different components of  $\langle N_1^m(x) \rangle$ . Clearly  $\text{dist}_G(x^{-(i)}, x^{-(j)}) = |i - j|$ , for  $i, j = 0, \dots, m$  where  $x^{-(0)} = x$ .

We define a colouring  $\chi$  of vertices of  $G$  in such a way that all the vertices  $x^{-(0)}, \dots, x^{-(t)}$  have distinct colours,  $\chi(x^{-(i)}) = \chi(x^{-(i+t+1)})$ ,  $i = 0, \dots, t$ , since  $|V(C)| \geq 2(t + 1)$ . If  $k > 1$ , then  $\chi(x^{-(i+j(t+1))}) = \chi(x^{-(i+(j-1)(t+1))})$  for  $i = 0, \dots, t$  and  $j = 1, \dots, 2k - 1$ . All the remaining vertices of  $G$  will be coloured with distinct unused colours. Clearly  $\chi$  is a  $t$ -vertex distance colouring in  $G$ .

We have coloured  $2k(t + 1)$  vertices of  $C$  with only  $t + 1$  colours. Since  $m = k(t + 1) - 1$ , we have coloured  $2m + 2$  vertices of  $C$  with only  $t + 1$  colours, implying that

$$\chi^{(t)}(G) \leq n - (2m + 2) + (t + 1) = n - (2k - 1)(t + 1),$$

a contradiction. ■

**Proof of Theorem 5.** We will use similar arguments as is the proof of Theorem 4. Suppose that  $G$  is not  $N_2^m$ -locally connected, i.e., there is a vertex  $x$  which is not  $N_2^m$ -locally connected. By Lemma 4 there is an induced cycle  $C$  containing  $x$ , such that  $|V(C)| \geq 2m + 3$ . Moreover  $\text{dist}_G(x, x^{-(i)}) = \text{dist}_G(x, x^{+(i)}) = i$  for  $i = 1, \dots, m + 1$ . Since  $x$  is not  $N_2^m$ -locally connected, the cycle  $C$  can be chosen such that  $x^{-(1)}$  and  $x^{+(1)}$  belong to different components of  $\langle N_2^m(x) \rangle$ . Clearly  $\text{dist}_G(x^{-(i)}, x^{-(j)}) = |i - j|$ , for  $i, j = 0, \dots, m + 1$  where  $x^{-(0)} = x$ .

We define a colouring  $\chi$  of vertices of  $G$  in such a way that all the vertices  $x^{-(0)}, \dots, x^{-(t)}$  have distinct colours,  $\chi(x^{-(i)}) = \chi(x^{-(i+t+1)})$ ,  $i = 0, \dots, t$ ,

since  $|V(C)| \geq 2(t + 1)$ . If  $k > 1$ , then  $\chi(x^{-(i+j(t+1))}) = \chi(x^{-(i+(j-1)(t+1))})$  for  $i = 0, \dots, t$  and  $j = 1, \dots, 2k$ , since  $|V(C)| \geq 2m + 3 = (2k + 1)(t + 1)$ . All the remaining vertices of  $G$  will be coloured with distinct unused colours. Clearly  $\chi$  is a  $t$ -vertex distance colouring in  $G$ .

Thus we can colour  $(2k + 1)(t + 1)$  vertices of  $C$  with only  $t + 1$  colours. Hence we have

$$\chi^{(t)}(G) \leq n - 2k(t + 1),$$

a contradiction. ■

Now we give an example which show that conditions of Theorem 3 are sharp. Let  $d$  and  $k \geq 2$  be two positive integers. Consider two vertices  $u$  and  $v$  and  $d - 1$  cliques  $K_1, \dots, K_{d-1}$  of order  $k$ . We construct a graph  $G$  by joining each vertex of  $K_1$  with  $u$ , each vertex of  $K_{d-1}$  with  $v$  and each vertex of  $K_i$  with each vertex of  $K_{i+1}$  for each  $i \in \{1, \dots, d - 2\}$ . The diameter of  $G$  is  $d$ , the graph  $G$  is  $k$ -connected and the  $t$ -distance chromatic number is equal to

$$\begin{cases} n - 1 & \text{if } t = d - 1, \\ n - (d - t - 2)k - 2 & \text{if } t < d - 1. \end{cases}$$

For the following two examples the conditions of Theorem 3 give better upper bound on the  $t$ -distance chromatic number than the conditions of Theorem B and C. Let  $d$  be a positive integer. Consider two vertices  $u, v$  and  $d - 1$  cliques  $K_1, \dots, K_{d-1}$  of order 3. Construct a graph  $G$  by joining each vertex of  $K_1$  with  $u$ , each vertex of  $K_{d-1}$  with  $v$ . Now pair vertices of  $K_i$  with vertices of  $K_{i+1}$ , for each  $i \in \{1, \dots, d - 2\}$ . The structure of  $G$  is shown in Figure 1.

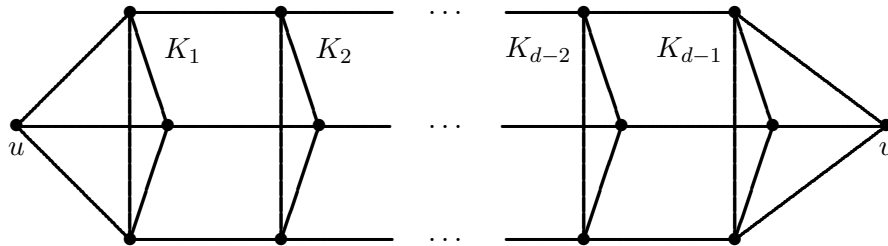


Figure 1

The graph  $G$  is 3-connected, the diameter of  $G$  is  $d$  and  $G$  is planar, because the graph on the following picture (Figure 2) is isomorphic with  $G$ .



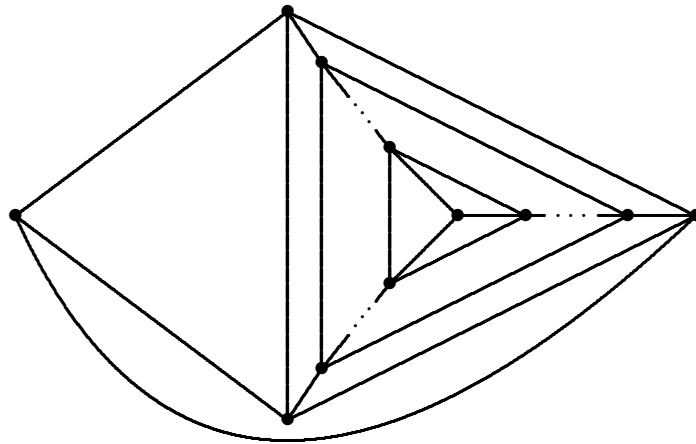


Figure 2

From Theorem 3 we obtain  $\chi^{(t)}(G) \leq 3(t + 1)$  and from Theorem B we get  $\chi^{(t)}(G) \leq \frac{9}{2}((7)^{t-1} - 1) + 6$ . For  $t \geq 2$  the upper bound of Theorem 3 is better.

For any positive integer  $d$ , consider two vertices  $u, v$ , and  $d - 1$  cliques  $K_1, \dots, K_{d-1}$ , such that  $K_1$  and  $K_{d-1}$  are triangles and  $K_1, \dots, K_{d-2}$  are alternatively cliques of orders 3 and 4. Construct a graph  $G$  in such a way that we join each vertex of  $K_1$  with  $u$ , each vertex of  $K_{d-1}$  with  $v$  and we pair vertices of  $K_i$  with vertices of  $K_{i+1}$ , for all  $i \in \{1, \dots, d - 2\}$ , in such a way that is shown in Figure 3.

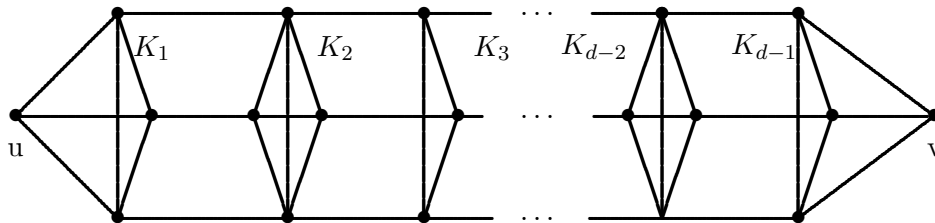


Figure 3

This graph  $G$  is 3-connected, the diameter of  $G$  is  $d$  and  $G$  is planar, because the graph on the following picture (Figure 4) is isomorphic with  $G$ .

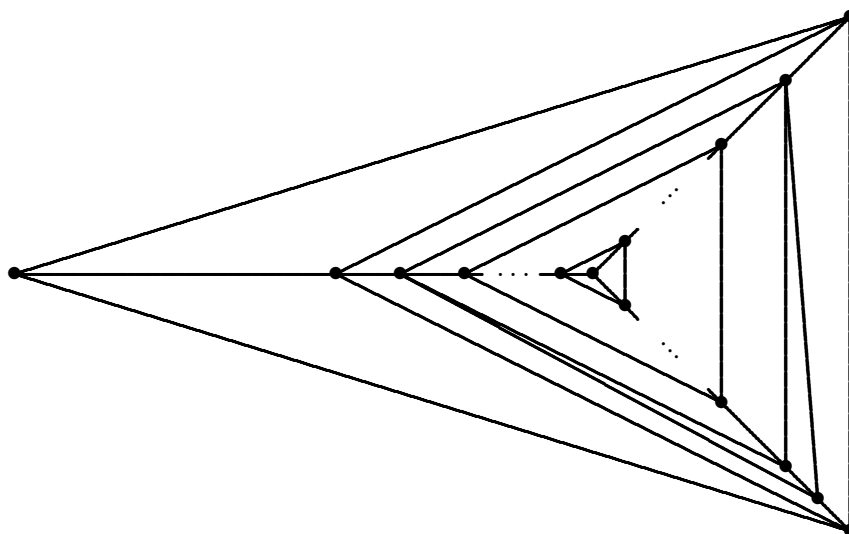


Figure 4

From Theorem 3 we get  $\chi^{(t)}(G) \leq 3(t+1) + 2 + \frac{d-1}{2}$ , and, from Theorem B we obtain  $\chi^{(t)}(G) \leq \frac{9}{2}((7)^{t-1} - 1) + 6$ . Comparing these two values, the upper bound of Theorem 3 is asymptotically better for  $t \geq 2$  and  $d \ll 7^t$ .

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