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# IMPROVED UPPER BOUNDS FOR NEARLY ANTIPODAL CHROMATIC NUMBER OF PATHS*

Yu-Fa Shen[a], Guo-Ping Zheng[a], Wen-Jie He[b]

[a]*Department of Mathematics and Physics*
*Hebei Normal University of Science and Technology*
*Qinhuangdao 066004, P.R. China*

[b]*Applied Mathematics Institute*
*Hebei University of Technology*
*Tianjin 300130, P.R. China*

**e-mail:** syf030514@163.com (Yu-Fa Shen).

**Abstract**

For paths $P_n$, G. Chartrand, L. Nebeský and P. Zhang showed that $\mathrm{ac}'(P_n) \le \binom{n-2}{2} + 2$ for every positive integer $n$, where $\mathrm{ac}'(P_n)$ denotes the nearly antipodal chromatic number of $P_n$. In this paper we show that $\mathrm{ac}'(P_n) \le \binom{n-2}{2} - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7$ if $n$ is even positive integer and $n \ge 10$, and $\mathrm{ac}'(P_n) \le \binom{n-2}{2} - \frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8$ if $n$ is odd positive integer and $n \ge 13$. For all even positive integers $n \ge 10$ and all odd positive integers $n \ge 13$, these results improve the upper bounds for nearly antipodal chromatic number of $P_n$.

**Keywords:** radio colorings, nearly antipodal chromatic number, paths.

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## 1. Introduction

*Radio k-colorings* are generalizations of ordinary colorings of graphs, which were inspired by (FM Radio) Channel Assignments Problem (see [5, 7]) and introduced by G. Chartrand, D. Erwan, F. Harary and P. Zhang [1]. For a connected graph $G$ of order $n$ and diameter $d$ and a integer $k$ with $1 \leq k \leq d$, a radio $k$-coloring of $G$ is a function $c$: $V(G) \to \mathbf{N}$, such that $d(u,v) + |c(u) - c(v)| \geq k+1$ for every pair $u$ and $v$ of distinct vertices of $G$, where $d(u,v)$ denotes the distance between $u$ and $v$ (the length of a shortest $u - v$ path) in $G$. Clearly, radio 1-colorings and ordinary colorings are synonymous. The *value* $\mathrm{rc}_k(c)$ of a radio $k$-coloring $c$ of $G$ is the maximum color assigned to a vertex of $G$; while the *radio k-chromatic number* $\mathrm{rc}_k(G)$ of $G$ is $\min\{\mathrm{rc}_k(c)\}$ taken over all $k$-coloring $c$ of $G$. In particular, radio $d$-colorings are referred to as *radio labelings* and the *radio d-chromatic number* is called the *radio number*. Radio $(d-1)$-colorings are referred to as *radio antipodal coloring* or, more simply, as an *antipodal coloring*, and the *radio* $(d-1)$-*chromatic number* is called the *antipodal chromatic number*, denoted by $\mathrm{ac}(G)$. Radio $k$-coloring and radio labeling of graphs were studied in [1, 2]. Radio antipodal coloring of paths were studied in [3, 4, 6].

Furthermore, G. Chartrand, L. Nebeský and P. Zhang gave the concepts of *nearly antipodal colorings* in [4]. For a connected graph $G$ of diameter $d$, a nearly antipodal coloring of $G$ is a function $c$: $V(G) \to \mathbf{N}$, such that $d(u,v) + |c(u) - c(v)| \geq d - 1$ for every two distinct vertices $u$ and $v$ of $G$. The *value* $\mathrm{ac}'(c)$ of a nearly antipodal coloring $c$ of $G$ is the maximum color assigned to a vertex of $G$. The *nearly antipodal chromatic number* $\mathrm{ac}'(G)$ of $G$ is $\min\{\mathrm{ac}'(c)\}$ taken over all nearly antipodal colorings of $G$ (In fact, for $d \geq 3$, a nearly antipodal coloring is a radio $(d-2)$-coloring).

Clearly, if $G$ is a connected graph of diameter 1 or 2, then $\mathrm{ac}'(G) = 1$; while if $\mathrm{diam}(G) = 3$, then $\mathrm{ac}'(G)$ is the chromatic number of $G$. Thus nearly antipodal colorings are most interesting for connected graphs of diameter 4 or more. For this reason, the nearly antipodal chromatic number of paths $P_n$ were investigated in [4] by G. Chartrand, L. Nebeský and P. Zhang. And they showed that $\mathrm{ac}'(P_5) = 5$, $\mathrm{ac}'(P_6) = 7$, $\mathrm{ac}'(P_7) = 11$ and $\mathrm{ac}'(P_8) = 16$. Moreover, they presented an upper bound for the nearly antipodal chromatic number of paths $P_n$ for every positive integer $n$ as follows.

**Theorem 1.1** ([4]). *If $n$ is a path of order $n \geq 1$, $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} + 2$.*

## 2. Our Rresults and the Idea of the Proof

In this paper we will provide an improved version for Theorem 1.1. We will show that

**Theorem 2.1.**

1. *If* $P_n$ *is even and* $n \geq 10$*, then* $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7$;
2. *If* $n$ *is odd and* $n \geq 13$*, then* $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8$.

Clearly, it holds that $-\frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7 \leq 1$ for all even integers $n \geq 10$, and $-\frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8 \leq 1$ for all odd integers $n \geq 13$. Thus, for all even integers $n \geq 10$ and all odd integers $n \geq 13$, Theorem 2.1 improves the upper bounds of $\mathrm{ac}'(P_n)$.

We will prove Theorem 2.1 in Section 3, and the proof will virtually provide a nearly antipodal coloring $c$ for paths $P_n$ with $\mathrm{ac}'(c)$ that is equal to the bound presented in Theorem 2.1. The idea of performing the coloring $c$ is based on pseudo greedy algorithm: Let $V(P_n) = \{p_1, p_2, \ldots, p_n\}$. At first, we use the color $c_1 = 1$ to color some vertex $p_{n_1} \in \{p_1, p_2, \ldots, p_n\}$, where $p_{n_1}$ is the (a) *central vertex* of $P_n$. Suppose that for $1 \leq i \leq n-1$ the vertices in $\{p_{n_1}, p_{n_2}, \ldots, p_{n_i}\} \subset \{p_1, p_2, \ldots, p_n\}$ have been colored with $c(p_{n_j}) = c_j$ for all $1 \leq j \leq i$, then we choose a color $c_{i+1} \in \mathbf{N}$ as small as possible to color one vertex $p_{n_{i+1}} \in V(P_n) \backslash \{p_{n_1}, p_{n_2}, \ldots, p_{n_i}\}$, such that $d(p_{n_{i+1}}, p_{n_j}) + |c(p_{n_{i+1}}) - c(p_{n_j})| \geq d - 1$ for all $1 \leq j \leq i$. And if there are two vertices can be chosen for $p_{n_{i+1}}$, then we take $p_{n_{i+1}}$ close to central vertices of $P_n$ as near as possible. Finally, we obtain that $\mathrm{ac}'(c) = c(p_{n_n})$ and hence $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c)$. In Section 4 we will give some examples which present the nearly antipodal coloring $c$ for some paths $P_n$ with $\mathrm{ac}'(c)$ showed in Theorem 2.1 by our methods.

## 3. Proof of Theorem 2.1

***Proof.*** 1. *$n$ is even and $n \geq 10$.* Firstly, we let $n \geq 12$, note that $-\lfloor \frac{10}{n} \rfloor = 0$, it suffices to show that $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} + 7$. Write $n = 2k = 10 + 2(4p + q)$, where $p \in \{0, 1, 2, \ldots\}$ and $q \in \{1, 2, 3, 4\}$. Then we have that $k = 5 + (4p + q)$ and $d - 1 = \mathrm{diam}(P_n) - 1 = 2k - 2$.

We denote the vertices of $P_n$ by $x'_1, x'_2, x'_3;\ v'_1, v'_2, \ldots, v'_{2p-1}, v'_{2p};$ $w_1, w_2 \ldots, w_q;\ v_{2p}, v_{2p-1}, \ldots, v_2,\ v_1;\ x_2, x_1;\ y_1, y_2;\ u_1, u_2, \ldots, u_{2p-1}, u_{2p};$

$z_q, \ldots, z_2, z_1$; $u'_{2p}, u'_{2p-1}, \ldots, u'_2, u'_1$; $y'_3, y'_2, y'_1$ (see Figure 1). And we write

$V_1 = \{x_1, x_2; y_1, y_2;\ x'_1, x'_2, x'_3; y'_1, y'_2, y'_3\}$,
$V_2 = \{v_1, u_2, v_3, u_4, \ldots, v_{2p-1}, u_{2p}; v'_1, v'_2, \ldots, v'_{2p-1}, v'_{2p}; u'_1, u'_2, \ldots, u'_{2p-1}, u'_{2p}\}$,
$V_3 = \{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q; v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}$.

In the following we will present a coloring $c$ for $P_n$ by three steps, such that

$$d(u,v) + |c(u) - c(v)| \geq d - 1 = 2k - 2 \tag{1}$$

holds for all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, and $\mathrm{ac}'(c) = \binom{n-2}{2} - \frac{n}{2} + 7$(note that $V_2 = \emptyset$ if $p = 0$, and it is easy to see that the following proof is also suitable for $V_2 = \emptyset$).

**Step 1.** Color the vertices in $V_1$ (see Figure 1).

Let

$c(x_1) = 1$ ($x_1$ is an central vertex of $P_n$);
$c(y'_1) = c(x_1) + (k-2) = k-1, \qquad c(x'_1) = c(x_1) + (k-1) = k;$
$c(y_1) = c(x'_1) + (k-2) = 2k-2;$
$c(x'_2) = c(y_1) + k - 1 = 3k-3, \qquad c(y'_2) = c(x'_2) + 1 = 3k-2;$
$c(x_2) = c(x'_2) + (k+1) = 4k-2;$
$c(y'_3) = c(x_2) + (k-1) = 5k-3, \qquad c(x'_3) = c(y'_3) + 3 = 5k;$
$c(y_2) = c(x'_3) + (k-1) = 6k-1.$

Then by the definition of $c$ and the value of $d(u,v)$ for $u, v \in V_1$, it is easy to verify that the following claim holds.

**Claim 3.1.** For all distinct vertices $u, v \in V_1$, the inequality (1) holds. At the same time, $\max_{v \in V_1} c(v) = c(y_2) = 6k-1$ and $\max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x'_3) = 5k$.

**Step 2.** Color the vertices in $V_2$ (see Figure 1).

For $i = 1, 2, \ldots, p$, let

$$\begin{aligned}
c(v'_{2i-1}) =&\ c(y_2) + (2i-1)k + 3(2i-2) + 2[1+2+\ldots+(2i-2)]\\
&+(2i-2)(k-1),\\
c(u'_{2i-1}) =&\ c(y_2) + (2i-1)k + 3(2i-1) + 2[1+2+\ldots+(2i-1)]\\
&+(2i-2)(k-1);\\
c(v_{2i-1}) =&\ c(y_2) + (2i-1)k + 3(2i-1) + 2[1+2+\ldots+(2i-1)]\\
&+(2i-1)(k-1);\\
c(u'_{2i}) =&\ c(y_2) + (2i)k + 3(2i-1) + 2[1+2+\ldots+(2i-1)]\\
&+(2i-1)(k-1),\\
c(v'_{2i}) =&\ c(y_2) + (2i)k + 3(2i) + 2[1+2+\ldots+(2i)] + (2i-1)(k-1);\\
c(u_{2i}) =&\ c(y_2) + (2i)k + 3(2i) + 2[1+2+\ldots+(2i)] + (2i)(k-1).
\end{aligned}$$

Then we have the following claim.

**Claim 3.2.** For all distinct vertices $u, v \in V_1 \cup V_2$, the inequality (1) holds. At the same time, it holds that $\max_{v\in V_1\cup V_2} c(v) = c(u_{2p}) = 6k - 1 + 2p(2k+2p+3)$ and $\max_{v\in (V_1\cup V_2)\setminus\{u_{2p}\}} c(v) = c(v'_{2p}) = 5k + 2p(2k+2p+3)$.

In fact, note $d-1 = 2k-2$. Since that $d(y_2, v'_1) = k-2$, $d(y_2, u'_1) = k-5$, $d(v'_1, u'_1) = 2k-7$, $c(v'_1) = c(y_2)+k$ and $c(u'_1) = c(y_2)+k+5$, then for all distinct vertices $u, v \in \{y_2, v'_1, u'_1\}$, the inequality (1) holds. As $\max_{v\in V_1\setminus\{y_2\}} c(v) = c(x'_3)$ by Claim 3.1, $c(v'_1) = c(y_2)+k = c(x'_3)+2k-1$ and $c(u'_1) > c(v'_1)$, we have that $c(v'_1) - c(x'_3) \geq d-1$ and $c(u'_1) - c(x'_3) \geq d-1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1\}$, the inequality (1) holds.

Since that $d(u'_1, v_1) = k-1$, $d(v_1, v'_1) = k-6$, and $c(v_1) = c(u'_1)+(k-1) = c(v'_1)+5+(k-1)$, then for all distinct vertices $u, v \in \{v_1, v'_1, u'_1\}$, the inequality (1) holds. As $\max_{v\in V_1} c(v) = c(y_2)$ by Claim 3.1, and $c(v_1) = c(y_2)+k+5+(k-1)$, we have that $c(v_1) - c(y_2) \geq d-1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\}$, the inequality (1) holds.

Note the fact that $d(v_1, u'_2) = k-2$, $d(v_1, v'_2) = k-5-2$, $d(u'_2, v'_2) = 2k-7-2$, $c(u'_2) = c(v_1)+k$, $c(v'_2) = c(v_1)+k+5+2$; and $d(v'_2, u_2) = k-1$, $d(u_2, u'_2) = k-6-2$, $c(u_2) = c(v'_2)+(k-1) = c(u'_2)+5+2+(k-1)$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\}$, the inequality (1) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\} \cup \ldots \cup \{v'_{2p-1}, u'_{2p-1}, v_{2p-1}\} \cup \{u'_{2p}, v'_{2p}, u_{2p}\} = V_1 \cup V_2$, the inequality (1) holds.

By the definition of $c$, it is easy to verify that $\max_{v\in V_1\cup V_2} c(v) = c(u_{2p}) = 6k-1+2p(2k+2p+3)$ and $\max_{v\in (V_1\cup V_2)\setminus\{u_{2p}\}} c(v) = c(v'_{2p}) = 5k+2p(2k+2p+3)$.

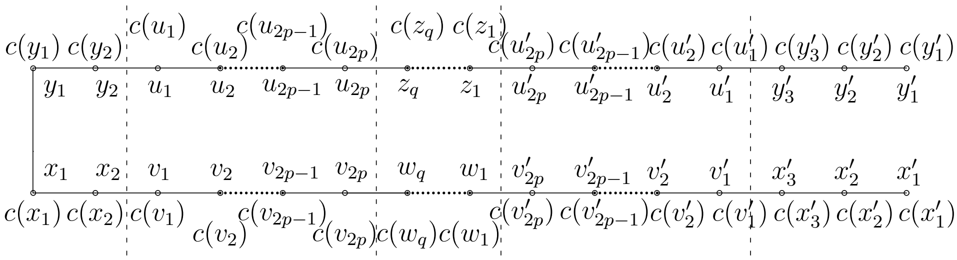


Figure 1: A nearly antipodal coloring for $P_n$ ($n = 2k \geq 10$).

**Step 3.** Color the vertices in $V_3$ (see Figure 1).

**Step 3.1.** Color the vertices in $\{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q\}$.

According the value of $q$, there are four cases.

*Case* 1. $q = 1$. Let

$$c(w_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3),$$
$$c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5).$$

*Case* 2. $q = 2$. Let

$$c(w_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3),$$
$$c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5),$$
$$c(w_2) = c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5),$$
$$c(z_2) = c(w_2) + 3 + 2(2p + 2) = 8k + 10 + 2p(2k + 2p + 7).$$

*Case* 3. $q = 3$. Let

$$c(w_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3),$$
$$c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5),$$
$$c(w_3) = c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5),$$
$$c(z_2) = c(w_3) + k = 9k + 3 + 2p(2k + 2p + 5),$$
$$c(w_2) = c(z_2) + 3 + 2(2p + 2) = 9k + 10 + 2p(2k + 2p + 7),$$
$$c(z_3) = c(w_2) + k = 10k + 10 + 2p(2k + 2p + 7).$$

*Case* 4. $q = 4$. Let

$$\begin{aligned}
c(w_1) &= c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3), \\
c(z_1) &= c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5), \\
c(w_4) &= c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5), \\
c(z_2) &= c(w_4) + k = 9k + 3 + 2p(2k + 2p + 5), \\
c(w_2) &= c(z_2) + 3 + 2(2p + 2) = 9k + 10 + 2p(2k + 2p + 7), \\
c(z_3) &= c(w_2) + (k - 1) = 10k + 9 + 2p(2k + 2p + 7), \\
c(w_3) &= c(z_3) + 3 + 2(2p + 3) = 10k + 18 + 2p(2k + 2p + 9), \\
c(z_4) &= c(w_3) + (k + 1) = 11k + 19 + 2p(2k + 2p + 9).
\end{aligned}$$

**Step 3.2.** Color the vertices in $\{v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}$.

For any case above ($q = 1, 2, 3, 4$), we let

$$\begin{aligned}
c(v_{2p}) &= c(z_q) + [(k + q) - 1], \\
c(u_{2p-1}) &= c(v_{2p}) + [(k + q - 1) + 2], \\
c(v_{2p-2}) &= c(u_{2p-1}) + [(k + q - 1) + 2 \cdot 2], \\
c(u_{2p-3}) &= c(v_{2p-2}) + [(k + q - 1) + 2 \cdot 3], \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
c(v_2) &= c(u_3) + [(k + q - 1) + 2(2p - 2)], \\
c(u_1) &= c(v_2) + [(k + q - 1) + 2(2p - 1)] \\
&= c(z_q) + 2p(k + q - 1) + 2 \cdot \tfrac{2p(2p-1)}{2} \\
&= c(z_q) + 2p(k + q + 2p - 2).
\end{aligned}$$

Then by a similar method to prove Claim 3.2, we can obtain the following claim.

**Claim 3.3.** For all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, the inequality (1) holds. And $\max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 2)$.

By Claim 3.3, we have shown that for all even integers $n \geq 12$, $c$ is a nearly antipodal coloring for $P_n$. Therefore $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c) = \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 2)$. To finish the proof of Theorem 2.1 for all even integers $n \geq 12$, it suffices to prove the following claim.

**Claim 3.4.** For any $p \in \{0, 1, 2, \ldots\}$ and any $q \in \{1, 2, 3, 4\}$, it holds that $c(u_1) = c(z_q) + 2p(k+q+2p-2) = \binom{n-2}{2} - \frac{n}{2} + 7$, where $n = 2k = 2(5+4p+q)$.

In fact, if $q = 1$, then $k = 4p + 6$, $2p = \frac{k-6}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_1) + 2p(k + q + 2p - 2) = 7k + 4 + 2p(2k + 2p + 5) \\ &\quad + 2p(k + 2p - 1) \\ &= 2k^2 - 6k + 10 = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 2$, then $k = 4p + 7$, $2p = \frac{k-7}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_2) + 2p(k + q + 2p - 2) = 8k + 10 + 2p(2k + 2p + 7) \\ &\quad + 2p(k + 2p) \\ &= 8k + 10 + 2p(3k + 4p + 7) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 3$, then $k = 4p + 8$, $2p = \frac{k-8}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_3) + 2p(k + q + 2p - 2) = 10k + 10 + 2p(2k + 2p + 7) \\ &\quad + 2p(k + 2p + 1) \\ &= 10k + 10 + 2p(3k + 4p + 8) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 4$, then $k = 4p + 9$, $2p = \frac{k-9}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_4) + 2p(k + q + 2p - 2) = 11k + 19 + 2p(2k + 2p + 9) \\ &\quad + 2p(k + 2p + 2) \\ &= 11k + 19 + 2p(3k + 4p + 11) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

Thus Claim 3.4 holds and hence $\mathrm{ac}'(P_n) \le \mathrm{ac}'(c) = \binom{n-2}{2} - \frac{n}{2} + 7$ for all even integers $n \geq 12$.

Secondly, for $n = 10$, in the above proof we take $p = 0$ and $q = 0$. Namely, $V_2 = V_3 = \emptyset$, $V(P_{10}) = V_1 = \{x'_1, x'_2, x'_3; x_2, x_1; y_1, y_2; y'_3, y'_2, y'_1\}$ (also see Figure 1 and let $p = q = 0$). Then coloring $c|_{v\in V_1}(v)$ is a nearly antipodal coloring for $P_{10}$. Thus by Claim 3.1, $\mathrm{ac}'(P_{10}) \le \mathrm{ac}'(c|_{v\in V_1}) = \max_{v\in V_1} c(v) = c(y_2) = (6k - 1)|_{k=5} = 29 = \binom{10-2}{2} + 1$. Since $-\lfloor \frac{10}{n} \rfloor = -1$ for $n = 10$, it follows that $\mathrm{ac}'(P_{10}) \le \mathrm{ac}'(c|_{v\in V_1}) = \binom{10-2}{2} + 1 = \binom{10-2}{2} - \frac{10}{2} - \lfloor \frac{10}{10} \rfloor + 7$.

Thus we complete the proof of assertion 1 in Theorem 2.1.

2. *$n$ is odd and $n \geq 13$.* Firstly, we let $n \geq 15$, note that $-\lfloor \frac{13}{n} \rfloor = 0$, it suffices to show that $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} + 8$. Write $n = 2k+1 = 13 + 2(4p+q)$, where $p \in \{0, 1, 2, \ldots\}$ and $q \in \{1, 2, 3, 4\}$. Then we have that $k = 6 + (4p+q)$ and $d - 1 = \mathrm{diam}(P_n) - 1 = 2k - 1$.

We denote the vertices of $P_n$ by $x'_1, x'_2, x'_3, x'_4$; $v'_1, v'_2, \ldots, v'_{2p-1}, v'_{2p}$; $w_1, w_2, \ldots, w_q$; $v_{2p}, v_{2p-1}, \ldots, v_2, v_1$; $x_2, x_1$; $x_0$; $y_1, y_2$; $u_1, u_2, \ldots, u_{2p-1}, u_{2p}$; $z_q, \ldots, z_2, z_1$; $u'_{2p}, u'_{2p-1}, \ldots, u'_2, u'_1$; $y'_4, y'_3, y'_2, y'_1$ (see Figure 2). And we write

$$V_1 = \{x_0; x_1, x_2; y_1, y_2; x'_1, x'_2, x'_3, x'_4; y'_1, y'_2, y'_3, y'_4\},$$

$$V_2 = \{v_1, u_2, v_3, u_4, \ldots, v_{2p-1}, u_{2p}; v'_1, v'_2, \ldots, v'_{2p-1}, v'_{2p}; u'_1, u'_2, \ldots, u'_{2p-1}, u'_{2p}\},$$

$$V_3 = \{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q; v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}.$$

Similar to the method of proof assertion 1, we will present a coloring $c$ for $P_n$ by three steps, such that

$$d(u,v) + |c(u) - c(v)| \geq d - 1 = 2k - 1 \tag{2}$$

holds for all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, and $\mathrm{ac}'(c) = \binom{n-2}{2} - \frac{n}{2} + 8$(note that $V_2 = \emptyset$ if $p = 0$, and it is easy to see that the following proof is also suitable for $V_2 = \emptyset$).

**Step 1.** Color the vertices in $V_1$ (see Figure 2).

Let

$c(x_0) = 1$ ($x_0$ is the central vertex of $P_n$);

$c(x'_1) = c(x_0) + (k-1) = k$, $\quad c(y'_1) = c(x_0) + (k-1) = k$;

$c(x_1) = c(x'_1) + k = 2k$;

$c(y'_2) = c(x_1) + (k-1) = 3k - 1$, $\quad c(x'_2) = c(x_1) + (k+1) = 3k + 1$;

$c(y_1) = c(y'_2) + (k+1) = 4k$;

$c(x'_3) = c(y_1) + k = 5k$, $\quad c(y'_3) = c(y'_3) + 3 = 5k + 3$;

$c(x_2) = c(x'_3) + (k+3) = 6k + 3$;

$c(y'_4) = c(x_2) + k = 7k + 3$, $\quad c(x'_4) = c(y'_4) + 5 = 7k + 8$;

$c(y_2) = c(x'_4) + k = 8k + 8$.

Then by the definition of $c$ and the value of $d(u,v)$ for $u, v \in V_1$, it is easy to verify that the following claim holds.

**Claim 3.5.** For all distinct vertices $u, v \in V_1$, the inequality (2) holds. At the same time, $\max_{v\in V_1} c(v) = c(y_2) = 8k + 8$ and $\max_{v\in V_1\setminus\{y_2\}} c(v) = c(x'_4) = 7k + 8$.

**Step 2.** Color the vertices in $V_2$ (see Figure 2).

For $i = 1, 2, \ldots, p$, let

$$\begin{aligned}
c(v'_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-2) + 2[1+2+\ldots+(2i-2)] \\
&\quad + (2i-2)k, \\
c(u'_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad + (2i-2)k; \\
c(v_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad + (2i-1)k; \\
c(u'_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad + (2i-1)k, \\
c(v'_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i) + 2[1+2+\ldots+(2i)] + (2i-1)k; \\
c(u_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i) + 2[1+2+\ldots+(2i)] + (2i)k.
\end{aligned}$$

Then we have the following claim.

**Claim 3.6.** For all distinct vertices $u, v \in V_1 \cup V_2$, the inequality (2) holds. At the same time, it holds that $\max_{v\in V_1\cup V_2} c(v) = c(u_{2p}) = 8k+8+2p(2k+2p+7)$ and $\max_{v\in(V_1\cup V_2)\setminus\{u_{2p}\}} c(v) = c(v'_{2p}) = 7k + 8 + 2p(2k + 2p + 7)$.

In fact, note $d - 1 = 2k - 1$. Since that $d(y_2, v'_1) = k - 2$, $d(y_2, u'_1) = k - 6$, $d(v'_1, u'_1) = 2k - 8$, $c(v'_1) = c(y_2) + (k + 1)$ and $c(u'_1) = c(y_2) + (k + 1) + 7$, then for all distinct vertices $u, v \in \{y_2, v'_1, u'_1\}$, the inequality (2) holds. As $\max_{v\in V_1\setminus\{y_2\}} c(v) = c(x'_4)$ by Claim 3.5, $c(v'_1) = c(y_2)+(k+1) = c(x'_4)+2k+1$ and $c(u'_1) > c(v'_1)$, we have that $c(v'_1)-c(x'_4) \geq d-1$ and $c(u'_1)-c(x'_4) \geq d-1$. Therefore for all distinct vertices $u, v \in V_1\cup\{v'_1, u'_1\}$, the inequality (2) holds.

Since that $d(u'_1, v_1) = k - 1$, $d(v_1, v'_1) = k - 7$, and $c(v_1) = c(u'_1) + k = c(v'_1) + 7 + k$, then for all distinct vertices $u, v \in \{v_1, v'_1, u'_1\}$, the inequality (2) holds. As $\max_{v\in V_1} c(v) = c(y_2)$ by Claim 3.5, and $c(v_1) = c(y_2)+(k+1)+7+k$, we have that $c(v_1) - c(y_2) \geq d-1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\}$, the inequality (2) holds.

Note the fact that $d(v_1, u'_2) = k - 2$, $d(v_1, v'_2) = k - 6 - 2$, $d(u'_2, v'_2) = 2k - 8 - 2$, $c(u'_2) = c(v_1) + (k + 1)$, $c(v'_2) = c(v_1) + (k + 1) + 7 + 2$; and

$d(v_2', u_2) = k-1$, $d(u_2, u_2') = k-7-2$, $c(u_2) = c(v_2')+k = c(u_2')+7+2+k$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_1 \cup \{v_1', u_1', v_1\} \cup \{u_2', v_2', u_2\}$, the inequality (2) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_1 \cup \{v_1', u_1', v_1\} \cup \{u_2', v_2', u_2\} \cup \ldots \cup \{v_{2p-1}', u_{2p-1}', v_{2p-1}\} \cup \{u_{2p}', v_{2p}', u_{2p}\} = V_1 \cup V_2$, the inequality (2) holds.
By the definition of $c$, it is easy to see that $\max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 8k+8+2p(2k+2p+7)$, and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v_{2p}') = 7k+8+2p(2k+2p+7)$.

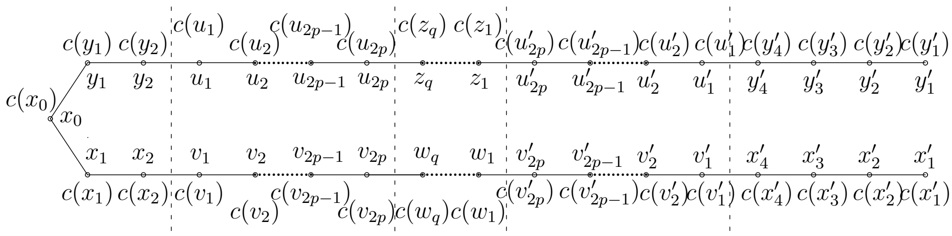

Figure 2. A nearly antipodal coloring for $P_n$ ($n = 2k+1 \geq 13$).

**Step 3.** Color the vertices in $V_3$ (see Figure 2).

**Step 3.1.** Color the vertices in $\{w_1, w_2, \ldots, w_q;\ z_1, z_2, \ldots, z_q\}$.

According the value of $q$, there are four cases.

*Case* 1. $q = 1$. Let

$$c(w_1) = c(u_{2p}) + (k+1) = 9k+9+2p(2k+2p+7),$$
$$c(z_1) = c(w_1) + 5 + 2(2p+1) = 9k+16+2p(2k+2p+9).$$

*Case* 2. $q = 2$. Let

$$c(w_1) = c(u_{2p}) + (k+1) = 9k+9+2p(2k+2p+7),$$
$$c(z_1) = c(w_1) + 5 + 2(2p+1) = 9k+16+2p(2k+2p+9),$$
$$c(w_2) = c(z_1) + k = 10k+16+2p(2k+2p+9),$$
$$c(z_2) = c(w_2) + 5 + 2(2p+2) = 10k+25+2p(2k+2p+11).$$

*Case* 3. $q = 3$. Let

$$\begin{aligned}
c(w_1) &= c(u_{2p}) + (k+1) = 9k + 9 + 2p(2k+2p+7),\\
c(z_1) &= c(w_1) + 5 + 2(2p+1) = 9k + 16 + 2p(2k+2p+9),\\
c(w_3) &= c(z_1) + k = 10k + 16 + 2p(2k+2p+9),\\
c(z_2) &= c(w_3) + (k+1) = 11k + 17 + 2p(2k+2p+9),\\
c(w_2) &= c(z_2) + 5 + 2(2p+2) = 11k + 26 + 2p(2k+2p+11),\\
c(z_3) &= c(w_2) + (k+1) = 12k + 27 + 2p(2k+2p+11).
\end{aligned}$$

*Case* 4. $q = 4$. Let

$$\begin{aligned}
c(w_1) &= c(u_{2p}) + (k+1) = 9k + 9 + 2p(2k+2p+7),\\
c(z_1) &= c(w_1) + 5 + 2(2p+1) = 9k + 16 + 2p(2k+2p+9),\\
c(w_4) &= c(z_1) + k = 10k + 16 + 2p(2k+2p+9),\\
c(z_2) &= c(w_4) + (k+1) = 11k + 17 + 2p(2k+2p+9),\\
c(w_2) &= c(z_2) + 5 + 2(2p+2) = 11k + 26 + 2p(2k+2p+11),\\
c(z_3) &= c(w_2) + k = 12k + 26 + 2p(2k+2p+11),\\
c(w_3) &= c(z_3) + 5 + 2(2p+3) = 12k + 37 + 2p(2k+2p+13),\\
c(z_4) &= c(w_3) + (k+2) = 13k + 39 + 2p(2k+2p+13).
\end{aligned}$$

**Step 3.2.** Color the vertices in $\{v_{2p}, u_{2p-1}, \dots, v_4, u_3, v_2, u_1\}$.

For each case above ($q = 1, 2, 3, 4$), we let

$$\begin{aligned}
c(v_{2p}) &= c(z_q) + (k+q),\\
c(u_{2p-1}) &= c(v_{2p}) + [(k+q) + 2],\\
c(v_{2p-2}) &= c(u_{2p-1}) + [(k+q) + 2 \cdot 2],\\
c(u_{2p-3}) &= c(v_{2p-2}) + [(k+q) + 2 \cdot 3],\\
&\cdots\cdots\cdots\cdots\cdots\cdots,\\
c(v_2) &= c(u_3) + [(k+q) + 2(2p-2)],\\
c(u_1) &= c(v_2) + [(k+q) + 2(2p-1)]\\
&= c(z_q) + 2p(k+q) + 2 \cdot \tfrac{2p(2p-1)}{2}\\
&= c(z_q) + 2p(k+q+2p-1).
\end{aligned}$$

Then by a similar method to prove Claim 3.6, we can obtain the following claim.

**Claim 3.7.** For all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, the inequality (2) holds. And $\max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 1)$.

By Claim 3.7, we have shown that for all odd integers $n \geq 15$, $c$ is a nearly antipodal coloring for $P_n$. Therefore $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c) = \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 1)$. To finish the proof of Theorem 2.1 for all odd integers $n \geq 15$, it suffices to prove the following claim.

**Claim 3.8.** For any $p \in \{0, 1, 2, \ldots\}$ and any $q \in \{1, 2, 3, 4\}$, it holds that $c(u_1) = c(z_q) + 2p(k + q + 2p - 1) = \binom{n-2}{2} - \frac{n-1}{2} + 8$, where $n = 2k + 1 = 13 + 2(4p + q)$.

In fact, if $q = 1$, then $k = 4p + 7$, $4p = k - 7$, $2p = \frac{k-7}{2}$. Thus

$$c(u_1) = c(z_1) + 2p(k + q + 2p - 1) = 9k + 16 + 2p(2k + 2p + 9) + 2p(k + 2p)$$
$$= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8.$$

If $q = 2$, then $k = 4p + 8$, $4p = k - 8$, $p = \frac{k-8}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_2) + 2p(k + q + 2p - 1) \\ &= 10k + 25 + 2p(2k + 2p + 11) + 2p(k + 2p + 1) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

If $q = 3$, then $k = 4p + 9$, $4p = k - 9$, $p = \frac{k-9}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_3) + 2p(k + q + 2p - 1) \\ &= 12k + 27 + 2p(2k + 2p + 11) + 2p(k + 2p + 2) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

If $q = 4$, then $k = 4p + 10$, $4p = k - 10$, $2p = \frac{k-10}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_4) + 2p(k + q + 2p - 1) \\ &= 13k + 39 + 2p(2k + 2p + 13) + 2p(k + 2p + 3) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

Thus Claim 3.8 holds and hence $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c) = \binom{n-2}{2} - \frac{n-1}{2} + 8$ for all odd integers $n \geq 15$.

Secondly, for $n = 13$, in the above proof we take $p = 0$ and $q = 0$. Namely, $V_2 = V_3 = \emptyset$, $V(P_{13}) = V_1 = \{x_1', x_2', x_3', x_4'; x_2, x_1; x_0; y_1, y_2; y_4', y_3', y_2', y_1'\}$ (also see Figure 2 and let $p = q = 0$). Then coloring $c|_{v \in V_1}(v)$ is a nearly antipodal coloring for $P_{13}$. Thus by Claim 3.5, $\mathrm{ac}'(P_{13}) \leq \mathrm{ac}'(c|_{v \in V_1}) = \max_{v \in V_1} c(v) = c(y_2) = (8k+8)|_{k=6} = 56 = \binom{13-2}{2} + 1$. Since $-\lfloor \frac{13}{n} \rfloor = -1$ for $n = 13$, it follows that $\mathrm{ac}'(P_{13}) \leq \mathrm{ac}'(c|_{v \in V_1}) = \binom{13-2}{2} + 1 = \binom{13-2}{2} - \frac{13-1}{2} - \lfloor \frac{13}{13} \rfloor + 8$.

Thus the assertion 2 in Theorem 2.1 holds. ■

# 4. Examples

In this section we give some examples which present the nearly antipodal coloring $c$ for some $P_n$ with $\mathrm{ac}'(c)$ presented in Theorem 2.1 by our methods.

**Example 4.1.** A nearly antipodal coloring $c$ for $P_{10}$ with $ac'(c) = \binom{10-2}{2} - \frac{10}{2} - \lfloor \frac{10}{10} \rfloor + 7 = \binom{10-2}{2} + 1 = 29$ (see Figure 3).

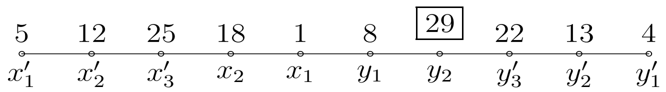


Figure 3. A nearly antipodal coloring for $P_{10}$.

**Example 4.2.** A nearly antipodal coloring $c$ for $P_{13}$ with $ac'(c) = \binom{13-2}{2} - \frac{13-1}{2} - \lfloor \frac{13}{13} \rfloor + 8 = \binom{13-2}{2} + 1 = 56$ (see Figure 4).

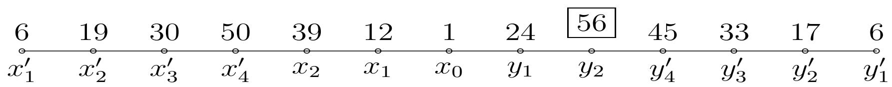


Figure 4. A nearly antipodal coloring $c$ for $P_{13}$.

**Example 4.3** A nearly antipodal coloring $c$ for $P_{32}$ with $ac'(c) = \binom{32-2}{2} - \frac{32}{2} + 7 = \binom{32-2}{2} - 9 = 426$ (see Figure 5).

Here $n = 2k = 10 + 2(4p + q) = 32$, then $k = 16$, $p = 2$ and $q = 3$.

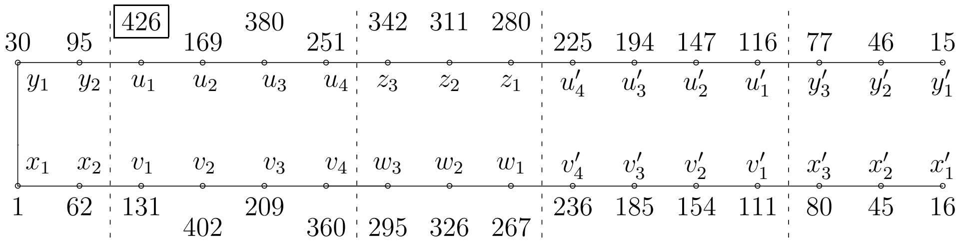


Figure 5. A nearly antipodal coloring for $P_{32}$.

**Example 4.4.** A nearly antipodal coloring $c$ for $P_{33}$ with $ac'(c) = \binom{33-2}{2} - \frac{33-1}{2} + 8 = \binom{33-2}{2} - 8 = 457$ (see Figure 6).

Here $n = 2k + 1 = 13 + 2(4p + q) = 33$, then $k = 16$, $p = 2$ and $q = 2$.

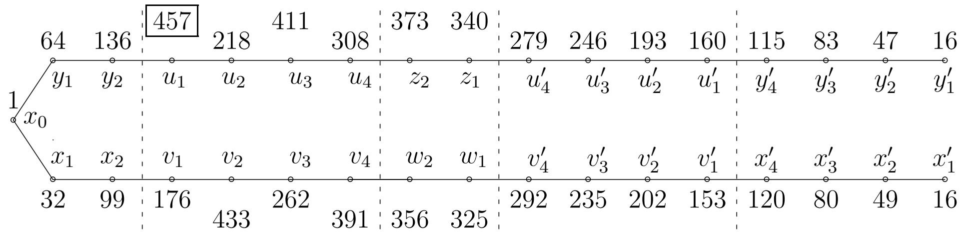


Figure 6. A nearly antipodal coloring $c$ for $P_{33}$.

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