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NONSINGULAR UNICYCLIC MIXED GRAPHS WITH AT MOST THREE EIGENVALUES GREATER THAN TWO*

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Abstract

This paper determines all nonsingular unicyclic mixed graphs on at least nine vertices with at most three Laplacian eigenvalues greater than two.

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1. Introduction

Let $G = (V, E)$ be a *mixed graph* with vertex set $V = V(G)$ and edge set $E = E(G)$, which is obtained from a simple graph by orienting some (possibly none or all) of its edges. For each $e \in E(G)$, we define the *sign* of e and denote by $\operatorname{sgn} e = 1$ if e is unoriented and $\operatorname{sgn} e = -1$ if e is oriented. Set $a_{ij} = \operatorname{sgn} e$ if there exists an edge e joining v_i and v_j, and $a_{ij} = 0$ otherwise. Then the resultant matrix $A = (a_{ij})$ is called the *adjacency matrix* of G. The *incidence matrix* of G is an $n \times m$ matrix $M = M(G) = (m_{ij})$ whose entries are given by $m_{ij} = 1$ if e_j is an unoriented edge incident to v_i or e_j is an oriented edge with head v_i, $m_{ij} = -1$ if e_j is an oriented edge with tail v_i, and $m_{ij} = 0$ otherwise. The *Laplacian matrix* of G is defined as $L(G) = MM^T$ (see [1] or [10]), where M^T denotes the transpose of M. Obviously $L(G)$ is symmetric and positive semi-definite, and $L(G) = D(G) + A(G)$ (or see [10, Lemma 2.1]), where $D(G) = \operatorname{diag}\{d(v_1), d(v_2), \ldots, d(v_n)\}$. Therefore the eigenvalues of $L(G)$ can be arranged as follows:

$$\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G) \geq 0.$$

We briefly called the eigenvalues and eigenvectors of $L(G)$ as those of G, respectively. G is called *singular* (or *nonsingular*) if $L(G)$ is singular (or nonsingular).

A mixed graph G is called *quasi-bipartite* if it does not contain a nonsingular cycle, or equivalently, G contains no cycles with an odd number of unoriented edges (see [1, Lemma 1]). Denote by $\overrightarrow{G}$ the all-oriented graph obtained from G by arbitrarily orienting every unoriented edge of G (if one exists), and D the *signature matrix* with 1 or -1 along its diagonal of a diagonal matrix. Then $D^T L(G) D$ is the Laplacian matrix of a graph with the same underlying graph as that of G. So each signature matrix of order n gives a re-signing of the edges of G (that is, some oriented edges of G may turn to be unoriented and vise versa), and preserves the spectrum and the singularity of each cycle of G. We now use the notation ${}^D G$ to denote the graph obtained from G by a re-signing under the signature D, and assume that the labelling of the vertices of ${}^D G$ is the same as that of G.

Lemma 1.1 ([10, Lemma 2.2], [4, Lemma 5]). *Let G be a connected mixed graph. Then G is singular if and only if G is quasi-bipartite.*

Theorem 1.2 ([1, Theorem 4]). *Let G be a connected mixed graph. Then G is quasi-bipartite if and only if there exists a signature matrix D such that $D^T L(G) D = L(\overrightarrow{G})$.*

If G is nonsingular, the number of edges of G is at least n (the number of vertices of G), since such graph G contains at least one nonsingular cycle, then nonsingular unicyclic mixed graphs may be considered as a class of mixed graphs whose edge number is minimal. By Lemma 1.1 and Theorem 1.2, the spectrum of a singular mixed graph is exactly that of a simple graph with the same underlying graph, one can refer to [10, 11, 3, 5]. So in this paper, we consider only the connected nonsingular unicyclic mixed graphs, and determine all those graphs G on at least 9 vertices with at most three eigenvalues greater than two, i.e., $\lambda_4(G) \le 2$. Then we could almost give all mixed graphs with at most three eigenvalues greater than two, since we can obtain the eigenvalues by mathematical softwares if G contains few vertices. A reason for our research can be explained as follows. Consider the *edge version* of the Laplacian matrix of G, $K(G) = M(G)^T M(G) = 2I + A(G^l)$ (see [2]), where G^l is the line graph of G (see [10]). Since $K(G)$ and $L(G)$ have the same nonzero eigenvalues, the distribution of eigenvalues of $L(G)$ greater than 2 is the same as that of $A(G^l)$ greater than 0.

2. Preliminaries

Lemma 2.1 ([11, Lemma 2.2]). *Let G be a mixed graph on n vertices and let e be an (oriented or unoriented) edge of G. Then*

$$\lambda_1(G) \ge \lambda_1(G-e) \ge \lambda_2(G) \ge \ldots \ge \lambda_n(G) \ge \lambda_n(G-e).$$

We now extend some known results on eigenvalues distribution of simple graphs to mixed graphs.

Theorem 2.2. *Let G be a connected mixed graph on n vertices, and let $\mu(G)$ be the matching number of G. Then*

(i) $m_G(2, +\infty) \ge \mu(G)$ *if* $n > 2\mu(G)$;

(ii) $m_G(2, +\infty) \ge \mu(G) - 1$ *if* $n = 2\mu(G)$.

Proof. Let $M \subseteq E(G)$ be a matching of G with maximum cardinality $\mu(G)$. There exists a spanning tree T of G which contains the matching M, and a signature matrix D such that ${}^D T$ is all-oriented in the graph ${}^D G$ by Theorem 1.2. Note that $\mu(T) = \mu(G)$. Then by Lemma 2.1 and the result of [8, Theorem 3], for the case of $n > 2\mu(G)$,

$$m_G(2, +\infty) = m_{{}^D G}(2, +\infty) \geq m_{{}^D T}(2, +\infty) \geq \mu(G).$$

For the case of $n = 2\mu(G)$, by [8, Theorem 2], $\lambda_{\mu(G)}({}^D T) = 2$, and $m_{{}^D T}(2, +\infty) = \mu(G) - 1$ from the fact that any integral eigenvalue greater than 1 of a tree has multiplicity one [7]. Then the result (ii) can be obtained similarly. ■

Corollary 2.3. *Let G be a nonsingular unicyclic mixed graph on at least 9 vertices. If $m_G(2, +\infty) \leq 3$, then G contains no cycles with length greater than 6.*

Proof. If follows from Theorem 2.2 that $\mu(G) \geq 4$ cannot happen. ■

A pendent vertex of G is a vertex of degree 1, a quasi-pendant vertex is a vertex adjacent to a pendant vertex. Denote by $\eta(G)$ the number of quasi-pendent vertices of G.

Lemma 2.4. *Let G be a connected mixed graph. Then $m_G[0, 1) \geq \eta(G)$ and $m_G(2, +\infty) \geq \eta(G)$.*

Proof. Let $v_1, v_2, \ldots, v_{\eta(G)}$ be all quasi-pendant vertices of G. By Lemma 2.1, there exists a signature matrix D such that the pendant edges of ${}^D G$ are all oriented. Let L_i $(i = 1, 2, \cdots, \eta(G))$ be the principal submatrix of $L({}^D G)$ corresponding to the vertex v_i and all pendent vertices incident to it, which permutes to the following matrix form:

$$L_i' = \begin{bmatrix} d(v_i) & -1 & \cdots & -1 \\ -1 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix}.$$

It is seen that the left-up 2×2 principal submatrix of L_i' has one eigenvalue less than 1 and one eigenvalue greater than 2 as $d(v_i) \geq 2$. By Courrant-Fischer interlacing theorem [9, Theorem 4.3.15], L_i' has at least one eigenvalue less than 1 and at least one eigenvalue greater than 2. As $\bigoplus_{i=1}^{\eta(G)} L_i'$

is a principal submatrix of $L(^DG)$, $L(G)$ has at least $\eta(G)$ eigenvalues less than 1 and at least $\eta(G)$ eigenvalues greater than 2 (including multiplicity). ■

3. Main Results

Let G be a connected graph with the property

$$\lambda_4(G) \le 2. \tag{3.1}$$

The property (3.1) is hereditary, because as a direct consequence of Lemma 2.1, for any (not necessarily induced) subgraph $U(\subseteq G)$ also satisfies (3.1). The inheritance(hereditary) of property (3.1) implies that there exist minimal connected graphs that do not obey (3.1); such graphs are called *forbidden subgraphs* for $\lambda_4(G) \le 2$. It is easy to verify that the graphs $H_1(1,2,2)$, $H_1(1,1,3)$, $H_2(3,3)$, $H_2(4,2)$, $H_3(2,3)$, $H_3(3,2)$ and H_4 listed in Figure 3.1 are forbidden subgraphs for $\lambda_4(G) \le 2$, where K_p^c (the complement graph of a complete graph on p vertices) is a graph consisting of p isolated vertices.

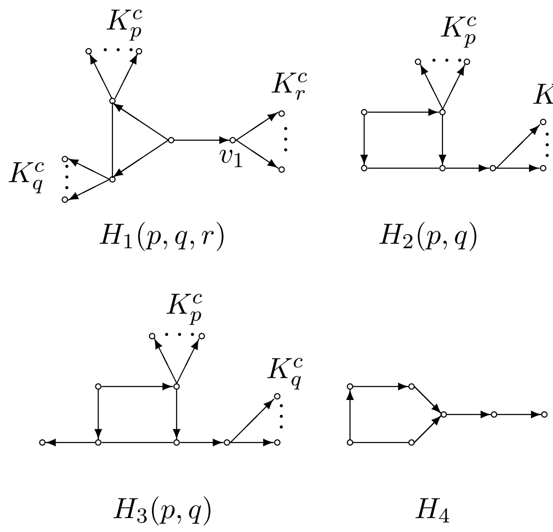


Figure 3.1. Forbidden Graphs

The nonsingular cycle of length k will be denoted by C_k, the set of neighbors of v in G will be denoted by $N(v)$, the cardinality of the set S will be denoted by $|S|$. If G is a connected nonsingular unicyclic mixed graph, by Lemma 1.1 and Theorem 1.2, the spectrum of G is exactly that of a graph G', which has the same underlying graph with G and which contains all oriented except an (arbitrary) unoriented edge on the cycle. So in the following, to convenience our discussion, we always consider the graph $G(|G| \geq 9)$ with all oriented except an (arbitrary) unoriented edge on the cycle.

Let $G_1 = G_1(p,q,r,s,t)$; $G_2 = G_2(p)$; $G_3 = G_3(p,q)$; $G_4 = G_4(p,q)$; $G_5 = G_5(p,q,r)$; $G_6 = G_6(p,q,r)$; $G_7 = G_7(p,q,r,s)$ listed in Figure 3.2 be unicyclic mixed graphs on at least nine vertices, where $p \geq 0, q \geq 0$, $r \geq 0, s \geq 0$ *and* $t \geq 0$. They will play an important role in our discussion.

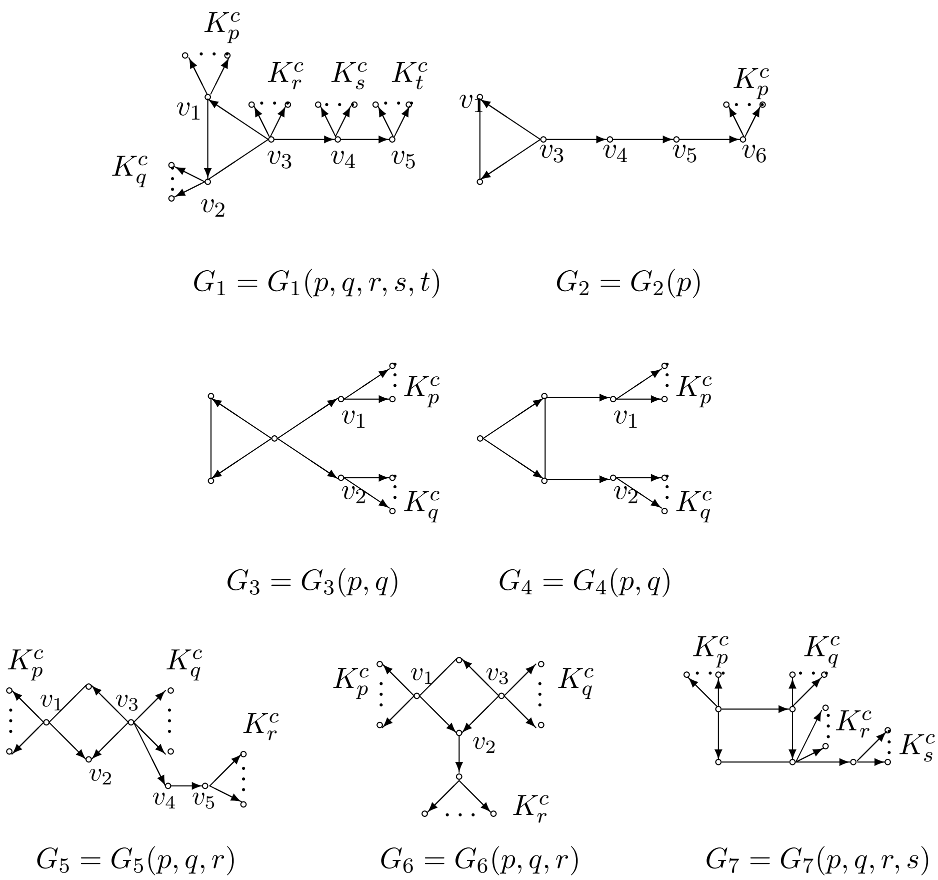


Figure 3.2

Lemma 3.1. *Let G be a connected nonsingular unicyclic mixed graph on at least 9 vertices. If $m_G(2, +\infty) \leq 3$, then G contains cycle of length less than 7, and G is one of the following types:*

(1) *types U_1 of Figure 3.3, $G_1(p, q, 0, 0, 0)$, $G_1(0, 0, 0, s, t)$, $G_1(p, 0, r, s, 0)$, $G_1(p, 0, r, 0, t)$, G_2, G_3 and G_4 of Figure 3.2, if G contains cycle C_3;*

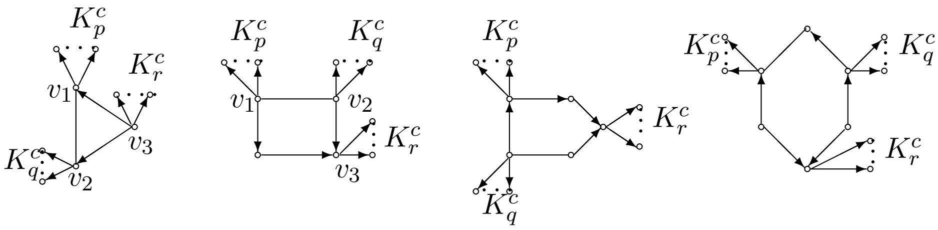
(2) *types U_2 of Figure 3.3, G_5, $G_6(p, q, 1)$, $G_6(1, 1, r)$, $G_7(p, 0, r, s)$, $G_7(0, q, 0, s)$ ($q \leq 2$ and $s \geq 0$, or $q \geq 0$ and $s \leq 1$), or graphs $G_6(2, 1, 2)$ and $G_7(0, 3, 0, 2)$ of Figure 3.2, if G contains cycle C_4;*

(3) *types U_3 (or U_4) of Figure 3.3, if G contains cycle C_5 (or C_6).*

Proof. By Theorem 2.2 and Corollary 2.3, we have

$$\mu(G) \leq 3, \tag{3.2}$$

and G contains exactly one cycle C_i for some i ($3 \leq i \leq 6$).



$U_1 = U_1(p, q, r)$ $U_2 = U_2(p, q, r)$ $U_3 = U_3(p, q, r)$ $U_4 = U_4(p, q, r)$

Figure 3.3. $p \geq 0, q \geq 0, r \geq 0$

We discuss the problem in the following cases.

Case 1. G contains cycle C_3. Let $U_1 = U_1(p, q, r)$ (assuming that $p \geq q \geq r$) be the subgraph of G induced by the vertices of C_3 together with those vertices incident to this cycle (see Figure 3.3).

(1.1) $r \geq 2$. Then $G = U_1$.

(1.2) $r = 1$. If $G \neq U_1$, by (3.2) G contains a subgraph isomorphic to $H_1(1, 1, 1)$, and each pendant vertex of $H_1(1, 1, 1)$ is also the pendant vertex of G. Without loss of generality, let $G_1(1, 1, 0, 0, 0)$ be the subgraph of G. By the fact that G contains at least 9 vertices and $H_1(1, 2, 2), H_1(1, 1, 3)$ of Figure 3.1 are forbidden subgraphs, G has the structure of type $G_1(p, q, 0, 0, 0)$ of Figure 3.2.

(1.3) $r = 0$, $q \geq 1$. If $p = 1$ (necessarily $q = 1$), then G has the structure of type $G_4(p, q)$ of Figure 3.2. If $p \geq 2$, by (3.2) there exists at most one pendant vertex adjacent to v_1 or v_2 in U_1, denote by v_4, which joins vertices of G except those of U_1. If v_4 joins exactly one vertex of $V(G)\backslash V(U_1)$, then G has the structure of type $G_1(p, 0, r, 0, t)$; otherwise G has the structure of type $G_1(p, 0, r, s, 0)$ of Figure 3.2.

(1.4) $r = 0, q = 0, p \geq 3$. Then there exists at most one pendant vertex adjacent to v_1 in U_1, also denoted by v_4, which joins vertices of G except those of U_1. If $|N(v_4)\backslash\{v_1\}| \geq 2$, then G has structure of type $G_1(0, 0, r, s, 0)$; and if $|N(v_4)\backslash\{v_1\}| = 1$, then G must be of the type $G_1(0, 0, r, 0, t)$.

(1.5) $r = 0, q = 0, p = 2$. If there exists at most one pendant vertex incident to v_1 in U_1, which joins vertices of G except those of U_1, then the discussion is similar to the case (4). Otherwise, the two pendant vertices incident to v_1 in U_1 have their own neighbors in G except v_1. Hence the structure of G must be of type $G_3(p, q)$.

(1.6) $r = 0, q = 0, p = 1$. Then the longest path P of the subgraph $G - U_1$ has length not greater than 2 by (3.2). If the length of P is 2, then G has the structure of type $G_2(p)$; and if the length of P is at most 1, then the structure of G must be of type $G_1(0, 0, 0, s, t)$.

Case 2. G contains cycle C_4. Let U_2 be the subgraph of G induced by the vertices of C_4 together with all vertices incident to the cycle, see Figure 3.3 (assuming that $p \geq r$). Note that by (3.2) there exists at most one pendant vertex of U_2 adjacent to vertices of $V(G)\backslash V(U_2)$. We discuss the problem in following subcases.

(2.1) Each of p, q, r is nonzero. If $q \geq 2$, then $G = U_2$ by (3.2). If $q = 1$ and $p \geq 2$, then, in the graph U_2, only the pendant vertex adjacent to v_2 has neighbors in $G - U_2$. As the graphs $H_2(3, 3)$, $H_2(4, 2)$ and $H_3(2, 3), H_3(3, 2)$ of Figure 3.1 are forbidden, G is of type $G_6(2, 1, r)$ with $r \leq 2$, or of type $G_6(p, q, 1)$. If $q = 1$ and $p = 1$ (necessarily $r = 1$), then, in the graph U_2, only the pendant vertex adjacent to v_2 or v_4 has neighbors in $G - U_2$, and G is of type $G_6(1, 1, r)$.

(2.2) Exactly two of p, q, r are nonzero. Then U_2 is of type $U_2(0, q, r)$ or $U_2(p, 0, r)$. For U_2 being the former type, as $H_2(3, 3)$ and $H_2(4, 2)$ are forbidden, G is of type $G_7(0, q, 0, s)$ ($q \leq 2$ and $s \geq 0$, or $q \geq 0$ and $s \leq 1$), or graph $G_7(0, 3, 0, 2)$; and for U_2 being the latter type, G is of type $G_7(p, 0, r, s)$ or $G_5(p, q, r)$.

(2.3) Exactly one of p, q, r is nonzero. Without loss of generality, let U_2 be the type $U_2(0, 0, r)$. Then G is of type $G_7(0, 0, r, s)$ or $G_5(0, q, r)$.

Case 3. G contains cycle C_5 or C_6. By (3.2) and the forbidden graph H_4 of Figure 3.1, the structure of G must be of type U_3 or U_4 of Figure 3.3. ■

Let $G = (V, E)$ be a mixed graph with $V = \{v_1, v_2, \ldots, v_n\}$, and let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be a real vector. It will be convenient to adopt the following terminology from [6]: x is said to give a valuation of the vertices of V, that is, for each vertex v_i of V, we associate the value x_i, i.e., $x(v_i) = x_i$. Then λ is an eigenvalue of G with the corresponding eigenvector $x = (x_1, x_2, \ldots, x_n)$ if and only if $x \neq 0$ and

$$(3.3) \qquad [\lambda - d(v_i)]x(v_i) = \sum_{e=\{v_i,v_j\}\in E} (\operatorname{sgn} e)x(v_j), \text{ for } i = 1, 2, \ldots, n.$$

Proposition 3.2. *Suppose G is a connected nonsingular unicyclic mixed graph on at least* 9 *vertices. If G is one of types U_1, $G_1(p, q, 0, 0, 0)$, $G_1(0, 0, 0, s, t)$, $G_1(p, 0, r, s, 0)$, $G_1(p, 0, r, 0, t)$, G_2, G_3 and G_4 as in Figure* 3.2 *or* 3.3, *where $p \geq 0$, $q \geq 0$, $r \geq 0$, $s \geq 0$ and $t \geq 0$, then $m_G(2, +\infty) \leq 3$.*

Proof. For the graph U_1, by Lemma 2.1, it suffices to prove the graph $U_1(m, m, m)$ (denoted still by U_1) holds $m_{U_1}(2, +\infty) \leq 3$, where $m = \max(p, q, r) \geq 1$, since $U_1(p, q, r) \subseteq U_1(m, m, m)$. By (3.3) and a direct calculation, we have that all eigenvalues of the graph U_1 distinct from 1 are determined by the equation

$$\Psi_{U_1}(\lambda) = (\lambda^2 - m\lambda - 5\lambda + 4)(\lambda^2 - m\lambda - 2\lambda - 1)^2.$$

Then, by Lemma 2.4, $m_{U_1}(0, 1) \geq 3$ as $\eta(U_1) = 3$. Hence, $m_{U_1}(2, +\infty) \leq 3$.

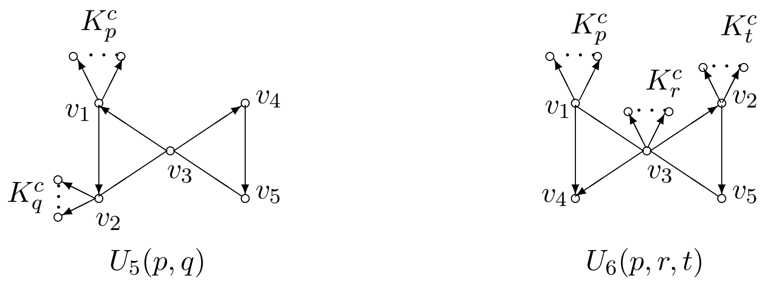


Figure 3.4

For the graphs of types $G_1(p, q, 0, 0, 0)$, $G_1(0, 0, 0, s, t)$ and G_3, by Lemma 2.1, it suffices to discuss the graph of type $U_5(p, q)$ ($p \geq 0, q \geq 0$ and

$p+q \geq 4$) of Figure 3.4, since $G_1(p,q,0,0,0) = U_5(p,q) - (v_3,v_5)$, $G_1(0,0,0,s,t) = U_5(p,q) - (v_2,v_3)$ and the spectrum of G_3 is same to that of the graph $U_5(p,q) - (v_1,v_2)$. For $U_5(p,q)$, let $m = \max(p,q) \geq 2$. Similarly, the eigenvalues of $U_5(m,m)$ (denoted by U_5) distinct from 1 are determined by the equation $\Psi_{U_5}(\lambda) = f_1(\lambda)g_1(\lambda) = 0$, where

$$f_1(\lambda) = \lambda^2 - m\lambda - 2\lambda + 1,$$

$$g_1(\lambda) = \lambda^4 - (m+11)\lambda^3 + (7m+39)\lambda^2 - (10m+53)\lambda + 24.$$

Then $m_{U_5}(1,2) \geq 1$, since $g_1(1) = -4m < 0$, $g_1(2) = 2 > 0$, and $m_{U_5}(0,1) \geq 1$ as $\eta(U_5) = 2$, by Lemma 2.4. Hence $m_{U_5}(2,+\infty) \leq 3$.

For the graphs of types $G_1(p,0,r,s,0)$ and $G_1(p,0,r,0,t)$, similar to above, it suffices to discuss the graph $U_6(p,r,t)$ ($p \geq 0, r \geq 0, t \geq 0$ and $p+r+t \geq 4$) of Figure 3.4, since $G_1(p,0,r,s,0) = U_6(p,r,t) - (v_3,v_5)$ and the spectrum of $G_1(p,0,r,0,t)$ is same to that of the graph $U_6(p,r,t) - (v_2,v_3)$. For $U_6(p,r,t)$, let $m = \max(p,r,t) \geq 2$. Then the eigenvalues of $U_6(m,m,m)$ (denoted by U_6) distinct from 1 are determined by the equation $\Psi_{U_6}(\lambda) = f_2(\lambda)g_2(\lambda) = 0$, where

$$f_2(\lambda) = \lambda^3 - (m+5)\lambda^2 + (2m+7)\lambda - 3,$$

and

$$g_2(\lambda) = \lambda^5 - (2m+10)\lambda^4 + (m^2+12m+32)\lambda^3 - (2m^2-19m+46)\lambda^2 + (9m+31)\lambda - 8.$$

Then, by a discussion similar to that of U_5, $m_{U_6}(2,+\infty) \leq 3$.

For the graph of type G_2, by a direct calculation, the eigenvalues of the graph G_2 distinct from 1 are determined by the equation $\Psi_{G_2}(\lambda) = (\lambda-1)g_3(\lambda) = 0$, where

$$g_3(\lambda) = \lambda^6 - (p+12)\lambda^5 + (10p+53)\lambda^4 - (33p+108)\lambda^3 + (41p+104)\lambda^2 - (15p+42)\lambda + 4.$$

Then $2 \leq m_{G_2}(2,+\infty) \leq 4$, since $g_3(1) = 2p > 0$, $g_3(2) = -2p < 0$ and $\eta(G_2) = 1$. If $m_{G_2}(2,+\infty) = 4$, then yield a contradiction to $\Psi_{G_2}(2) = (2-1) \times g_3(2) = -2p < 0$. Hence $m_{G_2}(2,+\infty) \leq 3$.

For the graph of type $G_4(p,q)$, by a similar method to that of U_1, it suffices to consider the graph $G_4(m,m)$, where $m = \max(p,q) \geq 2$. By a direct calculation, the eigenvalues of the graph $G_4(m, m)$ distinct from 1 are determined by the equation

$$\begin{aligned}\Psi_{G_4}(\lambda) &= \lambda^7 - (2m+12)\lambda^6 + (m^2+22m+54)\lambda^5 - (10m^2+88p+120)\lambda^4 \\ &\quad - (34m^2-160m-144)\lambda^3 - (48m^2+164m+94)\lambda^2 \\ &\quad - (24m^2-78m-31)\lambda - 14m - 4.\end{aligned}$$

Then $m_{G_4}(1,2) \geq 1$ as $\Psi_{G_4}(1) = m^2 > 0$ and $\Psi_{G_4}(2) = 2 - 2m < 0$. On the other hand, by Theorem 2.2(i) and Theorem 2.4, we have $m_{G_4}(2,\infty) \geq 3$ and $m_{G_4}[0,1) \geq 2$. Consequently, $m_{G_4}(2,+\infty) \leq 3$, otherwise, it will yield a contradiction to $\Psi_{G_4}(2) < 0$. ■

Proposition 3.3. *Suppose G is a connected nonsingular unicyclic mixed graph on at least 9 vertices. If G is one of types U_2, G_5, $G_6(p,q,1)$, $G_6(p,q,1)$, $G_6(1,1,r)$, $G_7(p,0,r,s)$, $G_7(0,q,0,s)$ ($q \leq 2$ and $s \geq 0$, or $q \geq 0$ and $s \leq 1$), $G_6(2,1,2)$ and $G_7(0,3,0,2)$ listed in Figure 3.2 or 3.3, where $p \geq 0, q \geq 0, r \geq 0, s \geq 0$, then $m_G(2,+\infty) \leq 3$.*

Proof. The result can be verified directly if G is $G_6(2,1,2)$ or $G_7(0,3,0,2)$. For the graph of type U_2, by (3.3), the eigenvalues of the graph $U_2(m,m,m)$ (still denoted by U_2) distinct from 1 are determined by the equation $\Psi_{U_2}(\lambda) = f_4(\lambda)g_4(\lambda) = 0$, where

$$f_4(\lambda) = \lambda^3 - (m+5)\lambda^2 + (2m+6)\lambda - 2,$$

$$g_4(\lambda) = \lambda^4 - (2m+6)\lambda^3 + (m^2+6m+11)\lambda^2 - (4m+8)\lambda + 2,$$

and $m = \max(p,q,r) \geq 1$. Then $m_{U_2}(2,+\infty) \leq 3$, since $f_4(1) = p > 0$, $f_4(2) = -2 < 0$, $g_4(1) = m^2 > 0$, $g_4(2) = 4m^2 - 2 > 0$ and $\eta(U_2) = 3$.

For the graph of type G_5, by Lemma 2.1, it suffices to discuss the graph G_5+e, where $e = (v_3, v), v \in K_r^c$, is unoriented. Let $m = \max(p,q,r) \geq 1$, by (3.3), the eigenvalues of the graph $G_5(m,m,m+1)+e$ (still denoted by G_5) distinct from 1 are determined by the equation $\Psi_{G_5}(\lambda) = (\lambda-2)f_5(\lambda)g_5^2(\lambda) = 0$, where

$$f_5(\lambda) = \lambda^3-(m+7)\lambda^2+(2m+10)\lambda-4, \quad g_5(\lambda) = \lambda^3-(m+5)\lambda^2+(2m+6)\lambda-2.$$

Then $m_{G_5}(1,2) \geq 3$, since $f_5(1) = m > 0$, $f_5(2) = -4 < 0$, $g_5(1) = m > 0$, $g_5(2) = -2 < 0$. And $m_{G_5}(0,1) \geq 3$ as $\eta(G_5) = 3$ by Lemma 2.4. Hence, $m_{G_5}(2,+\infty) \leq 3$.

For the graph of type $G_6(p,q,1)$, it suffices to discuss the graph $G_6(m,m,1)$ with $m=\max(p,q)\geq 1$, denoted by G_{61}. By (3.3), the eigenvalues of the graph G_{61} distinct from 1 are determined by the equation $\Psi_{G_{61}}(\lambda)=f_{61}(\lambda)g_{61}(\lambda)=0$, where

$$\begin{aligned} f_{61}(\lambda) &= \lambda^3-(m+5)\lambda^2+(2m+6)\lambda-2, g_{61}(\lambda) \\ &= \lambda^5-(m+9)\lambda^4+(6m+27)\lambda^3-(9m+33)\lambda^2+(2m+16)\lambda-2. \end{aligned}$$

Then $m_{G_{61}}(1,2)\geq 2$, since $f_{61}(1)=m>0$, $f_{61}(2)=-2<0$, $g_{61}(1)=-2m<0$, $g_{61}(2)=2>0$. And, by Lemma 2.3, $m_{G_{61}}(0,1)\geq 3$ as $\eta(G_{61})=3$. Hence, $m_{G_{61}}(2,+\infty)\leq 3$.

For the graph of type $G_6(1,1,r)$, denote by G_{62}. By (3.3), the eigenvalues of the graph G_{62} distinct from 1 are determined by the equation $\Psi_{G_{62}}(\lambda)=f(\lambda)g(\lambda)=0$, where

$$\begin{aligned} f_{62}(\lambda) &= \lambda^3-6\lambda^2+8\lambda-2, \\ g_{62}(\lambda) &= \lambda^5-(r+9)\lambda^4+(7r+26)\lambda^3-(12r+30)\lambda^2+(4r+14)\lambda-2. \end{aligned}$$

By a discussion similar to the graph G_{61}, we have $m_{G_{62}}(2,+\infty)\leq 3$.

For the graph of type $G_7(p,0,r,s)$, similarly, the eigenvalues of the graph $G_7(m,0,m,m)$ with $m=\max(p,r,s)\geq 1$ distinct from 1 are determined by the equation $\Psi_{G_7}(\lambda)=f_7(\lambda)g_7(\lambda)=0$, where

$$\begin{aligned} f_7(\lambda) &= \lambda^3-(m+5)\lambda^2+(2m+6)\lambda-2, \\ g_7(\lambda) &= \lambda^5-(2m+8)\lambda^4+(m^2+10m+21)\lambda^3 \\ &\quad -(2m^2+14m+24)\lambda^2+(6m+12)\lambda+4p-2. \end{aligned}$$

By a discussion similar to that of the graph G_{61}, we also have $m_{G_7}(2,+\infty)\leq 3$.

For the graph of type $G_7(0,q,0,s)$ ($q\leq 2$ and $s\geq 0$, or $q\geq 0$ and $s\leq 1$), it suffices to discuss the graphs $G_7(0,2,0,s)$ ($s\geq 2$) and $G_7(0,q,0,1)$ ($q\geq 3$), denoted respectively by G_{71} and G_{72}. The eigenvalues of G_{71} distinct from 1 are determined by the equation

$$\begin{aligned} \Psi_{71}(\lambda) &= \lambda^7-(s+14)\lambda^6+(73+12s)\lambda^5-(49s+180)\lambda^4 \\ &\quad +(80s+224)\lambda^3-(48s+140)\lambda^2+(8s+40)\lambda-4=0. \end{aligned}$$

Observe that $\Psi_{71}(1) = 2s > 0$ and $\Psi_{71}(2) = -4 < 0$ so that $m_{G_{71}}(1, 2) \geq 1$. By Lemma 2.4, $m_{G_{71}}(0, 1) \geq 2$ as $\eta(G_{71}) = 2$. Hence, $\Psi_{71}(\lambda) = 0$ has at most 4 roots greater than two. If it has exactly 4 roots greater than two, then $\Psi_{71}(2) > 0$ which yields a contradiction to $\Psi_{71}(2) = -4$. So we have $m_{G_{71}}(2, +\infty) \leq 3$. The eigenvalues of G_{72} distinct from 1 are determined by the equation

$$\Psi_{72}(\lambda) = \lambda^7 - (q + 13)\lambda^6 + (65 + 10q)\lambda^5 - (35q + 159)\lambda^4 + (51q + 202)\lambda^3 - (28q + 132)\lambda^2 + (4q + 40)\lambda - 4 = 0.$$

By discussion in a similar way we also have $m_{G_{72}}(2, +\infty) \leq 3$. ■

Proposition 3.4. *If G is of type $U_3(p, q, r)$ or $U_4(p, q, r)$ listed in Figure 3.3, where $p \geq 0$, $q \geq 0$, $r \geq 0$, then $m_G(2, +\infty) \leq 3$.*

Proof. Obviously, it suffices to discuss the graphs $U_3(m, m, m)$ and $U_4(m, m, m)$, where $m = max(p, q, r)$, still denoted respectively by U_3, U_4. By (3.3), the eigenvalues of the graphs U_3 and U_4 distinct from 1 are respectively determined by the equations $\Psi_{U_3}(\lambda) = 0$, $\Psi_{U_4}(\lambda) = 0$, where

$$\begin{aligned}\Psi_{U_3}(\lambda) &= (\lambda^3 - m\lambda^2 - 4\lambda^2 + 2m\lambda + 4\lambda - 1)\\ &\quad \times (\lambda^5 - 2m\lambda^4 - 9\lambda^4 + m^2\lambda^3 + 11m\lambda^3 + 28\lambda^3\\ &\quad -2m^2\lambda^2 - 16m\lambda^2 - 37\lambda^2 + 7m\lambda + 21\lambda - 4),\\ \Psi_{U_4}(\lambda) &= (\lambda - 2)(\lambda^2 - m\lambda - 3\lambda + 2)(\lambda^3 - m\lambda^2 - 5\lambda^2 + 2m\lambda + 5\lambda - 1)^2.\end{aligned}$$

By a discussion similar to the Proposition 3.2 and 3.3, the result follows. ■

By Proposition 3.1, 3.2, 3.3 and 3.4, we get the main result of this paper directly.

Theorem 3.5. *Let $G = (V, E)$ be a connected nonsingular unicyclic mixed graph on at least 9 vertices. Then $m_G(2, +\infty) = 3$ if and only if there exists a signature matrix D such that ${}^DG$ is one of the following types:*

(1) *types U_1, $G_1(p, q, 0, 0, 0)$, $G_1(0, 0, 0, s, t)$, $G_1(p, 0, r, s, 0)$, $G_1(p, 0, r, 0, t)$, G_2, G_3 and G_4 of Figure 3.2 and 3.3 if G contains the cycle C_3;*

(2) *types U_2, G_5, $G_6(p, q, 1)$, $G_6(1, 1, r)$, $G_7(p, 0, r, s)$, $G_7(0, q, 0, s)$ ($q \leq 2$ and $s \geq 0$, or $q \geq 0$ and $s \leq 1$), and $G_6(2, 1, 2)$, $G_7(0, 3, 0, 2)$ of Figure 3.2 or 3.3 if G contains the cycle C_4;*

(3) *type* U_3 (*and type* U_4*, respectively*) *of Figure* 3.3 *if* G *contains the cycle* C_5 (*and the cycle* C_6*, respectively*)*, where* $p \geq 0$*,* $q \geq 0$*,* $r \geq 0$*,* $s \geq 0$*,* $t \geq 0$*.*

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