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DIGRAPHS WITH ISOMORPHIC UNDERLYING AND DOMINATION GRAPHS: CONNECTED $UG^C(D)$

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Dedicated to the memory of Kenneth P. Bogart

Abstract

The domination graph of a directed graph has an edge between vertices x and y provided either (x, z) or (y, z) is an arc for every vertex z distinct from x and y. We consider directed graphs D for which the domination graph of D is isomorphic to the underlying graph of D. We demonstrate that the complement of the underlying graph must have k connected components isomorphic to complete graphs, paths, or cycles. A complete characterization of directed graphs where $k = 1$ is presented.

Keywords: domination graph, domination, graph isomorphism, underlying graph.

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1. Introduction

Domination graphs were first introduced by Fisher, Lundgren, Merz and Reid [15] to describe the structure of the domination graphs and competition graphs of tournaments. Further refinements were made on these characterizations for tournaments in later work, including Cho, Doherty, Kim, and Lundgren [4, 5], and Fisher *et al.* [11, 12, 13, 14, 16], but the characterization of the structure of domination graphs of arbitrary digraphs has proved elusive. Here we will examine digraphs D with the property that the underlying graph of D is isomorphic to its domination graph.

A *directed graph* $D = (V, A)$ will consist of a nonempty set of vertices V and a set of ordered pairs of vertices A. We do not permit loops, but we will allow for *bidirectional arcs*: That is, the pair of arcs (x, y), (y, x). The *complement of* D, D^c has the vertex set V and the set of ordered pairs not in A, although we still exclude loops. The *underlying graph* of D, $UG(D)$ has the same set of vertices with the set of edges $\{x, y\}$ where either (x, y) or (y, x) is in A. If D has no bidirectional arcs then D is an *orientation* of $UG(D)$.

The *union* of two graphs or directed graphs is the graph formed by the union of their vertices as well as their sets of edges or arcs. The *join* of two graphs G and H, $G + H$, is the graph that consists of $G \cup H$ and all edges joining a vertex in G and a vertex in H. We extend this definition to directed graphs as follows. The *join* of D_1 and D_2 consists of $D_1 \cup D_2$ together with all bidirectional arcs between any vertex of D_1 and any vertex of D_2.

The study of domination graphs in tournaments arose from the goal of characterizing competition graphs in tournaments. A competition graph of D has vertex set V with an edge between x and y if and only if there is a third vertex z with both arcs (x, z) and (y, z) in A. Complementarily, a pair of vertices x, y in D are a *dominating pair* if and only if for all other vertices z, either (x, z) is an arc or (y, z) is an arc. The *domination graph* of D, $dom(D)$ is the undirected graph consisting of vertex set V with an edge between every dominating pair.

In the second section of this paper, previous results are presented that build a foundation for the current work. We then examine the structural necessities in the underlying graph given that we have isomorphic underlying and domination graphs. Finally, a complete characterization of digraphs with $UG(D) \cong dom(D)$ is developed where the complement of the underlying graph, $UG^c(D)$ is connected.

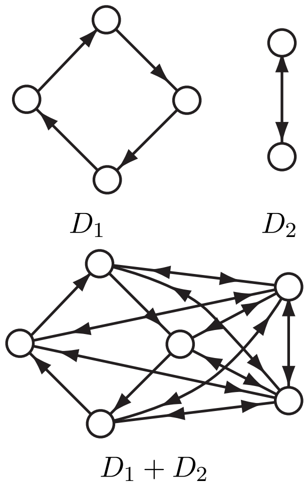


Figure 1. The join of two directed graphs.

2. Symmetric and Antisymmetric Digraphs

The underlying graph of a tournament is the complete graph on n vertices, K_n. For tournaments on more than three vertices, $dom(D)$ is not equal to K_n. A *semicomplete digraph* is a tournament with the possible addition of bidirectional arcs. Factor and Factor [8] characterize semicomplete digraphs which have domination graph equal to K_n. Factor and Langley [9] were able to fully characterize digraphs D with $UG(D) = dom(D)$. The underlying graph of such a digraph is a complete multipartite graph.

If D is an orientation of $UG(D)$ then D is also called an *antisymmetric* digraph. Bergstrand and Friedler [2] show that if $dom(D)$ is isomorphic to $UG(D)$ for an antisymmetric digraph, then $UG(D)$ must be a star or a collection of independent vertices.

If every arc in D is bidirectional, then D is a *symmetric* digraph. The characterization of $UG(D) \cong dom(D)$ for symmetric digraphs follows from a number of equivalences.

Notice that in a tournament, D^c is the reversal of D. That is, create a new tournament by reversing every arc. Through this definition for tournaments, Fisher *et al.* show that the competition graph of D is isomorphic to the domination graph of D^c [15].

The *neighborhood graph* or *two-step graph*, $N(G)$, of a graph G has an edge between vertices x and y provided both $\{x, z\}$ and $\{y, z\}$ exist in G for some third vertex z. Lundgren, Maybee and Rasmussen [18, 19] show that the competition graph of a symmetric digraph D is the neighborhood graph of $UG(D)$. Finally, Brigham and Dutton [3] characterize all graphs isomorphic to their own neighborhood graphs as follows:

Theorem 2.1 (Brigham and Dutton [3])**.** *$N(G) \cong G$ if and only if every component of G is either an odd cycle or a complete graph having other than two nodes.*

Following this chain of equivalences we have the following theorem,

Theorem 2.2 (Factor and Langley [9])**.** *Let D be a symmetric digraph. Then $UG(D) \cong dom(D)$ if and only if $UG(D)$ is the join of independent sets with other than two vertices and components that are the complements of odd cycles.*

For example, if C_n is the cycle on n vertices, where $n \geq 3$ and n is odd, by making D the complete biorientation of C_n^c, we obtain $UG(D) \cong dom(D)$.

In the following section we will consider what happens when some edges of an underlying graph are bioriented and some are not.

3. The Structure of the Underlying Graph

It is the nature of an underlying graph of a digraph D that $UG(D)$ will have many edges when $UG(D) \cong dom(D)$. This makes its complement a more desirable structure with which to work and to express results. Therefore, we refer to both the graphs of $UG(D)$ and $dom(D)$ as well as their complements throughout the course of this paper.

To begin, we show that if we have a set of underlying graphs that are isomorphic to their associated domination graphs, then their join retains that isomorphic property.

Theorem 3.1. *If $D_1, \ldots, D_k$ are directed graphs such that $UG(D_i) \cong dom(D_i)$ for $i = 1, \ldots, k$ and $D = D_1 + D_2 + \cdots + D_k$, then $UG(D) \cong dom(D)$. Also*

1. $UG(D) = \sum_{i=1}^{k} UG(D_i)$,
2. $dom(D) = \sum_{i=1}^{k} dom(D_i)$,
3. $UG^c(D) = \bigcup_{i=1}^{k} UG^c(D_i)$,
4. $dom^c(D) = \bigcup_{i=1}^{k} dom^c(D_i)$.

Proof. Let $D_1, \ldots, D_k$ be directed graphs such that $UG(D_i) \cong dom(D_i)$ for $i = 1, \ldots, k$. Let $D = \sum_{i=1}^{k} D_i$. Formula 1 follows directly from the definition of the join. Suppose $x \in D_i$ and $y \in D_j$. If $i = j$, and x and y are a dominating pair in D_i, then one or the other dominates every vertex in D_i. By the construction of the join, x and y dominate every vertex that is not in D_i, so they remain a dominating pair in D. If x and y are not a dominating pair, there must be some vertex z in D_i that both fail to dominate. They will not dominate z in the join, so they are not a dominating pair in D. If $i \neq j$, x dominates D_j, y dominates D_i and both dominate the remaining vertices, so x and y form a dominating pair in D. Since both formula 1 and 2 hold, it follows that $UG(D) \cong dom(D)$. Finally, formulas 3 and 4 follow immediately from the definition of complement. ∎

The remainder of the results in this section builds the types of components that are possible in the four graph structures. Specifically, we discuss possible adjacencies of the vertices. These adjacencies lead to cycles in the complements, independent sets in the underlying and domination graphs, and the corresponding cliques in $UG^c(D)$ and $dom^c(D)$. Using this foundation, the concluding theorem characterizes the possible structures for a graph where $UG(D)$ is isomorphic to $dom(D)$.

Lemma 3.2. *If a vertex y is adjacent to $x_1, x_2, \ldots, x_r$ in $UG^c(D)$ where $r \geq 2$, then x_i and x_j are adjacent in $dom^c(D)$, for all $1 \leq i < j \leq r$.*

Proof. Vertex y is adjacent to $x_1, \ldots, x_r$ in $UG^c(D)$ so y is not adjacent to those vertices in $UG(D)$. Thus there is no orientation of $UG(D)$ that allows x_i and x_j to dominate y. Since x_i and x_j are not adjacent in $dom(D)$, they must be adjacent in $dom^c(D)$. ∎

Corollary 3.3. *If a vertex y is not adjacent to $x_1, \ldots, x_r$ in $UG(D)$ where $r \geq 2$, then x_i and x_j are not adjacent in $dom(D)$, for all $1 \leq i < j \leq r$.*

Lemma 3.4. *If $x_1, x_2, x_3, \ldots, x_r, x_1$ is an odd length cycle in $UG^c(D)$ then $x_1, x_3, \ldots, x_r$, $x_2, x_4, \ldots, x_{r-1}, x_1$ is an odd length cycle in $dom^c(D)$.*

Proof. Consider vertices x_i, x_{i+1}, x_{i+2} of the cycle. By Lemma 3.2, x_i, x_{i+2} are adjacent in $dom^c(D)$. Since r is odd, we obtain the cycle $x_1, x_3, \ldots, x_r$, $x_2, x_4, \ldots, x_{r-1}, x_1$ in $dom^c(D)$. ∎

Corollary 3.5. *If x_1, x_2, x_3 is a 3-cycle in $UG^c(D)$, then x_1, x_2, x_3 is a 3-cycle in $dom^c(D)$.*

Corollary 3.6. *If x_1, x_2, x_3 is an independent set in $UG(D)$, then x_1, x_2, x_3 is an independent set in $dom(D)$.*

Lemma 3.7. *Let $UG(D)$ be isomorphic to $dom(D)$ and r be an integer, $r \geq 3$. Then $x_1, x_2, \ldots, x_r$ are independent in $dom(D)$ if and only if $x_1, x_2, \ldots, x_r$ are independent in $UG(D)$.*

Proof. By Corollary 3.6, if x_1, x_2, x_3 are independent in $UG(D)$ then x_1, x_2, x_3 are independent in $dom(D)$, so the set of independent triples of $UG(D)$ is a subset of the set of independent triples of $dom(D)$. Since $UG(D)$ is isomorphic to $dom(D)$, the number of independent triples of vertices is the same in both graphs. Therefore, the sets of independent triples must be the same as well. An independent set of vertices is completely determined by the independent triples contained within it. ∎

Corollary 3.8. *Let $UG(D)$ be isomorphic to $dom(D)$ and $r \geq 3$. Then $x_1, x_2, \ldots, x_r$ form a clique in $UG^c(D)$ if and only if $x_1, x_2, \ldots, x_r$ form a clique in $dom^c(D)$.*

Lemma 3.9. *If $UG(D)$ is isomorphic to $dom(D)$ and there is no edge between y and any of $x_1, x_2, \ldots, x_r$, $r \geq 3$ in $UG(D)$, then $y, x_1, x_2, \ldots, x_r$ are independent in $UG(D)$ and $dom(D)$, and form a clique in $UG^c(D)$ and $dom^c(D)$.*

Proof. It follows from Corollary 3.3 that $x_1, x_2, \ldots, x_r$ are an independent set in $dom(D)$ and thus by Lemma 3.7 must be independent in $UG(D)$. Since there are no arcs between y and x_i, y can be added to this independent set in $UG(D)$ and hence in $dom(D)$. Thus they form a clique in $UG^c(D)$ and $dom^c(D)$ as well. ∎

Lemma 3.10. *If $UG(D)$ is isomorphic to $dom(D)$ and $x_1, x_2, \ldots, x_r$, $r \geq 3$ form a maximal clique in $UG^c(D)$, then $x_1, x_2, \ldots, x_r$ form a connected component isomorphic to K_r in $UG^c(D)$.*

Proof. Let $x_1, x_2, \ldots, x_r$, $r \geq 3$ form a maximal clique in $UG^c(D)$. Suppose y is a vertex such that $y \neq x_i$ for $i = 1, \ldots, r$ and there is an edge between y and x_j in $UG^c(D)$. Without loss of generality, let $j = 1$. Then there is no edge between x_1 and $y, x_2, \ldots, x_r$ in $UG(D)$, so by Lemma 3.9, $y, x_1, x_2, \ldots, x_r$ form an independent set in $UG(D)$, contradicting the maximality assumption. Therefore, there are no edges between any vertex x_i and any vertex y not in the clique. Thus the clique is a connected component isomorphic to K_r in $UG^c(D)$. ∎

Theorem 3.11. *If $UG(D)$ is isomorphic to dom(D), then $UG^c(D)$ is comprised of one or more connected components, each either a complete graph, a path, or a cycle.*

Proof. It follows from Lemma 3.9 that any vertex of degree $r \geq 3$ or more in $UG^c(D)$ must be in a clique of 4 or more vertices and consequently, by Lemma 3.10 in a component isomorphic to K_{r+1}. So every vertex in $UG^c(D)$ that isn't in a component isomorphic to K_r must be degree 2 or less. It follows that every component of $UG^c(D)$ is isomorphic to a complete graph, path, or cycle. ∎

Corollary 3.12. *If $UG(D)$ is isomorphic to dom(D), then $UG(D)$ is the join of one or more independent sets, complements of paths, and complements of cycles.*

Since it is now shown that if $UG(D) \cong dom(D)$, each component of $UG^c(D)$ must be a complete graph, a cycle or path, we consider the question of which paths, cycles or complete graphs may actually be components of $UG^c(D)$. In partial answer to this question, Theorem 2.1 demonstrates that a complete biorientation of an underlying graph whose complement has components consisting of complete graphs on other than 2 vertices and odd cycles will work. What remains unanswered is whether there exist biorientations of $UG(D)$ where K_2, even cycles, and paths are structures which may be components of $UG^c(D)$.

In [9], the authors of the present paper find that a digraph D exists where $UG(D) = dom(D)$ and K_2 is a component of $UG^c(D)$.

Theorem 3.13 ([9]). *A biorientation D of a graph G on $n \geq 3$ vertices exists such that $UG(D) = dom(D)$ if and only if*

1. $G = \sum_{i=1}^{p} G_i$ *where* G_i, $i = 1, \ldots, p-1$ *are independent sets and* $G_p = K_m$ *for some* $m \geq 0$, *and*
2. *if we let* s *be the number of independent sets of size* 2, *then* $s \leq m$.

Theorem 3.13 guarantees that there are digraphs with $UG^c(D)$ containing K_2, where $UG(D) = dom(D)$ and, thus $UG(D) \cong dom(D)$.

Similarly for even cycles of the form C_{2^i}, $i \geq 2$, we are able to construct digraphs where $UG(D) \cong dom(D)$ and C_{2^i} is a component of $UG^c(D)$. However, not all graphs whose complements contain these cycles as components have a biorientation yielding $UG(D) \cong dom(D)$. As we see in the following lemma, the existence of an even length cycle on more than 4 vertices as a component in $UG^c(D)$ necessarily requires smaller cycles.

Lemma 3.14. *If* $UG^c(D)$ *contains a component isomorphic to a cycle on an even number of vertices,* C_{2k} *where* $k \geq 3$, *then* $dom^c(D)$ *contains at least two cycles of length* k.

Proof. Let $x_1, x_2, x_3, \ldots, x_{2k}, x_1$ form a cycle in $UG^c(D)$. By Lemma 3.2 x_i and $x_{i+2 \ (mod\ 2k)}$ will be adjacent in $dom^c(D)$. This means that $x_1, x_3, \ldots, x_{2k-1}, x_1$ and $x_2, x_4, \ldots, x_{2k}, x_2$ form two cycles of length k in $dom^c(D)$. ■

$UG^c(D) \cong C_8$ $dom^c(D)$ contains at least two 4-cycles.

Figure 2. $UG^c(D)$ is an even length cycle.

For existence of digraphs D_i where C_{2^i} is a component of $UG^c(D_i)$ and $UG(D) \cong dom(D)$, consider the following construction.

Define D_i to be a digraph with $(i-1)2^i + 2^{i-1}$ vertices. Begin by constructing $UG^c(D)$. Choose 2^i vertices to form one cycle of length 2^i in $UG^c(D_i)$, 2^i vertices to form two cycles of length 2^{i-1} in $UG^c(D_i)$ and so on, down to 2^{i-3} cycles of length 2^3. This leaves $2^i + 2^{i-1}$ vertices. Of these vertices, 2^i will be split into 4-cycles in $UG^c(D_i)$ and the rest will be isolated vertices. For $1 \leq j \leq 4$ and $1 \leq k \leq 2^{i-2}$, label 2^i of the remaining vertices $x_{j,k}$, and label the last 2^{i-1} vertices $y_{1,k}$, $y_{2,k}$. Form the 4-cycles in $UG^c(D_i)$ as $x_{1,k}, x_{2,k}, x_{3,k}, x_{4,k}, x_{1,k}$ for $k = 1, \ldots, 2^{i-2}$. Of course any edge between two vertices in $UG^c(D_i)$ means there will be no arcs between those vertices in D_i. Next, to continue the construction of D_i, place single arcs $(y_{1,k}, x_{2,k}), (y_{1,k}, x_{1,k+1}), (y_{2,k}, x_{3,k})$, and $(y_{2,k}, x_{4,k+1})$, for $k = 1, \ldots, 2^{i-2} - 1$ as well as the single arcs $(y_{1,2^{i-2}}, x_{1,1})$, $(y_{1,2^{i-2}}, x_{4,1})$, $(y_{2,2^{i-2}}, x_{2,2^{i-2}})$, and $(y_{2,2^{i-2}}, x_{3,2^{i-2}})$. Place double arcs between all pairs of vertices not otherwise accounted for, so $UG^c(D_i)$ contains one cycle of length 2^i, two cycles of length 2^{i-1} and so on down to 2^{i-2} cycles of length 4, and 2^{i-1} components isomorphic to K_1.

By the proof of Lemma 3.14 each cycle of length 2^r, $r \geq 3$ in $UG^c(D_i)$ will form two cycles of length 2^{r-1} in $dom^c(D_i)$. So, $dom^c(D_i)$ will contain the correct number of cycles of length less than 2^i. The collection of vertices $x_{j,k}$, $y_{1,k}$, $y_{2,k}$ will form the missing cycle of length 2^i on the $x_{j,k}$ and the $y_{1,k}, y_{2,k}$ will remain components isomorphic to K_1 in $dom^c(D_i)$.

We illustrate the construction of D_3 in Figure 3 where C_{2^3} is a component of $UG^c(D_3)$. Dotted lines represent pairs of vertices with no arcs between them, hence the dotted lines are edges in $UG^c(D_3)$. Single arcs are shown. All other pairs of vertices have double arcs, but these are not shown in the figure. The vertices $v_1, v_2, \ldots, v_8, v_1$ form the single 8-cycle in $UG^c(D_3)$. The remaining vertices are labeled as in the construction to form two 4-cycles and four independent vertices in $UG^c(D_3)$.

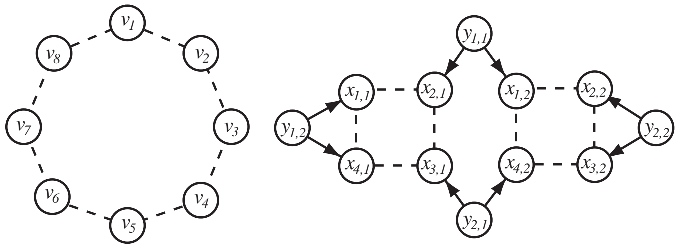


Figure 3. Construction of D_3.

The graph $dom^c(D_3)$ is not shown, but it will contain two components isomorphic to 4-cycles: v_1, v_3, v_5, v_7, v_1 and v_2, v_4, v_6, v_8, v_2 and one 8-cycle: $x_{1,1}$, $x_{3,1}$, $x_{4,2}$, $x_{2,2}$, $x_{3,2}$, $x_{1,2}$, $x_{2,1}$, $x_{4,1}$, $x_{1,1}$. The vertices $y_{1,1}, y_{1,2}, y_{2,1}$, and $y_{2,2}$ will remain independent in $dom^c(D_3)$ as well, so $dom^c(D_3) \cong UG^c(D_3)$.

To conclude our discussion on cycles for this paper, we now consider cycles $C_r, r = 2^l k$, where $k \geq 3$ is an odd integer, and l is a positive integer. Cycles of this form cannot be components of $UG^c(D)$ when $UG(D) \cong dom(D)$.

Lemma 3.15. *If $UG(D) \cong dom(D)$, and $x_1, \ldots, x_r$ form an odd length cycle in $dom^c(D)$, then $x_1, \ldots, x_r$ are vertices of an odd length cycle in $UG^c(D)$.*

Proof. From Lemma 3.4, every odd length cycle in $UG^c(D)$ generates an odd length cycle on the same set of vertices in $dom^c(D)$. If $UG^c(D) \cong dom^c(D)$, each graph must contain the same number of odd cycles. Thus, there can be no odd length cycles that are not generated as described in Lemma 3.4. ∎

Theorem 3.16. *Let $r = 2^l k$ where $k \geq 3$ is an odd integer and l is a positive integer. If $UG(D)$ is isomorphic to $dom(D)$, then no component of $UG^c(D)$ is isomorphic to C_r, a cycle of length r.*

Proof. This follows by induction on l. Suppose $l = 1$. Then $r = 2k$ where k is odd. Suppose $x_1, x_2, x_3, \ldots, x_r, x_1$ form a component isomorphic to C_r in $UG^c(D)$. Consider the indices mod r. Since r is even, by the proof of Lemma 3.14 $x_1, x_3, \ldots, x_{r-1}, x_1$ form a cycle of length k in $dom^c(D)$, (and $x_2, x_4, \ldots, x_r, x_2$ form a cycle of length k in $dom^c(D)$ as well). However, it follows by Lemma 3.15 that $x_1, x_3, \ldots, x_{r-1}, x_1$ are vertices of an odd cycle in $UG^c(D)$ which contradicts the fact that these vertices are part of a component isomorphic to C_r.

Suppose $l \geq 2$ and the theorem holds for cycles of length $2^{l-1}k$. Suppose $r = 2^l k$ where k is odd and $x_1, x_2, x_3, \ldots, x_r, x_1$ form a component isomorphic to C_r in $UG^c(D)$. It follows that there are two cycles of length $2^{l-1}k$ in $dom^c(D)$. Since $dom^c(D)$ is isomorphic to $UG^c(D)$, these two cycles must also be in $UG^c(D)$. By Theorem 3.11, they are either connected components isomorphic to $C_{2^{l-1}k}$ or contained within components isomorphic to K_m, where $m \geq 2^{l-1}k$. However, the first case contradicts the inductive hypothesis. The second case is contradicted by Lemma 3.7 as each set would be independent in $UG(D)$ and so could not be in C_r in $UG^c(D)$. ∎

Finally, we discuss two basic results related to the structure of the underlying graph when $UG^c(D) = P_n$. The actual construction of a path in $dom^c(D)$ relies upon careful orientation of specific edges in $UG(D)$. Thus that portion of the characterization is in Section 4, where the complete characterization of the digraph D, with $UG^c(D)$ connected, is developed.

Lemma 3.4 describes the odd length cycle that is created in $dom^c(D)$ when $UG^c(D)$ is an odd cycle. Here, we make a similar observation for the structure $UG^c(D) = P_n$. Unlike the case of the cycle, the path can have an odd or even number of vertices.

Lemma 3.17. *If $UG^c(D) = P_n = x_1, x_2, \ldots, x_n$ for $n \geq 3$, then*

1. *if n is odd, $x_1, x_3, \ldots, x_n$ and $x_2, x_4, \ldots, x_{n-1}$ are paths in $dom^c(D)$, and*
2. *if n is even, $x_1, x_3, \ldots, x_{n-1}$ and $x_2, x_4, \ldots, x_n$ are paths in $dom^c(D)$.*

Proof. Vertices x_{i-1} and x_{i+1}, $i = 2, \ldots n-1$, are not adjacent to vertex x_i in $UG(D)$, so cannot dominate x_i. Thus $\{x_{i-1}, x_{i+1}\}$ is not an edge of $dom(D)$, but is an edge of $dom^c(D)$. This implies that $x_1, x_3, \ldots, x_n$ and $x_2, x_4, \ldots, x_{n-1}$ are paths in $dom^c(D)$ when n is odd, while $x_1, x_3, \ldots, x_{n-1}$ and $x_2, x_4, \ldots, x_n$ are paths in $dom^c(D)$ when n is even. ■

4. Characterization of D where $UG^c(D)$ is Connected

Now that we know what each component of $UG^c(D)$ must be, we focus our attention on the case where $UG^c(D)$ is a single component.

What exact form do the digraphs take where $UG(D)$ is isomorphic to $dom(D)$ and $UG^c(D)$ is connected? To answer this question, we first introduce a simple result that links the degree of vertices in an underlying graph of any digraph to the existence of a K_3 in $dom^c(D)$. Although similar to the results in Section 2 regarding K_3 in $UG^c(D)$ and $dom^c(D)$, it is not identical. Here, the use of an orientation of an existing edge of $UG(D)$ does not translate into an adjacency issue in $UG^c(D)$. Thus, we approach the K_3 in $dom^c(D)$ through $dom(D)$.

Lemma 4.1. *Let D be a digraph on n vertices, and (u, v) be an arc in D where (v, u) is not an arc in D and $deg(u) = k$ in $UG(D)$. If $k < n-2$, then K_3 is a subgraph of $dom^c(D)$.*

Proof. Suppose $deg(u) < n-2$ in $UG(D)$. This implies that there are two vertices x_1, x_2 that are not adjacent to u in $UG(D)$. Since in D there is no arc from v to u, then v, x_1, x_2 do not dominate u, so form an independent set in $dom(D)$. Thus, they create a copy of K_3 in $dom^c(D)$. ■

Using the preceding result, we introduce the following two lemmas, which examine the existence of arcs that are not in a 2-cycle in some biorientation D of C_n^c or of P_n^c where $UG(D) \cong dom(D)$.

Lemma 4.2. *Let D be a directed graph on $n \geq 5$ vertices, where n is odd and $UG(D) = C_n^c$. Then, $UG(D) \cong dom(D)$ if and only if D is symmetric.*

Proof. Let D be a directed graph with $UG^c(D) = C_n$, $n \geq 5$. As $UG^c(D) = C_n$, all degrees of the vertices of $UG(D)$ are $n - 3$. Suppose that $UG(D) \cong dom(D)$. Then $dom^c(D)$ contains no K_3, and by Lemma 4.1, if (u, v) is an arc in D, then (v, u) must also be an arc in D. That is, D must by symmetric. On the other hand, if D is symmetric, we know by Theorem 2.2 that $UG(D) \cong dom(D)$. ■

Lemma 4.3. *Let D be a directed graph on $n \geq 3$ vertices and $UG^c(D) = P_n = x_1, \ldots, x_n$. Then, $dom^c(D) \cong P_n$ if and only if every arc of D is in a two-cycle except*

1. *if n is odd, exactly one of the following sets of arcs are in D but not in a two-cycle:*
   (a) (x_1, x_n),
   (b) (x_n, x_1),
   (c) (x_1, x_n) *and* (x_n, x_{n-3}), *or*
   (d) (x_n, x_1) *and* (x_1, x_4), *and*
2. *if n is even, exactly one of the following sets of arcs are in D but not in a two-cycle:*
   (a) (x_1, x_{n-1}),
   (b) (x_n, x_2),
   (c) (x_1, x_{n-1}) *and* (x_n, x_2),
   (d) (x_n, x_2) *and* (x_1, x_4), *or*
   (e) (x_1, x_{n-1}) *and* (x_n, x_{n-3}).

Proof. Let D be a directed graph such that $UG^c(D) = P_n = x_1, \ldots, x_n$ for $n \geq 3$. Lemma 3.17 shows that $dom^c(D)$ contains the edges $\{x_1, x_3\}, \ldots,$ $\{x_{n-2}, x_n\}$, $\{x_2, x_4\}, \ldots, \{x_{n-3}, x_{n-1}\}$ if n is odd, or the edges $\{x_1, x_3\}, \ldots,$

$\{x_{n-3}, x_{n-1}\}$, $\{x_2, x_4\}, \ldots, \{x_{n-2}, x_n\}$ if n is even. Thus $dom^c(D)$ contains at least $n-2$ edges.

($\Rightarrow$) Suppose $dom^c(D) \cong P_n$. Thus $UG^c(D) \cong dom^c(D)$ and $UG(D) \cong dom(D)$. By Theorem 2.2, since $UG^c(D) \cong P_n$, D cannot be symmetric. Thus, at least one arc must not be in a two-cycle of D.

Since $dom^c(D) \cong P_n$, $dom^c(D)$ contains no subgraph isomorphic to K_3. Thus, for any arc (u, v) in D where (v, u) is not in D, Lemma 4.1 states $deg(u) \geq n-2$ in $UG^c(D)$. Vertices x_1 and x_n are the only vertices that meet this criterion. This indicates that if (u, v) is an arc in D but (v, u) is not, $u = x_1$ or $u = x_n$.

Suppose that (x_1, x_i) is an arc, but (x_i, x_1) is not. Then, $\{x_2, x_i\}$ is not an edge in $dom(D)$, since neither dominates x_1 in D. This ensures that $\{x_2, x_i\}$ is an edge in $dom^c(D)$. If $i = 3$ or $5 \leq i \leq n-2$, then x_i will be adjacent to three vertices, x_{i-2}, x_{i+2} and x_2 in $dom^c(D)$, but P_n has no vertices of degree three. Also, by the structure of $UG^c(D)$, there are no arcs between x_1 and x_2. Consequently, there are three possibilities to consider: $i = 4$, $i = n-1$, or $i = n$. Likewise if (x_n, x_i) is an arc but (x_i, x_n) is not, then $i = 1$, $i = 2$, or $i = n-3$.

Suppose that both (x_1, x_i) and (x_1, x_j) are arcs in D, with $i \neq j$, but neither (x_i, x_1) nor (x_j, x_1) is an arc in D. Then, x_i, x_j, and x_2 all fail to dominate x_1. Consequently, x_i, x_j, and x_2 form a K_3 in $dom^c(D)$, which is impossible if $dom^c(D)$ is isomorphic to a path. Thus at most one of (x_4, x_1), (x_{n-1}, x_1), (x_n, x_1) is missing from D. Similarly, at most one of (x_1, x_n), (x_2, x_n), (x_{n-3}, x_n) is missing from D. All other arcs must be in two-cycles.

If (x_1, x_i) is an arc in D, but (x_i, x_1) is not, x_i and x_2 fail to dominate x_1, so $\{x_2, x_i\}$ must be an edge in $dom^c(D)$. If $i = 4$, by Lemma 3.17, $\{x_2, x_i\}$ is already an edge in $dom^c(D)$. If n is odd, $\{x_2, x_{n-1}\}$ cannot be an edge in $dom^c(D)$, since $x_2, \ldots, x_{n-1}$ would form a cycle, which is impossible. Thus if n is odd, $i \neq n-1$. If n is even, $\{x_2, x_n\}$ cannot be an edge in $dom^c(D)$, otherwise $x_2, \ldots, x_n$ would form a cycle. Thus if n is even, $i \neq n$. Similarly, if we assume (x_n, x_i) is an arc but (x_i, x_n) is not, $\{x_i, x_{n-1}\}$ is an edge in $dom^c(D)$, and if n is odd, $i \neq 2$, and if n is even $i \neq 1$.

Since we need at least one additional edge in $dom^c(D)$ to form a path, at least one of (x_n, x_1), (x_{n-1}, x_1), (x_1, x_n), (x_2, x_n) must be missing from D. Finally, notice that, since $\{x_1, x_n\}$ is an edge in $UG(D)$, at least one of (x_1, x_n), (x_n, x_1) is an arc in D.

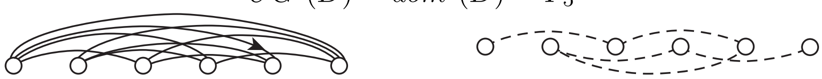
Consequently, every arc in D will be in a two cycle, except for one of the following cases: If n is odd, either (x_1, x_n) is not in D, (x_1, x_n) and (x_4, x_1)

are not in D, (x_n, x_1) is not in D, or (x_n, x_1) and (x_{n-3}, x_n) are not in D. If n is even, either (x_2, x_n) is not in D, (x_2, x_n) and (x_4, x_1) are not in D, (x_{n-1}, x_1) is not in D, (x_{n-1}, x_1) and (x_{n-3}, x_n) is not in D, or (x_{n-1}, x_1) and (x_2, x_n) is not in D.

($\Leftarrow$) Suppose D has one of the patterns in the previous paragraph. If n is odd, and (x_1, x_n) is an arc in D, but (x_n, x_1) is not, then $\{x_2, x_n\}$ will be an edge in $dom^c(D)$ and thus $x_1, x_3, \ldots, x_n, x_2, x_4, \ldots, x_{n-1}$ is a path in $dom^c(D)$. Similarly, if (x_n, x_1) is an arc in D, but (x_1, x_n) is not, then $\{x_1, x_{n-1}\}$ is an edge in $dom^c(D)$. Thus $dom^c(D)$ contains the path $x_2, x_4, \ldots, x_{n-1}, x_1, \ldots, x_n$. If n is even, since either (x_1, x_{n-1}) is an arc but (x_{n-1}, x_1) is not, or (x_n, x_2) is an arc but (x_2, x_n) is not an arc in D, $\{x_2, x_{n-1}\}$ is an edge in $dom^c(D)$. Thus $\{x_1, x_3, \ldots, x_{n-1}, x_2, \ldots, x_n\}$ is a path in $dom^c(D)$.

Now consider any pair of vertices x_i, x_j, not in a path described above, with $i < j$. Note that $j \neq i + 2$. By the construction of D, every vertex x_i has an arc to every vertex x_k with the following possible exceptions: $k = i, i-1, i+1, 1, n$. Suppose that there is an edge between x_i and x_j in $dom^c(D)$. There must be some $k \neq i, j$ so that neither (x_i, x_k) is an arc, nor (x_j, x_k) is an arc. Since $i + 1 \neq j - 1$, this means $k = 1$ or $k = n$. Suppose $k = 1$. By the way D is constructed, there are only a limited number of possibilities. i must equal 2. If n is odd, j might equal n, in which case $\{x_2, x_n\}$ will already be in $dom^c(D)$. If n is even, j might equal $n - 1$, in which case $\{x_2, x_{n-1}\}$ is already in $dom^c(D)$. In either case no additional edges are in $dom^c(D)$. Similar arguments suffice for $k = n$. In either case $dom^c(D) = P_n$. ■

$UG^c(D) \cong dom^c(D) \cong P_5$



$UG^c(D) \cong dom^c(D) \cong P_6$

Figure 4. Example biorientations of the complement of paths.

It is now possible to completely characterize all digraphs with isomorphic underlying graphs and domination graphs where $UG^c(D)$ is connected.

Theorem 4.4. *For any digraph D, $UG(D) \cong dom(D)$ and $UG^c(D)$ is connected if and only if*

1. *D is a digraph of k isolated vertices other than $k = 2$, or*
2. *D is a complete biorientation of C_n^c where n is odd, or*
3. *D is the biorientation of P_n^c where $P_n = x_1, \ldots, x_n$ with $n \geq 3$, and*
   (a) *if n is odd, exactly one of the following sets of arcs are in D but not in a two-cycle:*
      (i) (x_1, x_n),
      (ii) (x_n, x_1),
      (iii) (x_1, x_n) *and* (x_n, x_{n-3}), *or*
      (iv) (x_n, x_1) *and* (x_1, x_4), *and*
   (b) *if n is even, exactly one of the following sets of arcs are in D but not in a two-cycle:*
      (i) (x_1, x_{n-1}),
      (ii) (x_n, x_2),
      (iii) (x_1, x_{n-1}) *and* (x_n, x_2),
      (iv) (x_n, x_2) *and* (x_1, x_4), *or*
      (v) (x_1, x_{n-1}) *and* (x_n, x_{n-3}).
   (c) *for all other arcs (x, y) of D, (y, x) is also an arc.*

Proof. Recall that Theorem 3.11 states that $UG^c(D)$ must be a complete graph, a cycle, or a path, so D must be an orientation of the complement of such graphs. We know from Theorem 2.2 that D exists with $UG^c(D) = K_n$ for $n \neq 2$ or C_n, where n is odd. Lemma 4.3 provides constructions when $UG^c(D) \cong P_n$, $n \geq 3$.

If $UG^c(D)$ is isomorphic to K_n then D has no arcs so there is no choice of orientation. If $n = 2$, then $dom(D) = K_2 \ncong UG(D)$. In all other cases $dom(D)$ will be isomorphic to k isolated vertices.

Suppose $UG^c(D)$ is an odd length cycle. If $n = 3$ then D is isomorphic to 3 isolated vertices listed in the previous case. If $n \geq 5$, Lemma 4.2 states that D must be a complete biorientation of $UG(D)$, the complement of C_n. It follows from Lemma 3.14 that if $UG^c(D)$ is an even length cycle on more than 4 vertices then $dom^c(D)$ will contain two smaller cycles which will result in a disconnected graph. Finally, let x_1, x_2, x_3, x_4, x_1 be a 4 cycle in $UG^c(D)$, so $UG(D)$ has two edges $\{x_1, x_3\}$ and $\{x_2, x_4\}$. There are only three non-isomorphic orientations of two edges. Each of these orientations results in $dom^c(D) \ncong C_4$.

The final case, where $UG^c(D)$ is isomorphic to a path is fully described in Lemma 4.3 ■

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