

COMBINATORIAL LEMMAS FOR POLYHEDRONS I

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Abstract

We formulate general boundary conditions for a labelling of vertices of a triangulation of a polyhedron by vectors to assure the existence of a balanced simplex. The condition is not for each vertex separately, but for a set of vertices of each boundary simplex. This allows us to formulate a theorem, which is more general than the Sperner lemma and theorems of Shapley; Idzik and Junosza-Szaniawski; van der Laan, Talman and Yang. A generalization of the Poincaré-Miranda theorem is also derived.

Keywords: b -balanced simplex, labelling, polyhedron, simplicial complex, Sperner lemma.

2000 Mathematics Subject Classification: 05B30, 47H10, 52A20, 54H25.

1. PRELIMINARIES

For $n \in \mathbb{N}$, let $N = \{1, \dots, n\}$ and $N_0 = \{0, \dots, n\}$. By a *polyhedron* we understand the convex hull of a finite set of \mathbb{R}^n . Let $P \subset \mathbb{R}^n$ be a polyhedron of dimension n . A *face* of the polyhedron P is the intersection of P with some of its supporting hyperplanes. Denote the set of all k -dimensional faces of the polyhedron P by $\mathbf{F}_k(P)$ ($k < n$), the set of all faces of the polyhedron P by $\mathbf{F}(P)$ (hence $\mathbf{F}(P) = \bigcup_{k=0}^{n-1} \mathbf{F}_k(P)$) and the set of all vertices of the polyhedron P by $V(P)$ ($V(P) = \mathbf{F}_0(P)$). The maximal dimension proper faces of the polyhedron P are called *facets*. For a finite set $A = \{a_0, \dots, a_m\} \subset \mathbb{R}^n$ a set $\text{co } A = \{\alpha_0 a_0 + \dots + \alpha_m a_m : a_i \in A, \sum_{i=0}^m \alpha_i = 1, \alpha_i \geq 0 \text{ for } i \in \{0, \dots, m\}\}$ is the *convex hull* of A , $\text{aff } A = \{\alpha_0 a_0 + \dots + \alpha_m a_m : \sum_{i=0}^m \alpha_i = 1, a_i \in A, \alpha_i \in \mathbb{R} \text{ for } i \in \{0, \dots, m\}\}$ is the *affine hull* of A . And if for a finite set $A = \{a_0, \dots, a_m\} \subset \mathbb{R}^n$ ($m \in \{0, \dots, n\}$) the dimension of $\text{aff } A$ is equal to m , then $\text{co } A$ is called a *simplex* (precisely an *m-simplex*). Let Tr_n be a finite family of n -simplexes such that $P = \bigcup_{\delta \in Tr_n} \delta$ and for any $\delta_1, \delta_2 \in Tr_n$, $\delta_1 \cap \delta_2$ is the empty set or their common face. A *triangulation* of the polyhedron P (we denote it by Tr) is a family consisting of simplexes of Tr_n and all their faces. Let Tr_m ($m \in N_0$) denote the family of m -simplexes belonging to a triangulation Tr . Hence $Tr = \bigcup_{i=0}^n Tr_i$. Let $V = Tr_0$ be the set of vertices of all simplexes of Tr . Notice, that $V = \bigcup_{\delta \in Tr_n} V(\delta)$. An $(n-1)$ -simplex of Tr_{n-1} is a *boundary (n-1)-simplex* if it is a facet of exactly one n -simplex of Tr_n . For a triangulation Tr^P of the polyhedron P and a triangulation Tr^Q of a polyhedron Q a function $f : V(Tr^P) \rightarrow V(Tr^Q)$ is a *simplicial function* if for every $\sigma \in Tr^P$ there exists $\delta \in Tr^Q$ such that $f(V(\sigma)) = V(\delta)$.

2. MAIN RESULT

We start with the following

Definition 2.1. Let $\sigma \subset \mathbb{R}^n$ be a simplex, $l : V(\sigma) \rightarrow \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $Z \subset \mathbb{R}^n$. A simplex σ is *b-balanced* if the point b belongs to $\text{co}(l(V(\sigma)))$ and *b-subbalanced with respect to Z*, if the point b belongs to $\text{co}(l(V(\sigma)) \cup Z)$. If $Z = \{x\}$, then we write *b-subbalanced with respect to x* instead of with respect to $\{x\}$. For $b = 0$ we say *balanced* and *subbalanced* instead of *b-balanced* and *b-subbalanced*, respectively.

Notice that in the case Z is a polyhedron, a simplex σ is b -subbalanced with respect to Z if and only if σ is b -subbalanced with respect to $V(Z)$.

Lemma 2.2. *Let $P \subset \mathbb{R}^n$ be a polyhedron of dimension n , Tr be a triangulation of the polyhedron P , $l : Tr_0 \rightarrow \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. If the triangulation Tr contains neither a b -balanced simplex of dimension less than n nor a simplex of dimension less than $n - 1$ which is b -subbalanced with respect to x , then the number of b -balanced simplexes in Tr is congruent modulo 2 to the number of b -subbalanced with respect to x boundary simplexes in Tr .*

Proof. For this proof by a b -subbalanced simplex we understand a b -subbalanced simplex with respect to x . Consider a graph $G = (W, E)$ where W is the set of b -balanced n -simplexes and b -subbalanced $(n - 1)$ -simplexes in Tr and there is an edge between two different simplexes $\sigma_1, \sigma_2 \in W$ if and only if there exists a simplex $\sigma \in Tr$ containing σ_1 and σ_2 (in particular $\sigma = \sigma_1$). We will show that

$$\deg_G(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is a } b\text{-balanced or a boundary } b\text{-subbalanced simplex,} \\ 2 & \text{if } \sigma \text{ is a } b\text{-subbalanced simplex not in the boundary.} \end{cases}$$

Let σ be a b -balanced simplex of Tr . By our assumption σ is an n -dimensional simplex. Let $V(\sigma) = \{v_0, \dots, v_n\}$, $u_i = l(v_i)$ for $i \in N_0$ and let $A_i = \text{co}\{u_0, \dots, u_{i-1}, x, u_{i+1}, \dots, u_n\}$ for $i \in N_0$. There is at least one $j \in N_0$ such that $b \in A_j$ since $b \in \text{co}\{u_0, \dots, u_n\} \subseteq \bigcup_{i=0}^n A_i$. If there exists $j, k \in N_0$, $j < k$ such that $b \in A_j$ and $b \in A_k$, then it is easy to show that $b \in \text{co}\{x, u_0, \dots, u_{j-1}, u_{j+1}, \dots, u_{k-1}, u_{k+1}, \dots, u_n\}$, so that the simplex $\text{co}\{v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}$ is b -subbalanced and of dimension less than $n - 1$. This contradicts our assumption.

Now let σ be a b -subbalanced simplex in Tr of dimension $n - 1$ and let σ_1 be an n -simplex containing σ , $V(\sigma) = \{v_1, \dots, v_n\}$, $V(\sigma_1) \setminus V(\sigma) = \{v_0\}$, $u_i = l(v_i)$ for $i \in N_0$, $B_0 = \text{co}\{u_0, u_1, \dots, u_n\}$, $B_i = \text{co}\{x, u_1, \dots, u_{i-1}, u_0, u_{i+1}, \dots, u_n\}$ for $i \in N_0$. Since $b \in \text{co}\{x, u_1, \dots, u_n\} \subseteq \bigcup_{i=0}^n B_i$, then there exists $i \in N_0$ such that $b \in B_i$. If $b \in B_0$, then σ_1 is b -balanced and σ and σ_1 form an edge in G . If $b \in B_i$ for some $i \in N$, then $\sigma_2 = \text{co}\{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ is b -subbalanced and σ and σ_2 form an edge in G . If $b \in B_0 \cap B_j$ for some $j \in N$, then the simplex $\text{co}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$ is b -subbalanced of dimension less than $n - 1$, but this is impossible. If $b \in B_j \cap B_k$ for some $j, k \in N$, $j < k$, then the simplex

$\text{co}\{v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}$ is b -subbalanced of dimension less than $n - 1$, but this is also impossible. In all cases, σ_1 defines an adjacent edge to σ in G . Hence, if σ is a boundary simplex (it is a face of exactly one n -simplex), then $\deg_G(\sigma) = 1$ and if σ is not a boundary simplex (it is a face of exactly two n -simplexes), then $\deg_G(\sigma) = 2$.

Graph G has vertices of degree one and two only. Thus the number of vertices of degree one is even and hence the number of b -balanced simplexes in Tr is congruent modulo two to the number of b -subbalanced with respect to x boundary simplexes in Tr . ■

Remark 2.3. Let $S \subset \mathbb{R}^n$ be a polyhedron, \widetilde{Tr} be a triangulation of $\text{bd } S$ and $p \in \text{ri } S$, then $Tr = \{\text{co}(\{p\} \cup \sigma) : \sigma \in \widetilde{Tr}\} \cup \widetilde{Tr} \cup \{p\}$ is a triangulation of the polyhedron S .

Definition 2.4. Two n -dimensional polyhedrons P and Q are *dual to each other through* ψ if $\psi : \mathbf{F}(P) \rightarrow \mathbf{F}(Q)$ is a one-to-one inclusion-reversing mapping, i.e., $F_1 \subset F_2$ if and only if $\psi(F_1) \supset \psi(F_2)$ for any $F_1, F_2 \in \mathbf{F}(P)$. Polyhedrons P and Q are dual to each other if there exists $\psi : \mathbf{F}(P) \rightarrow \mathbf{F}(Q)$ such that P and Q are dual to each other through ψ .

A simplex of any dimension is dual to itself and a 3-dimensional cube and octahedron are dual to each other. For more examples and properties of dual polyhedrons see Grunbaum [4], pp. 46–48. Notice that $\dim F + \dim \psi(F) = n - 1$ for any $F \in \mathbf{F}(P)$.

Duality of polyhedrons may be defined in many ways (see e.g. Alexandrov [1], pp. 49):

Definition 2.5. Two n -dimensional polyhedrons P and Q are *dual to each other through* ϕ , if $\phi : \mathbf{F}_0(P) \rightarrow \mathbf{F}_{n-1}(Q)$ fulfils the following condition: for $v_1, v_2 \in \mathbf{F}_0(P)$, $\text{co}\{v_1, v_2\}$ is a face of P if and only if $\phi(v_1)$ and $\phi(v_2)$ have a common $(n - 2)$ -dimensional face.

Observe that both definitions are equivalent.

Theorem 2.6. Let $P, Q \subset \mathbb{R}^n$ be n -dimensional polyhedrons, dual to each other through a mapping ψ , Tr be a triangulation of the polyhedron P , $V = Tr_0$, $b \in \text{ri } Q$ and $l : V \rightarrow \mathbb{R}^n$ be a labelling. If for every $G \in \mathbf{F}(P)$ and every simplex $\sigma \in Tr$ and $\sigma \subseteq G$, σ is not b -subbalanced with respect to the set $\psi(G)$, then there exists a b -balanced simplex in Tr .

Proof. For $n = 1$ the boundary condition implies that the labels of two vertices of P lie on opposite sides of the point b . Thus there is a vertex $v \in Tr_0$ such that $l(v) = b$ or the number of b -balanced simplexes in Tr is odd.

Consider the case $n > 1$. Assume that there is no b -balanced simplex in Tr of dimension less than n . We show that there exists a b -balanced simplex of dimension n in Tr .

We define a triangulation Tr^Q of $\text{bd } Q$. For every face $H \in \mathbf{F}_1(Q)$ we choose a point $u_H \in \text{ri } H$ and apply Remark 2.3 to get a triangulation of the face H . Then inductively for $k = 2, \dots, n - 1$: for every face $H \in \mathbf{F}_k(Q)$ we choose a point $u_H \in \text{ri } H$ and apply Remark 2.3 to get a triangulation of the face H . Finally we obtain a triangulation of $\text{bd } Q$.

Let $V(P) = \{a_0, \dots, a_k\}$ ($k \geq n$). For $i \in \{0, \dots, k\}$ and $c \in \text{ri } P$, let $a'_i = 2a_i - c$ and $P' = \text{co}\{a'_0, \dots, a'_k\}$. Notice that $P \subset P'$.

Now we define a triangulation of P' , which is an extension of the triangulation Tr of the polyhedron P . We define a triangulation of the set $P' \setminus \text{ri } P$.

For every face $F = \text{co}\{a_{i(0)}, \dots, a_{i(l)}\}$ (defined by some: $\{a_{i(0)}, \dots, a_{i(l)}\} \subset V(P)$) of the polyhedron P we denote $F' = \text{co}\{a'_{i(0)}, \dots, a'_{i(l)}\}$. Every face F of P has one-to-one correspondence to the face F' of P' .

Let us denote $FF' = \text{co}\{F \cup F'\}$. Thus $P' \setminus \text{ri } P = \bigcup_{F \in \mathbf{F}_{n-1}(P)} FF'$.

For every face $F_1 \in \mathbf{F}_1(P)$ we choose a point $v_{F'_1} \in \text{ri } F'_1$. By Remark 2.3 we receive a triangulation of F'_1 . Then for every face $F_1 \in \mathbf{F}_1(P)$ we choose a point $v_{F_1 F'_1} \in \text{ri } F_1 F'_1$. By Remark 2.3 we receive a triangulation of $F_1 F'_1$.

Now we apply the induction for $k \in \{2, \dots, n - 1\}$: for any face $F_k \in \mathbf{F}_k(P)$ we choose a point $v_{F'_k} \in \text{ri } F'_k$ and by Remark 2.3 we get a triangulation of the face F'_k . Analogously we choose a point $v_{F_k F'_k} \in \text{ri } F_k F'_k$ and get a triangulation of $F_k F'_k$.

Finally we obtain a triangulation of $P' \setminus \text{ri } P$ and denote it by Tr'' . Hence $Tr' = Tr \cup Tr''$ is a triangulation of P' , which is an extension of the triangulation Tr on P .

Let $V' = Tr'_0$. If $v \in V' \setminus V$, then $v \in V(P') \cup \{v_{GG'}, v_{G'} : G \in \mathbf{F}_k(P), k \in \{1, \dots, n - 1\}\}$. For $G' \in \mathbf{F}_0(P')$ we also denote $v_{G'} := G'$.

Now, we define a labelling $l' : V' \rightarrow \mathbb{R}^n$:

$$l'(v) = \begin{cases} l(v) & \text{for } v \in V, \\ u_{\psi(G)} & \text{for } v = v_{GG'} \text{ or } v = v_{G'} . \end{cases}$$

We prove that there is no b -balanced simplex in Tr'' . Consider an n -simplex $\sigma \in Tr''$. If $\sigma \cap P = \emptyset$, then there is exactly one vertex v of σ , which is also a vertex of P' . Let $v = a'_j \in V(P')$ ($j \in \mathbb{N}$) and thus $l'(V(\sigma)) \subset \psi(a_j)$, where $\psi(a_j)$ is a facet of Q so σ is not b -balanced. Now consider the case $\sigma \cap P \neq \emptyset$: let $\tau = \sigma \cap P$, G_τ be the smallest face of P (in the sense of inclusion) containing τ . Let $v \in V(\sigma) \setminus V(\tau)$, thus $v = v_G$ or $v = v_{G'}$ for some $G \in \mathbf{F}(P)$. Notice that $G_\tau \subseteq G$ and thus $\psi(G_\tau) \supseteq \psi(G)$. From definition of Tr^Q we have $u_H \in H$ for any $H \in \mathbf{F}(Q)$ and from definition of labelling l' we have $l'(v) = u_{\psi(G)} \in \psi(G) \subseteq \psi(G_\tau)$. Thus $l'(V(\sigma) \setminus V(\tau)) \subseteq \psi(G_\tau)$ and $l'(V(\sigma)) = l'((V(\sigma) \setminus V(\tau)) \cup V(\tau)) = l'((V(\sigma) \setminus V(\tau))) \cup l'(V(\tau)) \subseteq \psi(G_\tau) \cup l'(V(\tau)) = \psi(G_\tau) \cup l(V(\tau))$.

From the assumption $b \notin \text{co}(l(V(\tau)) \cup \psi(G_\tau))$. Therefore $b \notin \text{co} l'(V(\sigma))$ and σ is not a b -balanced simplex.

Let σ be an $(n - 1)$ -simplex, $V(\sigma) = \{v_1, \dots, v_n\}$, $v'_i = 2b - v_i$ ($i \in N$), $C(\sigma) = \text{cone}(\{v'_1, \dots, v'_n\}, b)$. An $(n - 1)$ -simplex σ is b -subbalanced with respect to x if and only if $x \in C(\sigma)$. The set $C(\sigma)$ is an $(n - 1)$ -dimensional set and the union $\bigcup_{\sigma \in Tr'_{n-1}, \sigma \subset \text{bd} P'} C(\sigma)$ is also an $(n - 1)$ -dimensional set. Hence, we can choose $x \in \mathbb{R}^n$, $x \neq b$ in such a way that Tr' does not contain a b -subbalanced simplex with respect to x of dimension smaller than $n - 1$. Consider a line going through x and b . This line meets $\text{bd} Q$ in two points. By x' we denote the common point of this line and $\text{bd} Q$ such that $b \in \text{co}\{x, x'\}$ and by $\sigma_Q \in Tr^Q$ we denote the $(n - 1)$ -dimensional boundary simplex containing x' . The function l' restricted to the set $\text{bd} P' \cap V'$ is a one-to-one simplicial function. The simplex $\sigma_P := \text{co} l'^{-1}(V(\sigma_Q))$ is b -subbalanced with respect to x and it is the only such simplex on $\text{bd} P$.

Now, from Lemma 2.2 it follows that the number of b -balanced simplexes in Tr' is odd. Since $Tr' = Tr \cup Tr''$ and there is no b -balanced simplex in Tr'' , there exists b -balanced simplex in Tr . ■

3. COROLLARIES AND APPLICATIONS

In this section we present corollaries to Theorem 2.6 in order to show the strength of this theorem. First we apply Theorem 2.6 to the simplex:

Corollary 3.1. *Let $P = \text{co}\{d_0, \dots, d_n\} \subset \mathbb{R}^n$ be an n -dimensional simplex, $m_{F_i} = \sum_{j \neq i} \frac{d_j}{n}$ be the gravity center of a facet $F_i = \text{co}\{d_0, \dots, d_{i-1}, d_{i+1}, \dots, d_n\}$, $m_P = \sum_{j=0}^n \frac{d_j}{n+1}$ be the gravity center of P , Tr be a triangulation*

of the simplex P , $V = Tr_0$ and $l : V \rightarrow \mathbb{R}^n$ be a labelling. If for every face $F = \text{co}\{d_i : i \in M\}$ and every simplex $\sigma \subset F$, $\sigma \in Tr$, σ is not m_P -subbalanced with respect to the set $\{m_{F_i} : i \in M\}$, then there exists an m_P -balanced simplex in Tr .

Corollary 3.1 is more general than the Sperner lemma [12] and the Shapley lemma (Lemma 7.2 in [11]).

Corollary 3.2. *Let $P = \text{co}\{d_0, \dots, d_n\} \subset \mathbb{R}^n$ be an n -dimensional simplex, $m_P = \sum_{j=0}^n \frac{d_j}{n+1}$ be the gravity center of P , Tr be a triangulation of the simplex P , $V = Tr_0$ and $l : V \rightarrow \mathbb{R}^n$ be a labelling. If for every face $F = \text{co}\{d_i : i \in M\}$ ($M \subset N_0$) and every simplex $\sigma \subset F$, $\sigma \in Tr$, σ is not m_P -subbalanced with respect to the set $\{d_i : i \notin M\}$, then there exists an m_P -balanced simplex in Tr .*

Corollary 3.2 is more general than the Scarf lemma ([10]; see also Theorem 3.4 in [9]) and the Garcia lemma ([3], see also Theorem 3.6 in [9]).

The next result is on an n -dimensional cube. Let $I^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : -1 \leq x_i \leq 1, i \in N\}$ be an n -dimensional cube and for $k \in N$, $i_1, \dots, i_k \in N$, $i_1 < i_2 < \dots < i_k$, $s_{i_1}, \dots, s_{i_k} \in \{-1, 1\}$ let $I(s_{i_1}i_1, \dots, s_{i_k}i_k) = \{(x_1, \dots, x_n) \in I^n : x_{i_j} = s_{i_j}, j \in \{1, \dots, k\}\}$ be an $(n - k)$ -dimensional face of I^n .

Corollary 3.3. *Let Tr be a triangulation of the cube I^n , $V = Tr_0$ and $l : V \rightarrow \mathbb{R}^n$ be a labelling. If for all $k \in N$, $i_1, \dots, i_k \in N$, $i_1 < i_2 < \dots < i_k$, $s_{i_1}, \dots, s_{i_k} \in \{-1, 1\}$ and every simplex $\sigma \in Tr$ and $\sigma \subseteq I(s_{i_1}i_1, \dots, s_{i_k}i_k)$, σ is not subbalanced with respect to the set $\{s_{i_j}e_{i_j} : j \in \{1, \dots, k\}\}$, then there exists a balanced simplex in Tr .*

Proof. It follows directly from Theorem 2.6 for $P = I^n$, $Q = \text{co}\{e_i, -e_i : i \in N\}$ and $\psi(\{s_{i_j}e_{i_j} : j \in \{1, \dots, k\}\}) = \text{co}\{s_{i_j}e_{i_j} : j \in \{1, \dots, k\}\}$ for all $k \in N$, $i_1, \dots, i_k \in N$, $i_1 < i_2 < \dots < i_k$, $s_{i_1}, \dots, s_{i_k} \in \{-1, 1\}$. ■

Corollary 3.3 is more general than the Freund lemma (Lemma 1 in [2], see also Lemma 3.7 in [9]). Our next result is a generalization of the Poincaré-Miranda theorem [8]:

Theorem 3.4. *Let $f : I^n \rightarrow \mathbb{R}^n$ be a continuous function, such that for all $k \in N$, $i_1, \dots, i_k \in N$, $i_1 < i_2 < \dots < i_k$, $s_{i_1}, \dots, s_{i_k} \in \{-1, 1\}$*

$$f(I(s_{i_1}i_1, \dots, s_{i_k}i_k)) \cap \text{cone}\{-s_{i_j}e_{i_j} : j \in \{1, \dots, k\}\} = \emptyset,$$

then there exists $x \in I^n$ such that $f(x) = 0$.

Proof. Consider a sequence of triangulations Tr^m of I^n ($m \in \mathbb{N}$) with mesh tending to zero, when m tends to infinity. Let $V^m = V(Tr^m)$ and $l^m = f|_{V^m}$. We show that the labelling l^m fulfils the condition of Corollary 3.3. Let $I(s_{i_1}i_1, \dots, s_{i_k}i_k)$ be a face of I^n for $k \in \mathbb{N}$, $i_1, \dots, i_k \in \mathbb{N}$, $i_1 < i_2 < \dots < i_k$, $s_{i_1}, \dots, s_{i_k} \in \{-1, 1\}$. Take $\sigma^m \subseteq I(s_{i_1}i_1, \dots, s_{i_k}i_k)$. Because f is continuous and for sufficiently large m the mesh of Tr^m is small enough, the condition $f(I(s_{i_1}i_1, \dots, s_{i_k}i_k)) \cap \text{cone}\{-s_{i_j}e_{i_j} : j \in \{1, \dots, k\}\} = \emptyset$ implies $l^m(V(\sigma^m)) \cap \text{cone}\{-s_{i_j}e_{i_j} : j \in \{1, \dots, k\}\} = \emptyset$. This is equivalent to the condition that σ^m is not subbalanced with respect to the set $\{s_{i_j}e_{i_j} : j \in \{1, \dots, k\}\}$. Corollary 3.3 implies that there exists a balanced simplex in Tr^m . Now, if the mesh of Tr^m tends to zero, the sequence of simplexes σ^m tends to a point z . But each σ^m is a balanced simplex so we have $f(z) = 0$. ■

Theorem 2.6 is more general than our previous result:

Corollary 3.5 (Theorem 3.4 in [6]). *Let $P \subset \mathbb{R}^n$ be a polyhedron of dimension n , Tr be a triangulation of the polyhedron P , $V = Tr_0$, $b \in \text{ri } P$ and $l : V \rightarrow \mathbb{R}^n$ be a labelling. If for every facet F of the polyhedron P there exists an $(n-1)$ -dimensional hyperplane h_b^F containing the point b and disjoint with F such that $l(V \cap F) \subset H_b^F$, where H_b^F is an open halfspace containing F such that h_b^F is in its boundary, then there exists a b -balanced n -simplex in the triangulation Tr .*

Proof. For every polyhedron P and any point $b \in P$ there exists a dual polyhedron Q such that every face of Q is perpendicular to the ray issuing from b through the vertices of P and whose vertices lie on the rays issuing from b and perpendicular to the faces of P (for the proof see [1], pp. 45). Hence P, Q and l fulfil the condition of Theorem 2.6 and we get our corollary. ■

For every n -dimensional polyhedron $P \subset \mathbb{R}^n$ and any point $x_0 \in \text{ri } P$ there exist vectors a_i ($i \in I \subset \mathbb{N}$), such that $P = \{x \in \mathbb{R}^n : a_i x \leq 1 + a_i x_0, i \in I\}$. Let $\text{car } B = \{i \in I : a_i x = 1 + a_i x_0 \text{ for all } x \in B\}$ for $B \subset P$.

Two theorems below, proved by van der Laan, Talman and Yang, follow also from our Theorem 2.6:

Corollary 3.6 (Theorem 4.1 in [9]). *Let $P = \{x \in \mathbb{R}^n : a_i x \leq 1 + a_i x_0, i \in I\} \subset \mathbb{R}^n$ be an n -dimensional polyhedron, $Q = \text{co}\{a_j \in \mathbb{R}^n : j \in I\}$, $b \in \text{ri} Q$, Tr be a triangulation of P , $V = Tr_0$ and $l : V \rightarrow \mathbb{R}^n$ be a labelling. There exists a simplex $\sigma \in Tr$ such that $b \in \text{co}(l(V(\sigma)) \cup \{a_i : i \in \text{car } \sigma\})$.*

Proof. For a polyhedron $P = \{x \in \mathbb{R}^n : a_i x \leq 1 + a_i x_0, i \in I\} \subset \mathbb{R}^n$ the polyhedron $Q = \text{co}\{a_j \in \mathbb{R}^n : j \in I\}$ is dual to P through a mapping $\psi : \mathbf{F}(P) \rightarrow \mathbf{F}(Q)$ defined by $\psi(F) = \text{co}\{a_i \in \mathbb{R}^n : i \in \text{car } F\}$, (for the proof see Grunbaum [4] pp. 46–49).

Notice that for any boundary simplex $\sigma \in Tr$ the condition $b \in \text{co}(l(V(\sigma)) \cup \{a_i : i \in \text{car } \sigma\})$ says that σ is b -subbalanced with respect to $\text{co}\{a_i \in \mathbb{R}^n : i \in \text{car } \sigma\}$. Hence, if there is no boundary simplex σ such that $b \in \text{co}(l(V(\sigma)) \cup \{a_i : i \in \text{car } \sigma\})$, then the assumptions of Theorem 2.6 are satisfied and we get the thesis. ■

For any k -dimensional polyhedron $P \subset \mathbb{R}^n$ there exists m vectors $a_i \in \mathbb{R}^n$, m real numbers $\alpha_i \in \mathbb{R}$ ($m > k, i \in I \subset \mathbb{N}$), and $n - k$ vectors $d_h \in \mathbb{R}^n$, $n - k$ real numbers $\delta_h \in \mathbb{R}$ ($h \in N_k \subset \mathbb{N}$) such that $P = \{x \in \mathbb{R}^n : a_i x \leq \alpha_i \text{ for } i \in I, d_h x = \delta_h \text{ for } h \in N_k\}$.

Corollary 3.7 (Theorem 3.1 in [9]). *Let $P = \{x \in \mathbb{R}^n : a_i x \leq \alpha_i \text{ for } i \in I, d_h x = \delta_h \text{ for } h \in N_k\}$ be a k -dimensional polyhedron, $W = \text{aff}\{d_h : h \in N_k\}$, $W^* = \{x \in \mathbb{R}^n : xy = 0 \text{ for all } y \in W\}$, Tr a triangulation of P , $l : V \rightarrow \mathbb{R}^n$ be a labelling such that $(\text{col}(V)) \cap W = \{0\}$. If for every $F \in \mathbf{F}(P)$ and every simplex $\sigma \in Tr, \sigma \subseteq F$ the intersection $(\text{col}(V(\sigma))) \cap (\text{cone}(0, \{a_i : i \in \text{car } (F)\}) + W)$ is empty or contains the point $0 \in \mathbb{R}^n$, then there exists a balanced simplex in Tr .*

Proof. Without loss of generality we may assume that the vectors a_i for $i \in I$ are parallel to the hyperplane W^* . We can consider projection of the polyhedron P , labels $l(V)$ and vectors a_i for $i \in I$ on the hyperplane W^* parallel to the hyperplane W . Hence, we reduce this theorem to the full-dimensional case. Analogously vectors a_i ($i \in I$) can be scaled in such a way that $Q = \text{co}\{a_i : i \in I\}$ is a polyhedron dual to P through a mapping $\psi : \mathbf{F}(P) \rightarrow \mathbf{F}(Q)$ defined by $\psi(F) = \text{co}\{a_i \in \mathbb{R}^n : i \in \text{car } F\}$. If there exists a simplex σ such that $\text{col}(V(\sigma))$ contains $0 \in \mathbb{R}^n$, then σ is a balanced simplex. If for any $F \in \mathbf{F}(P)$ and any simplex $\sigma \in Tr, \sigma \subseteq F$ the intersection $\text{col}(V(\sigma)) \cap \text{cone}(0, \{a_i : i \in \text{car } (F)\})$ is empty, then σ is not a subbalanced simplex with respect to the set $\psi(F)$ and by Theorem 2.6 we get our theorem. ■

Acknowledgment

The authors thank the anonymous referee for many helpful comments and suggestions.

REFERENCES

- [1] A.D. Alexandrov, *Convex Polyhedra* (Springer, Berlin, 2005).
- [2] R.W. Freund, *Variable dimension complexes Part II: A unified approach to some combinatorial lemmas in topology*, *Math. Oper. Res.* **9** (1984) 498–509.
- [3] C.B. Garcia, *A hybrid algorithm for the computation of fixed points*, *Manag. Sci.* **22** (1976) 606–613.
- [4] B. Grunbaum, *Convex Polytopes* (Wiley, London, 1967).
- [5] A. Idzik and K. Junosza-Szaniawski, *Combinatorial lemmas for nonoriented pseudomanifolds*, *Top. Meth. in Nonlin. Anal.* **22** (2003) 387–398.
- [6] A. Idzik and K. Junosza-Szaniawski, *Combinatorial lemmas for polyhedrons*, *Discuss. Math. Graph Theory* **25** (2005) 95–102.
- [7] B. Knaster, C. Kuratowski and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n -dimensionale Simplexe*, *Fund. Math.* **14** (1929) 132–137.
- [8] W. Kulpa, *Poincaré and Domain Invariance Theorem*, *Acta Univ. Carolinae - Mathematica et Physica* **39** (1998) 127–136.
- [9] G. van der Laan, D. Talman and Z. Yang, *Existence of balanced simplices on polytopes*, *J. Combin. Theory (A)* **96** (2001) 25–38.
- [10] H. Scarf, *The approximation of fixed points of a continuous mapping*, *SIAM J. Appl. Math.* **15** (1967) 1328–1343.
- [11] L.S. Shapley, *On balanced games without side payments*, in: T.C. Hu and S.M. Robinson (eds.), *Mathematical Programming*, New York: Academic Press (1973) 261–290.
- [12] E. Sperner, *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebiets*, *Abh. Math. Sem. Univ. Hamburg* **6** (1928) 265–272.

Received 2 December 2005

Revised 3 October 2006