

**A LOWER BOUND ON THE INDEPENDENCE NUMBER
OF A GRAPH IN TERMS OF DEGREES**

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Abstract

For a connected and non-complete graph, a new lower bound on its independence number is proved. It is shown that this bound is realizable by the well known efficient algorithm MIN.

Keywords: independence, stability, algorithm.

2000 Mathematics Subject Classification: 05C69, 05C85.

1. INTRODUCTION AND THEOREM

Let G be a finite, undirected, simple, non-complete, and connected graph on its vertex set $V(G) = \{1, 2, \dots, n\}$. For a subgraph H of G and for a vertex $i \in V(H)$ let $d_H(i)$ be the degree of i in H , i.e., the cardinality of the neighbourhood $N_H(i) \subset V(H)$ of i in H , and let $\delta(H)$ be the minimum degree of H . A subset I of $V(G)$ is called *independent* if the subgraph of G spanned by I is edgeless. The *independence number* $\alpha(G)$ is the largest cardinality $|I|$ among all independent sets I of G . The following algorithm MIN (cf. [8]) is a well known procedure to construct an independent set of a graph G .

Algorithm MIN:

1. $G_1 := G, j := 1$
 2. while $V(G_j) \neq \emptyset$ do
begin
choose $i_j \in V(G_j)$ with $d_{G_j}(i_j) = \delta(G_j)$, delete $\{i_j\} \cup N_{G_j}(i_j)$ to obtain G_{j+1} and set $j := j + 1$;
end;
 3. $k := j - 1$
- STOP

Obviously, the set $\{i_1, i_2, \dots, i_k\} \subset V(G)$ is an independent set of G and therefore $\alpha(G) \geq k$ for every output k of algorithm MIN. Let k_{MIN} be the smallest k Algorithm MIN provides for a fixed graph G . In the following Theorem a new lower bound on k_{MIN} is established.

Theorem. *Let G be a finite, simple, connected, and non-complete graph on n vertices with maximum degree Δ , n_j be the number of vertices of degree j in G , and*

$$x(j) = \frac{j(j+1)}{j(j+1)-1} \left[\left(\frac{1}{j+1} - (\Delta-j) \right) n_\Delta + \left(\frac{1}{j+1} - (\Delta-j-1) \right) n_{\Delta-1} \right. \\ \left. + \dots + \left(\frac{1}{j+1} - 1 \right) n_{j+1} + \frac{n_j}{j+1} + \frac{n_{j-1}}{j} + \dots + \frac{n_1}{2} - 1 \right]$$

for $j \in \{\Delta, \Delta-1, \dots, 1\}$.

- (i) *Then there is a unique $j_0 \in \{\Delta, \Delta-1, \dots, 1\}$ such that $0 \leq x(j_0) < n_\Delta + \dots + n_{j_0}$ and*

$$(ii) \quad k_{MIN} \geq \left(\sum_{j=1}^{\Delta} \frac{n_j}{j+1} \right) + \frac{n_\Delta}{\Delta(\Delta+1)} + \frac{n_\Delta + n_{\Delta-1}}{(\Delta-1)\Delta} \\ + \dots + \frac{n_\Delta + \dots + n_{j_0+1}}{(j_0+2)(j_0+1)} + \frac{x(j_0)}{(j_0+1)j_0} \\ = 1 + x(j_0) + n_{j_0+1} + 2n_{j_0+2} + \dots + (\Delta - j_0)n_\Delta.$$

2. PROOF

Let $d_i = d_G(i), i = 1, \dots, n$ and for $1 \leq k \leq d_1 + \dots + d_n + 1$ let $f(k) = \min \sum_{i=1}^n \frac{1}{d_i+1-x_i}$, where the minimum is taken over integers x_i with $0 \leq x_i \leq d_i$ and $\sum_{i=1}^n x_i = k - 1$. Lemma 1 and Lemma 2 are proved in [7].

Lemma 1. $k_{MIN} \geq f(k_{MIN})$.

Lemma 2. *The following algorithm A calculates $f(k)$:*

Input: $F = \{d_1, d_2, \dots, d_n\}, k \in \{1, 2, \dots, d_1 + \dots + d_n + 1\}, j := 0;$
while $j < k - 1$ *do begin* $F := (F \setminus \{\max(F)\}) \cup \{\max(F) - 1\}; j := j + 1$
end. *Output:* $f(k) = \sum_{f \in F} \frac{1}{f+1}$.

Note that F is a family, i.e., a member of F may occur more than once. Given $k \in \{1, 2, \dots, d_1 + \dots + d_n + 1\}$, in each of the $k - 1$ steps of algorithm A a maximum member f of the current family F is replaced by $f - 1$.

If $k = d_1 + \dots + d_n + 1$ then $f(k) = n$. If $1 \leq k \leq d_1 + \dots + d_n = n_1 + 2n_2 + \dots + \Delta n_\Delta$ then there are unique integers j and x with $j \in \{\Delta, \Delta - 1, \dots, 1\}$ and $0 \leq x < n_\Delta + \dots + n_j$ such that $k - 1 = x + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta = n_\Delta + (n_\Delta + n_{\Delta-1}) + \dots + (n_\Delta + n_{\Delta-1} + \dots + n_{j+1}) + x$. With this expression for $k - 1$ the part cut away by algorithm A is illustrated in Figure 1.

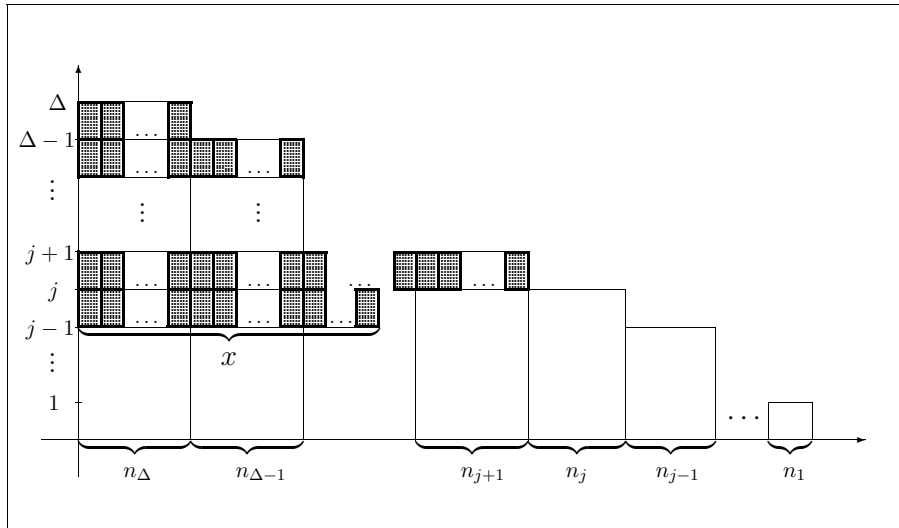


Figure 1

Hence, after applying algorithm A , the family F contains the member $j - 1$ exactly $x + n_{j-1}$ times, the member j exactly $n_\Delta + \dots + n_j - x$ times, and all other members of F being smaller than $j - 1$ at the beginning remain unchanced. Thus, the following Lemma 3 is proved.

Lemma 3.

- (i) Given $k \in \{1, \dots, d_1 + \dots + d_n\}$, there are unique integers j and x with $j \in \{\Delta, \Delta - 1, \dots, 1\}$ and $x \in \{0, \dots, n_\Delta + \dots + n_j - 1\}$ such that

$$\begin{aligned} k - 1 &= n_\Delta + (n_\Delta + n_{\Delta-1}) + \dots + (n_\Delta + n_{\Delta-1} + \dots + n_{j+1}) + x \\ &= x + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta \end{aligned}$$

and

- (ii) $f(k) = (n_\Delta + \dots + n_j - x) \frac{1}{j+1} + \frac{x}{j} + \frac{n_{j-1}}{j} + \dots + \frac{n_1}{2}$
 $= (n_\Delta + \dots + n_j) \frac{1}{j+1} + \frac{x}{j(j+1)} + \frac{n_{j-1}}{j} + \dots + \frac{n_1}{2}$ for that k .

Lemma 4. *If $k = 1 + x + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta$ with $j \in \{\Delta, \Delta - 1, \dots, 1\}$ and $x \in \{0, \dots, n_\Delta + \dots + n_j - 1\}$, then $f(k+1) - f(k) = \frac{1}{j(j+1)}$.*

Proof of Lemma 4. If $x \leq n_\Delta + \dots + n_j - 2$ then $k + 1 = 1 + (x + 1) + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta$ and if $x = n_\Delta + \dots + n_j - 1$ then $k + 1 = 1 + n_j + 2n_{j+1} + \dots + (\Delta - j + 1)n_\Delta$. In both cases Lemma 3 implies Lemma 4. ■

Using Lemma 3, the calculation of $f(k)$ is possible now without taking a minimum and without using algorithm A. In the sequel, we will define the function f for real $k \in [1, d_1 + \dots + d_n + 1)$ and show that the function $g(k) = k - f(k)$ is continuous and strictly increasing on $[1, d_1 + \dots + d_n + 1)$. Finally, using $g(1) < 0$ and $g(k_{MIN}) \geq 0$, the lower bound k_0 on k_{MIN} is the unique solution of the equation $k = f(k)$.

Thus, for given integer $j \in \{\Delta, \Delta - 1, \dots, 1\}$ and real number x with $0 \leq x < n_\Delta + \dots + n_j$ let the real numbers k and $f(k)$ (implicitly) be defined as $k = 1 + x + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta$ and $f(k) = (n_\Delta + \dots + n_j) \frac{1}{j+1} + \frac{x}{j(j+1)} + \frac{n_{j-1}}{j} + \dots + \frac{n_1}{2}$.

Lemma 5. *The function g with $g(k) = k - f(k)$ is continuous and strictly increasing on $[1, d_1 + \dots + d_n + 1)$.*

Proof of Lemma 5. First, let $j \in \{\Delta, \Delta - 1, \dots, 1\}$ be fixed. Then $k = 1 + x + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta$ with $0 \leq x < n_\Delta + \dots + n_j$ belongs to the interval $I(j) = [1 + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta, 1 + n_j + 2n_{j+1} + \dots + (\Delta - j + 1)n_\Delta)$. Obviously g is continuous on $I(j)$ and, because $g(k + \epsilon) - g(k) = \epsilon - \frac{\epsilon}{j(j+1)}$ and $j(j + 1) \geq 2$, g is strictly increasing on $I(j)$.

Now consider g on $[1, \dots, d_1 + \dots + d_n + 1)$ and note that $I(\Delta) \cup \dots \cup I(1) = [1, \dots, d_1 + \dots + d_n + 1)$ and $I(j) \cap I(j') = \emptyset$ if $j \neq j'$. It is easy to see that g is also continuous in $k = 1 + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta$ for $j \in \{\Delta - 1, \Delta - 2, \dots, 2\}$ and we are done. ■

In [2, 12] the well known Caro-Wei-bound $CW = \sum_{j=1}^\Delta \frac{n_j}{j+1}$ is proved to be a lower bound on $\alpha(G)$ and being tight if and only if G is complete. With our assumption that G is non-complete, $g(1) = 1 - \sum_{j=1}^\Delta \frac{n_j}{j+1} < 0$ and $g(k_{MIN}) \geq 0$ by Lemma 1. As a consequence of Lemma 5 there is a unique zero $k_0 = 1 + x(j_0) + n_{j_0+1} + 2n_{j_0+2} + \dots + (\Delta - j_0)n_\Delta$ of g with $1 < k_0 \leq k_{MIN}$ and $0 \leq x(j_0) < n_\Delta + \dots + n_{j_0}$. Considering the equation $f(k) = k$ we obtain

Lemma 6. *If $j \in \{\Delta, \Delta - 1, \dots, 1\}$ and $k = 1 + x + n_{j+1} + 2n_{j+2} + \dots + (\Delta - j)n_\Delta$ with $0 \leq x < n_\Delta + \dots + n_j$, then $f(k) = k$ if and only if*

$$x = \frac{j(j+1)}{j(j+1)-1} \left[\left(\frac{1}{j+1} - (\Delta - j) \right) n_\Delta + \dots + \left(\frac{1}{j+1} - 1 \right) n_{j+1} + \frac{n_j}{j+1} + \dots + \frac{n_1}{2} - 1 \right].$$

Now we complete the proof of the Theorem. Assume there is $j_1 \in \{\Delta, \Delta - 1, \dots, 1\}$ with $j_1 \neq j_0$, $x = \frac{j_1(j_1+1)}{j_1(j_1+1)-1} \left[\left(\frac{1}{j_1+1} - (\Delta - j_1) \right) n_\Delta + \dots + \left(\frac{1}{j_1+1} - 1 \right) n_{j_1+1} + \frac{n_{j_1}}{j_1+1} + \dots + \frac{n_1}{2} - 1 \right]$, and $0 \leq x < n_\Delta + \dots + n_{j_1}$. Then $k_1 = 1 + x(j_1) + n_{j_1+1} + 2n_{j_1+2} + \dots + (\Delta - j_1)n_\Delta$ is a solution of the equation $f(k) = k$ by Lemma 6 and $k_0 \neq k_1$ by Lemma 3 (i) contradicting the uniqueness of k_0 .

With $k_0 = f(k_0) = f(1) + (f(2) - f(1)) + \dots + (f(\lfloor k_0 \rfloor) - f(\lfloor k_0 \rfloor - 1)) + (f(k_0) - f(\lfloor k_0 \rfloor))$ and Lemma 4 we have $f(k_0) = \left(\sum_{j=1}^\Delta \frac{n_j}{j+1} \right) + \frac{n_\Delta}{\Delta(\Delta+1)} + \frac{n_\Delta + n_{\Delta-1}}{(\Delta-1)\Delta} + \dots + \frac{n_\Delta + \dots + n_{j_0+1}}{(j_0+2)(j_0+1)} + \frac{x(j_0)}{(j_0+1)j_0}$ and the Theorem is proved. ■

Many lower bounds on $\alpha(G)$ are known (cf. [1, 2, 3, 4, 5, 6, 8, 9, 10, 11]). If we compare them with k_0 , let us remark here that, by the Theorem,

$$\begin{aligned} k_0 &= CW + \frac{n_\Delta}{\Delta(\Delta+1)} + \frac{n_\Delta + n_{\Delta-1}}{(\Delta-1)\Delta} + \dots + \frac{n_\Delta + \dots + n_{j_0+1}}{(j_0+2)(j_0+1)} + \frac{x(j_0)}{(j_0+1)j_0} \\ &\geq CW + \frac{n_\Delta}{\Delta(\Delta+1)} + \frac{n_\Delta + n_{\Delta-1}}{\Delta(\Delta+1)} + \dots + \frac{n_\Delta + \dots + n_{j_0+1}}{\Delta(\Delta+1)} \\ &\quad + \frac{x(j_0)}{\Delta(\Delta+1)} = CW + \frac{k_0 - 1}{\Delta(\Delta+1)}. \end{aligned}$$

This implies $k_0 \geq CW + \frac{CW-1}{\Delta(\Delta+1)-1}$ improving the well known lower bound $CW + \frac{CW-1}{\Delta(\Delta+1)}$ on $\alpha(G)$ by O. Murphy ([8]).

In [6] it was established $\alpha \geq \frac{CW^2}{CW - \sum_{ij \in E(G)} (d_i - d_j)^2 q_i^2 q_j^2}$, and S.M. Selkow ([9]) proved $\alpha \geq \sum_{i=1}^n q_i (1 + \max\{0, d_i q_i - \sum_{ij \in E(G)} q_j\})$, where $q_i = \frac{1}{d_i+1}$ and $E(G)$ is the edge set of G . Both bounds equal CW if the graph is regular, however, Murphy's bound and therefore also k_0 are considerably larger in that case. For a star $K_{1,p}$ on $p+1$ vertices we have the converse situation, i.e., k_0 is not comparable with these bounds in [6, 9].

Acknowledgement

The authors want to thank Olga Gross (Ilmenau) and Martin Sonntag (Freiberg) for their hints and remarks.

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Received 28 November 2005

Revised 28 June 2006