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SOME RECENT RESULTS ON DOMINATION IN GRAPHS

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Abstract

In this paper, we survey some new results in four areas of domination in graphs, namely:

(1) the toughness and matching structure of graphs having domination number 3 and which are "critical" in the sense that if one adds any missing edge, the domination number falls to 2;

(2) the matching structure of graphs having domination number 3 and which are "critical" in the sense that if one deletes any vertex, the domination number falls to 2;

(3) upper bounds on the domination number of cubic graphs; and

(4) upper bounds on the domination number of graphs embedded in surfaces.

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1. Introduction

Let G be a finite undirected graph. A set of vertices $S \subseteq V(G)$ *dominates* the graph G if every vertex in $V(G)$ either belongs to S or is adjacent to a vertex in S. The size of any smallest dominating set for G is denoted by $\gamma(G)$. A graph G will be said to be k*-critical* if $\gamma(G) = k$, but $\gamma(G+e) = k-1$, for every edge $e \in \overline{G}$. The structure of k-critical graphs remains far from completely understood when $\gamma(G) \geq 3$. In Section 2, we discuss recent results about the ratio $|S|/\omega(G-S)$, where S is a vertex cutset in G and

$\omega(G-S)$ denotes the number of components in $G-S$. For obvious reasons, we refer to these results as "toughness-like" results. We then use these toughness-like results to obtain new theorems about matchings in 3-critical graphs.

In Section 3, we turn to vertex-criticality with respect to domination number. A graph G is *k-vertex-critical* if $\gamma(G) = k$, but $\gamma(G-v) = k-1$, for every vertex $v \in V(G)$. As in the case of edge-criticality, little has been known heretofore about vertex-criticality when $\gamma \geq 3$. Although it has long been known that 3-critical graphs of even order must have perfect matchings, this is not necessarily true for 3-vertex-critical graphs. We will present some sufficient conditions for 3-vertex-critical even (resp. odd) graphs to contain perfect (resp. near-perfect) matchings. Additional matching results for these graphs are also discussed.

In Section 4, we examine domination in cubic graphs; i.e., graphs in which each vertex has degree 3. Reed [34] conjectured that a connected cubic graph with n vertices can always be dominated using no more than $\lceil (n+1)/3 \rceil$ vertices. This ten year old conjecture was only very recently found to be false by Kostochka and Stodolsky [23]. However, we present some positive results related to Reed's conjecture; namely, we have shown [22] that with sufficiently large girth, cubic graphs will have domination number arbitrarily close to Reed's bound.

Finally, in Section 5, we investigate some domination questions for graphs embedded in surfaces. Matheson and Tarjan [30] showed that every planar triangulation (i.e., planar graph in which every face is a triangle) of order n can be dominated using no more than $\lfloor n/3 \rfloor$ vertices. In this final section, we show that the slightly weaker bound $\lceil n/3 \rceil$ holds for triangulations of the torus and of the Klein bottle.

Other terminology and notation will be introduced below as needed.

2. 3-Critical Graphs

The first so-called "toughness result" for 3-critical graphs is due to Sumner and Blitch [36].

Theorem 2.1. *Let G be a connected 3-critical graph. Then if S is a vertex cutset in G, $G-S$ has at most $|S|+1$ components.* ■

The next theorem represents a sharpening the Sumner-Blitch result. Let $\omega(G-S)$ denote the number of components in the graph $G-S$.

Theorem 2.2. *Let G be a connected 3-critical graph and let S be a vertex cutset in G. Then*

(a) *if $|S| \geq 6, \omega(G-S) \leq |S|-2$;*

(b) *if $|S| \geq 4, \omega(G-S) \leq |S|-1$;*

(c) *if $|S| = 3$, then $\omega(G-S) \leq 3$, and if $G-S$ has exactly three components, each component is complete and at least one of them is a singleton;*

(d) *if $|S| = 2$, then $\omega(G-S) \leq 3$, and if $G-S$ has exactly three components, then G must the the graph shown in Figure 2.1 with $n = 2$; and*

(e) *if $|S| = 1$, then $\omega(G-S) = 2$, and exactly one of the components of $G-S$ is a singleton. Furthermore, G has at most three cutvertices. If it has two, then G is a graph of the type shown in Figure 2.1 with $n \geq 2$, while if it has three, it is the graph shown in Figure 2.1 with $n = 1$.* ■

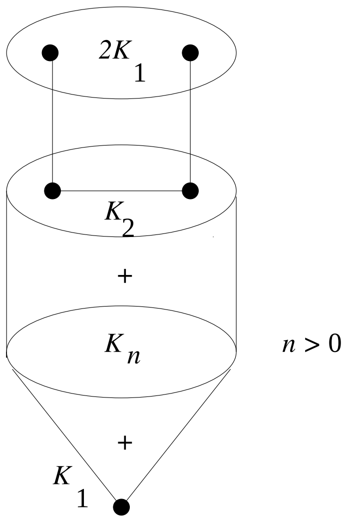

Figure 2.1

Part (a) of the preceeding theorem is Theorem 2.1 of [2]. Part (b) follows immediately from Lemma 6 of [36] and Lemma 3 of [10]. Parts (c) and (d) are proved in Theorem 2.1 of [1]. As for the "missing case" in the Theorem 2.2 above, we have the following result (Theorem 2.2, [2]).

Theorem 2.3. *If G is a connected 3-critical graph, S is a vertex cutset with $4 \leq |S| \leq 5$, and if each component of $G-S$ has at least three vertices, then $\omega(G-S) \leq |S|-2$.* ■

Pertaining to part (c) of Theorem 2.2 above, it is not possible to say more about the number of singleton components, for in Figure 2.2 we present examples of 3-critical graphs in which $G-S$ has three, two and one singleton component respectively.

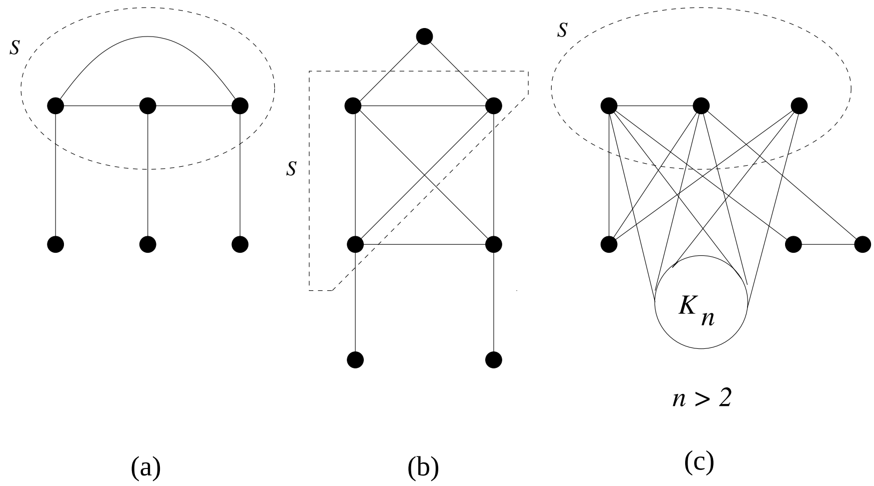

Figure 2.2

As for extending the above results to cutsets of arbitrary size, we offer only the following conjecture.

Conjecture 2.4. Let G be a connected 3-critical graph, S, a vertex cutset in G and t, a positive integer. Then if $|S| \geq 2(t+1)$, $\omega(G-S) \leq |S| - t$.

The conjecture is true for $t = 1$ by part (b) of Theorem 2.2 and for $t = 2$ by part (a) of Theorem 2.2. However, the approach used in proving these two special cases does not readily extend to the case $t \geq 3$ and we believe that a new approach must be found.

The next theorem, which has come to be called the "arrow theorem", is used extensively in proving the toughness results above and also the matching theorems to follow. To state this important result, we introduce some notation. If u, v and w are vertices of graph G and u and v dominate $G-w$, we will write $[u,v] \longrightarrow w$. Note that if G is 3-critical, and u and v are non-adjacent vertices of G, then $\gamma(G+uv) = 2$ and therefore there must be a vertex $x \in V(G)$ such that either $[u,x] \longrightarrow v$ or $[v,x] \longrightarrow u$.

Theorem 2.5. *Let G be a connected 3-critical graph and let S be an independent set of $n \geq 2$ vertices in $V(G)$.*

(i) *Then the vertices of S can be ordered as $a_1, a_2, \ldots, a_n$ in such a way that there exists a sequence of distinct vertices $x_1, x_2, \ldots, x_{n-1}$ so that $[a_i, x_i] \longrightarrow a_{i+1}$ for $i = 1, 2, \ldots, n-1$.*

(ii) *If, in addition, $n \geq 4$, then the x_i's can be chosen so that $x_1 x_2 \cdots x_{n-1}$ is a path and $S \cap \{x_1, \ldots, x_{n-1}\} = \emptyset$.* ∎

The case $n \geq 4$ was proved in [36]; the cases $n = 2$ and 3 were proved in [18].

We now turn our attention to the matching properties of 3-critical graphs. Part (a) of the next theorem is historically the first result on matchings in 3-critical graphs and is due to Sumner and Blitch [36]. It follows immediately from Theorem 2.1 above and Tutte's theorem on perfect matchings. Part (b) is an easy consequence of the Gallai-Edmonds theorem on matchings.

Theorem 2.6. *Let G be a connected 3-critical graph. Then*

(a) *if $|V(G)|$ is even, G contains a perfect matching, while*

(b) *if $|V(G)|$ is odd, G contains a near-perfect matching.* ∎

Over the past thirty years or so, considerable attention has been paid to the matching structure of graphs and, in particular, a canonical decomposition theory of graphs has been developed. Not surprisingly, much subsequent attention has fixed on the "atoms" or indecomposable structures in this theory. Among these are the factor-critical graphs and the bicritical graphs. A graph G of odd order is said to be *factor-critical* if $G - v$ contains a perfect matching, for all $v \in V(G)$ and a graph G of even order is called *bicritical* if $G - u - v$ contains a perfect matching, for all pairs of distinct vertices $u, v \in V(G)$. A 3-connected bicritical graph is called a *brick*. The structure of bicritical graphs in general, and of bricks in particular, has turned out to be quite complicated. (See [27].) For several recent papers on bicritical graphs and bricks, we refer the reader to [14, 28, 12, 13, 31, 32].

Ananchuen and the author proved the following ([3], Theorem 2.1).

Theorem 2.7. *If G is an even 3-connected 3-critical graph with* mindeg $G \geq 4$*, then G is a brick.* ∎

The minimum degree bound in the above theorem is best possible as there are 3-connected 3-critical graphs having minimum degree 3 which are not bicritical. Two such graphs are shown in Figure 2.3. The first is due to Sumner and Blitch [36]. (In each graph, $G-x-y$ has no perfect matching.)

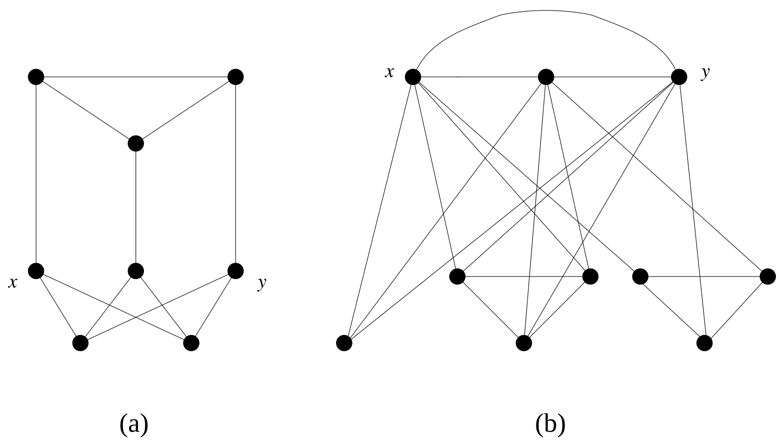

Figure 2.3

If the graph under consideration is claw-free, we can relax both the connectivity and minimum degree hypotheses somewhat as seen in the next theorem ([3], Theorem 3.3).

Theorem 2.8. *Let G be a 3-critical 2-connected claw-free graph of even order. Then if* mindeg $G \geq 3$*, G is bicritical.* ■

Note that both the connectivity condition and the minimum degree condition are sharp in the above theorem as it is clear that every bicritical graph must be 2-connected and have minimum degree 3.

For graphs of odd order, we have the following result ([3], Theorem 2.4).

Theorem 2.9. *Let G be a 2-connected 3-critical graph of odd order. Then G is factor-critical.* ■

The lower bound on connectivity in the above theorem is best possible as the 3-critical graphs shown in Figure 2.1 (with n even) are not factor-critical. The concepts of factor-criticality and bicriticality have been generalized as follows. Let G be a graph on n vertices and suppose k is a positive integer with $k < n/2$. Then G is *k-factor-critical* if $G - S$ has a perfect matching for every set of vertices $S \subset V(G)$ with $|S| = k$. The next theorem is found in [4].

Theorem 2.10. *Let G be a 4-connected 3-critical graph of odd order and suppose* mindeg $G \geq 5$. *Then G is 3-factor-critical.* ∎

It should be noted that the minimum degree bound stated as an hypothesis in the preceding theorem is best possible. In Figure 2.4 we exhibit a 3-critical 4-connected graph having minimum degree 4 and odd order, but which is not 3-factor-critical.

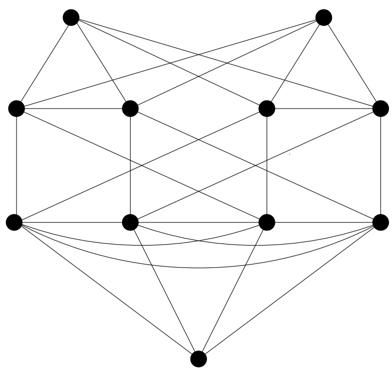

Figure 2.4

If the graph G is claw-free, we can relax both the connectivity and the minimum degree hypotheses of Theorem 2.10 slightly and still guarantee 3-factor-criticality.

Theorem 2.11 ([4], Theorem 3.4)**.** *Let G be a 3-connected claw-free 3-critical graph of odd order. Then if* mindeg $G \geq 4$, *G is 3-factor-critical.* ∎

Both the connectivity and minimum degree bounds stated as hypotheses in the preceding theorem are best possible. Indeed, Favaron has proved ([15], Theorems 2.5 and 2.6) that for all $k \geq 0$, every k-factor-critical graph of order $n > k$ is k-(vertex)-connected and for all $k \geq 1$, every k-factor-critical graph of order $n > k$ is $(k+1)$-edge-connected (and hence has minimum degree at least $k + 1$).

An infinite family of graphs satisfying the hypotheses of Theorem 2.11 is presented in [4] along with two conjectures concerning analogs of Theorems 2.10 and 2.11 for k-factor-critical graphs for $k \geq 4$.

3. 3-Vertex-Critical Graphs

A second kind of domination criticality may be defined in terms of vertex deletion. Let us say that a graph G is *k-vertex-critical* if $\gamma(G) = k$, but $\gamma(G - v) = k - 1$, for every vertex $v \in V(G)$. Brigham *et al.* [8, 9] and Fulman *et al.* [19, 20] were the first to undertake the study of vertex-criticality. From the point of view of matchings, the properties of 3-vertex-critical graphs differ quite dramatically from those of the 3-critical graphs treated in Section 2. For example, by Theorem 2.6 above, every connected even 3-critical graph must have a perfect matching and every connected odd 3-critical graph must have a near-perfect matching. Such is not true for connected 3-vertex-critical graphs, as will be seen below.

So what might be some reasonable conditions one might place on a 3-vertex-critical graph sufficient to guarantee the existence of a perfect (or near-perfect) matching? One of the classical theorems about matchings is the following, due independently to Sumner [35] and Las Vergnas [26].

Theorem 3.1. *Every connected claw-free graph of even order has a perfect matching.* ■

Our next two results [5, 6] can be considered as a variations on this classical result.

Theorem 3.2. *If G is a $K_{1,5}$-free 3-vertex-critical graph of even order, then G has a perfect matching.* ■

Theorem 3.3. *If G is a $K_{1,5}$-free 3-vertex-critical graph of odd order at least 11 with* mindeg $G > 0$*, then G has a near-perfect matching.* ■

The extra assumption that the order be at least 11 in Theorem 3.3 is necessary, for in Figure 3.1 we exhibit a 3-vertex-critical $K_{1,5}$-free of odd order 9 which does not have a near-perfect matching.

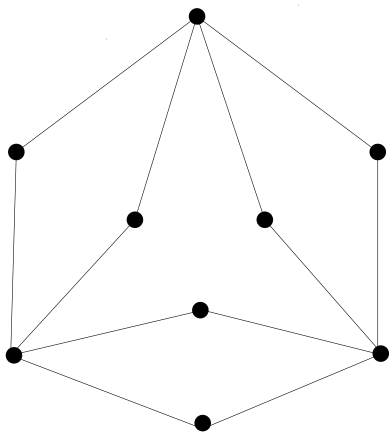

Figure 3.1

In the odd order case, we also have the following result [6].

Theorem 3.4. *If G is a $K_{1,4}$-free 3-vertex-critical graph of odd order with minimum degree at least 3, then G is factor-critical.* ■

We now present a new family of 3-vertex-critical graphs. (See [5].) Let k be any integer with $k \geq 5$. We proceed to construct a graph which we will call $H_{k,\binom{k}{2}-k}$. The vertex set will consist of two disjoint subsets of vertices called *central* and *peripheral* respectively. Let $\{v_1, \ldots, v_k\}$ denote the set of central vertices. The subgraph induced by these central vertices will be the complete graph K_k with the Hamilton cycle $v_1 \cdots v_k$ deleted. The peripheral vertices will be $\binom{k}{2}$ in number and will be denoted by the symbol $\sim \{i,j\}$, where the (unordered) pair $\{i,j\}$ $(i \neq j)$ ranges over all the $\binom{k}{2}$ subsets of size 2 of the set $1, \ldots, k$, *except* those having $j = i+2$ where $i+2$ is read modulo k. The neighbor set of peripheral vertex $\sim \{i,j\}$ will be precisely the set of all central vertices, except i and j. Figure 3.2 shows the graph $H_{6,9}$.

Note that for $k \geq 6$, the graph $H_{k,\binom{k}{2}-k}$ is $(k-2)$-connected, but does not contain a perfect matching, even when $\binom{k}{2}$ is even.

At this point, we offer the following two conjectures. (See [5] and [6] respectively.)

Conjecture 3.5. If G is a 3-vertex-critical graph of even order and $K_{1,7}$-free, then G contains a perfect matching.

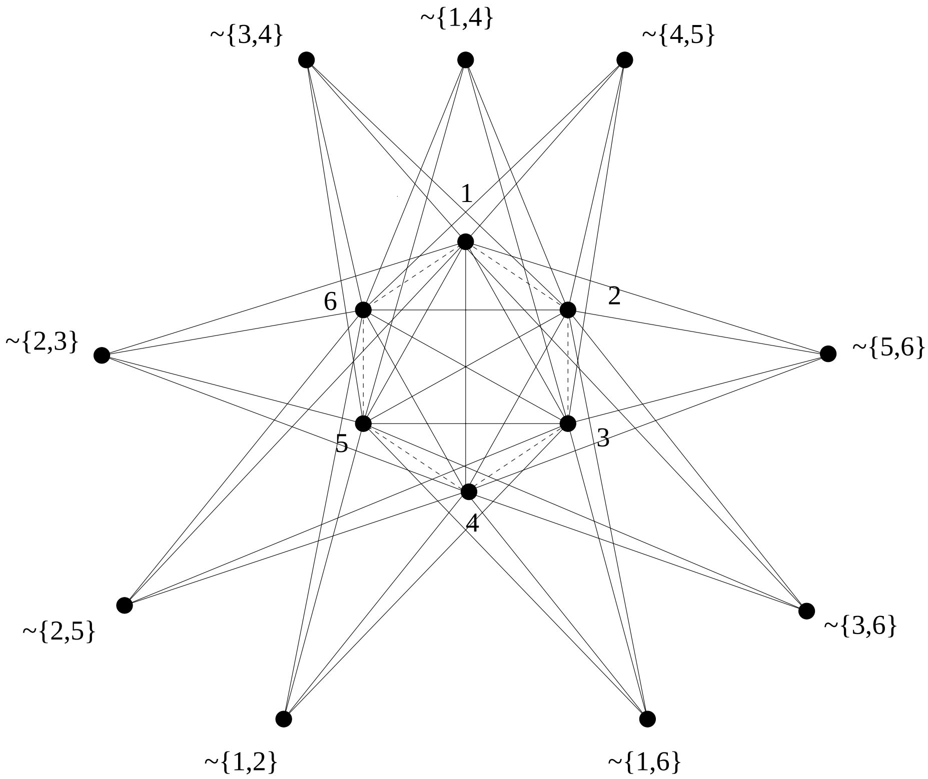

Figure 3.2

Conjecture 3.6. If G is a 2-connected 3-vertex-critical graph of odd order with minimum degree at least 3, then G is factor-critical.

We remark that the hypothesis that G be 2-connected in Conjecture 3.6 is necessary, for we have examples of connected 3-vertex-critical graphs of odd order and minimum degree at least 3 which have cutvertices and which are not factor-critical.

If a 3-vertex-critical graph is also claw-free, more can be said. The following theorem, due independently to Favaron, Flandrin and Ryjáček [16] and to Liu and Yu [27], is useful here.

Theorem 3.7. *If G is a $(k+1)$-connected claw-free graph of order n, and if $n-k$ is even, then G is k-factor-critical.* ■

In [7] the following result is found.

Theorem 3.8. *Let G be a connected claw-free* 3*-vertex-critical graph. Then*

(a) *G is* 2*-connected;*

(b) *if G is of even order or if* mindeg $G \geq 3$*, then G is* 3*-connected; and*

(c) *if* mindeg $G \geq 5$*, then G is* 4*-connected.* ∎

The next result is then immediate by Theorems 3.7 and 3.8.

Theorem 3.9. *Let G be a connected claw-free* 3*-vertex-critical graph. Then*

(a) *if G has odd order, G is factor-critical;*

(b) *if G has even order, G is bicritical; and*

(c) *if G has odd order and* mindeg $G \geq 5$*, G is* 3*-factor-critical.* ∎

4. Efficient Domination of Cubic Graphs

Some ten years ago, Reed [Re] made the following interesting conjecture.

Conjecture 4.1. If G is a connected cubic graph on n vertices, then $\gamma(G) \leq \lceil n/3 \rceil$.

This conjecture attracted the attention of a number of graph theorists until, very recently, it was shown to be false by A.V. Kostochka and B.Y. Stodolsky [23] who produced an infinite family of counterexamples, the smallest of which has 60 vertices and $\gamma = 21$. All of these counterexamples have cutvertices. Even more recently, I have learned from these two authors [24] that they have now constructed 2-connected counterexamples as well. At present, it seems that Reed's conjecture remains unsettled when the graphs are 3-connected, however.

So what is the correct upper bound for γ for the family of cubic graphs? In the same paper containing Conjecture 4.1, Reed proved that if G has mindeg $G \geq 3$, then $\gamma(G) \leq (3/8)|V(G)|$. In a more recent paper, Kawarabayashi, Saito and the author [22] have obtained another result related to this question for graphs of large girth.

Theorem 4.2. *Let G be a connected graph with a* 2*-factor F and let k be any positive integer. If F has at least two components and the order of each component is at least $3k$, then $\gamma(G) \leq \frac{3k+2}{9k+3}|V(G)|$.* ∎

Every 2-edge-connected cubic graph has a 1-factor by Petersen's classical result and hence has a 2-factor. There are then two cases to handle in order to derive the next theorem. If the 2-factor has two components, then the conclusion of Theorem 4.2 is immediate. If, however, the 1-factor has only one component, i.e., the 1-factor is a Hamilton cycle, a separate argument can be made. Hence we have the next result.

Theorem 4.3. *Let k be any positive integer. Then every 2-edge-connected cubic graph G of girth at least $3k$ satisfies $\gamma(G) \leq \frac{3k+2}{9k+3}|V(G)|$.* ■

Note that for 2-edge-connected cubic graphs of girth at least nine, the bound of Theorem 4.3 is better than Reed's 3/8 bound for minimum degree at least 3.

If a cubic graph has a Hamilton cycle, then it trivially satisfies Reed's bound of $\lceil n/3 \rceil$. But recently the author asked whether or not it is true that if G is a cubic Hamiltonian graph on n vertices, then $\gamma(G) \leq \lfloor n/3 \rfloor$. Cropper, Greenwell, Hilton and Kostochka [11] have now shown that the answer is "yes" in exactly two out of three cases, modulo 3.

Theorem 4.4. *Let G be a cubic graph on $n \geq 4$ vertices which has a Hamilton cycle. Then it is true that $\gamma(G) \leq \lfloor n/3 \rfloor$, when $n \equiv 0 (\mathrm{mod}\ 3)$ and when $n \equiv 1 (\mathrm{mod}\ 3)$, but not necessarily true when $n \equiv 2 (\mathrm{mod}\ 3)$.* ■

Note added in proof. As this paper was going to press, the author learned of the following two even more recent results by Kostochka and Stodolsky [25].

Theorem 4.5. *If G is a connected cubic graph with $n > 8$ vertices, then $\gamma(G) \leq 4n/11$.* ■

Theorem 4.6. *If G is a connected cubic graph with n vertices and girth at least g, then*

$$\gamma(G) \leq n\left(\frac{1}{3} + \frac{8}{3g^2}\right).$$

■

It is interesting to compare Theorems 4.2 and 4.6. For graphs of large girth, the bound in Theorem 4.6 is better than that in Theorem 4.2. However Theorem 4.2 is more general in its application than Theorem 4.6 in that in

Theorem 4.2 it is only the girth of a 2-factor which is required to be large. Short cycles, on the other hand, are permitted to exist.

5. Domination of Graphs Embedded in Surfaces

In this final section, we investigate some problems involving the domination of graphs embedded in surfaces. It is well-known that the problem of determining the domination number of a graph is NP-complete, even when the graph is planar. (See [21].) A *triangulated disc* is a graph embedded in the plane such that all faces of the embedding are triangles, with possibly one exception. A *triangulation* (of the plane or sphere) is a graph embedded in the plane such that all faces are triangles. In [30], Matheson and Tarjan proved that a triangulated disc of order n can be dominated using no more than $\lfloor n/3 \rfloor$ vertices. They also provide an infinite family of extremal graphs to show that this upper bound is tight. In [33], Zha and the present author investigate the extension of this result to the projective plane, torus and Klein bottle.

An embedding of graph G on a surface S, denoted $\Phi : G \longrightarrow S$, is a triple $(S, G, V(G))$ such that G is a closed subset of the surface S and $V(G)$ is a finite subset of G such that the connected components of $G - V(G)$ are a finite number of open 1-cells. As usual, an element of $V = V(G)$ is called a *vertex*, the closure of each 1-cell of $G - V(G)$ is called an *edge* and a connected component of $S - G$ is called a *face* of $\Phi(G)$. The *representativity* (or *face-width*) of a graph embedded on a surface S is the smallest number k such that S contains a non-contractible closed curve that intersects the graph in k vertices. If Φ is an embedding of graph G on surface S, and there exists a subset K of Φ which contains all the vertices of G and which is bounded by a subgraph of G, we call K a *spanning subset* of Φ. If the spanning subset is homeomorphic to a topological disc, we call it a *spanning disc* of Φ. Note that if an embedded graph of order n has a spanning disc, then one can immediately conclude via Matheson-Tarjan, that it can be dominated using no more than $\lfloor n/3 \rfloor$ of its vertices. In [17] it is shown that every 3-connected graph in the projective plane has a spanning disc, and hence again by Matheson-Tarjan, it follows that every triangulation G on n vertices embedded in the projective plane has $\gamma(G) \leq \lfloor n/3 \rfloor$.

However, when one considers embeddings on the torus, the situation changes. If one considers the dual embedding of K_7 on the torus (said embedding is necessarily unique under surface homomeorphism), this embedding contains no spanning disc. (The underlying graph of this dual embedding is sometimes known as the *Heawood graph.*) Hence one cannot apply Matheson-Tarjan to bound the domination number from above.

Hence we must take a different approach. In order to state the next theorem, we present some notation and terminology. Let H be a subgraph of a graph G. A *bridge* B is a subgraph of G which is either an edge with both endvertices in H or the union of a connected component K of $G-H$ together with the edges which join K to H and their incident vertices. If B is a bridge of H, then we call each vertex of $B \cap H$ a *vertex of attachment* of B on H.

Next, let C be a cycle in graph G and let u and v be two vertices of C. Assign a clockwise orientation to C. Then $C[uv]$ denotes the path of C from u to v and similarly, $[vu]$ denotes the path of C from v to u, where both paths are chosen to follow the clockwise orientation.

Theorem 5.1. *Let G be a 3-connected graph and $\Phi : G \longrightarrow T$ be a 3-representative embedding of G on the torus T. Let f be any face of the embedding. Then $\Phi(G)$ contains a spanning cylinder Y which contains face f and is bounded by two disjoint homotopic cycles C_1 and C_2 such that*

(1) *$T-Y$ only contains edges each of which joins a vertex on C_1 to a vertex on C_2.*
(2) *Y contains a closed disc D_f such that D_f contains f and is bounded by a null homotopic cycle $P = C_1[x_2x_1] \cup P_{x_1y_1} \cup C_2[y_2y_1] \cup P_{x_2y_2}$, where x_1 and x_2 are two vertices on C_1, y_1 and y_2 are two vertices on C_2 and $P_{x_1y_1}$ and $P_{x_2y_2}$ are paths joining C_1 and C_2. All bridges of D_f in Y are either edges contained in $Y - D_f$ joining $P_{x_1y_1}$ to $P_{x_2y_2}$ or possibly in D_1 or D_2 where D_1 has $\{x_1, x_2\}$ as its vertices of attachment and D_2 has $\{y_1, y_2\}$ as its vertices of attachment. Moreover, both D_i's are discs and each is bounded by a null homotopic cycle of G.*
(3) *The underlying graph of Y is 2-connected.*
(4) *If, in addition, Φ is a triangulation, then $Y - D_f$ contains edges x_1x_2 and y_1y_2.* ■

It is helpful here to consider Figure 5.1.

In essence, what is guaranteed by Theorem 5.1 is that the graph G embedded on the torus has a spanning subgraph consisting of three discs attached as shown in Figure 5.1 (where one or both of D_1 and D_2 may be degenerate, i.e., may consist of a single edge).

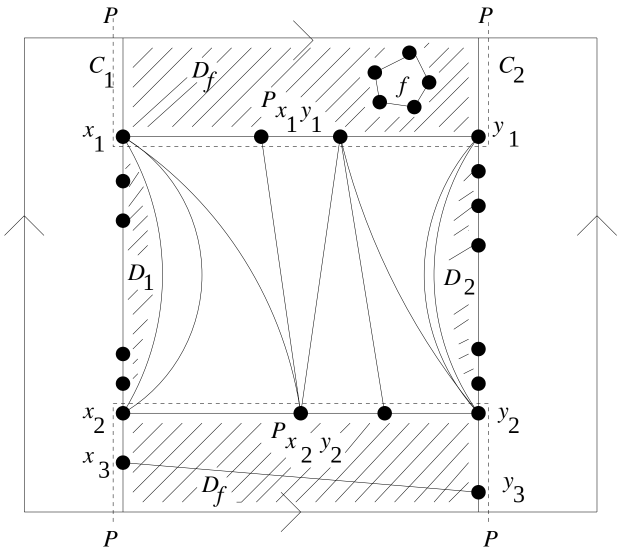

Figure 5.1

Clearly, any set dominating this spanning subgraph will automatically dominate the entire graph G.

The $\lfloor n/3 \rfloor$ bound of Matheson and Tarjan is an immediate corollary to the following result also found in their paper.

Theorem 5.2. *Every triangulated disc on n vertices has a partition of its vertex set into three dominating sets* (*and hence can be dominated using no more than* $\lfloor n/3 \rfloor$ *vertices*)*. Moreover, every two consecutive vertices on the boundary cycle belong to different sets of this partition.* ∎

So we proceed to dominate each of D_1, D_2 and D_f separately, with the help of Theorem 5.2, and combine the three dominating sets in a suitable way so as to dominate the whole of G.

An analog of Theorem 5.1 for the Klein bottle is also proved in [33]. One then combines these two theorems with Theorem 5.2 to prove the following.

Theorem 5.3. *If G is a triangulation of the torus or of the Klein bottle, then* $\gamma(G) \leq \lceil |V(G)|/3 \rceil$. ∎

Actually, Matheson and Tarjan did not think their bound best possible for triangulations of the plane, and ventured the following conjecture.

Conjecture 5.4. If n is sufficiently large and G is a planar triangulation of order n, then $\gamma(G) \leq \lfloor n/4 \rfloor$.

We end with an extension of their conjecture.

Conjecture 5.5. If $|V(G)|$ is sufficiently large and G triangulates some surface (orientable or non-orientable), then $\gamma(G) \leq \lfloor |V(G)|/4 \rfloor$.

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