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THE UPPER DOMINATION RAMSEY NUMBER $u(4,4)$

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Abstract

The upper domination Ramsey number $u(m, n)$ is the smallest integer p such that every 2-coloring of the edges of K_p with color red and blue, $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$, where B and R is the subgraph of K_p induced by blue and red edges, respectively; $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of a graph G. In this paper, we show that $u(4, 4) \leq 15$.

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1. Introduction

Our notation comes from [6] and [7]. Let $G = (V(G), E(G))$ be a graph with a vertex set $V(G)$ of order $p = |V(G)|$ and an edge set $E(G)$. If v is a vertex in $V(G)$, then the open neighborhood of v is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is $N_G[v] = \{v\} \cup N_G(v)$. The external private neighborhood of v relative to $S \subseteq V(G)$ is $epn(v, S) =$

$N(v)-N[S-\{v\}]$. The open neighborhood of a set S of vertices is $N_G(S) = \bigcup_{v\in S} N_G(v)$, and the closed neighborhood is $N_G[S] = N_G(v) \cup S$.

A set $S \subseteq V(G)$ is a *dominating set* in S if each vertex v of G belongs to S or is adjacent to some vertex in S. A set $S \subseteq V(G)$ is an *irredundant set* if for each $s \in S$ there is a vertex w in G such that $N_G[w] \cap S = \{s\}$. A set $S \subseteq V(G)$ is *independent* in G if no two vertices of S are adjacent in G. If S is an irredundant set in G and $v \in S$, then the set $N[v] - N[S - \{v\}]$ is nonempty and is called the set of *private neighbors of* v in G (relative to S), denoted by $pn_G(v, S)$ or simply by $pn(v, S)$. The *upper domination number* of G, denoted by $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G. The *upper irredundance number* of G, denoted by $IR(G)$, is the maximum cardinality of an irredundant set of G. The *independence number* of G, denoted by $\beta(G)$, is the maximum cardinality among all independent sets of vertices of G. A minimal dominating set of cardinality $\Gamma(G)$ is called a $\Gamma(G)$-set. Similarly, an irredundant set of cardinality $IR(G)$ is called an $IR(G)$-set.

It is apparent that irredundance is a hereditary property.

Remark 1. Any independent set is also irredundant.

Remark 2. Every minimal dominating set is an irredundant set. Consequently, we have $\Gamma(G) \leq IR(G)$ for every graph G.

Remark 3 [5]**.** A set $D \subseteq V(G)$ is a minimal dominating set if and only if it is dominating and irredundant, and therefore, if $\Gamma(G) < IR(G)$, then no IR-set is dominating.

Remark 4. Every maximum independent set is also a dominating set, thus we have $\beta(G) \leq \Gamma(G)$ for every graph G.

Hence the parameters $\beta(G), \Gamma(G), IR(G)$ are related by the following inequalities which were observed by Cockayne and Hedetniemi [3].

Theorem 1 [3]**.** *For every graph G, $\beta(G) \leq \Gamma(G) \leq IR(G)$.*

Let $G_1, G_2, \ldots, G_t$ be an arbitrary t-edge coloring of K_n, where for each $i \in \{1, 2, \ldots, t\}$, G_i is the spanning subgraph of K_n whose edges are colored with color i. The classical *Ramsey number* $r(n_1, n_2, \ldots, n_t)$ is the smallest value of n such that for every t-edge coloring $G_1, G_2, \ldots, G_t$ of K_n, there is an $i \in \{1, 2, \ldots, t\}$ for which $\beta(\overline{G_i}) \geq n_i$, where $\overline{G}$ is the complement of G. The *irredundant Ramsey number* denoted by $s(n_1, n_2, \ldots, n_t)$, is the smallest n such that for every t-edge coloring $G_1, G_2, \ldots, G_t$ of K_n, there is at least one $i \in \{1, 2, \ldots, t\}$ for which $IR(\overline{G_i}) \geq n_i$. The irredundant Ramsey numbers exist by Ramsey's theorem, and by Remark 1 $s(n_1, n_2, \ldots, n_t) \leq r(n_1, n_2, \ldots, n_t)$ for all n_i, where $i = 1, 2, \ldots, t$. The *upper domination Ramsey number* $u(n_1, n_2, \ldots, n_t)$ is defined as the smallest n such that for every t-edge coloring $G_1, G_2, \ldots, G_t$ of K_n, there is at least one $i \in \{1, 2, \ldots, t\}$ for which $\Gamma(\overline{G_i}) \geq n_i$.

In the case $t = 2$, $r(m, n)$ is the smallest integer p such that for every 2-coloring of the edges of K_p with colors red (R) and blue (B), $\beta(B) \geq m$ or $\beta(R) \geq n$. Similarly, the irredundant Ramsey number $s(m, n)$ is the smallest integer p such that every 2-coloring of the edges of K_p with colors red (R) and blue (B) satisfies $IR(B) \geq m$ or $IR(R) \geq n$. Finally, the upper domination Ramsey number $u(m, n)$ is the smallest integer p such that every 2-coloring of the edges of K_p with colors red (R) and blue (B) satisfies $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$.

It follows from Theorem 1 that for all m, n,

$$s(m, n) \leq u(m, n) \leq r(m, n),$$

and for the purpose of our proof of the main result, let us recall the following results.

Theorem 2 [2]**.** $s(4, 4) = 13$.

Theorem 3 [4]**.** $r(3, 4) = 9$.

Theorem 4 [4]**.** $r(4, 4) = 18$.

2. Main Result

First we state the following

Lemma 5. *Let (R, B) be a 2-edge coloring of K_n such that $\Gamma(B) \leq 3$, $IR(B) \geq 4$ and $\beta(R) \leq 3$. Then there exists an irredundant set X of B such that $|X| = 4$ and $epn(x, X) \neq \emptyset$ for each $x \in X$.*

Proof. Let Y be an IR-set of B and X the subset of Y such that $epn(x, Y) \neq \emptyset$ for all $x \in X$. Suppose firstly that $|X| = 3$; say $X = \{x_1, x_2, x_3\}$ and let $X' = \{x'_1, x'_2, x'_3\}$, where $x'_i \in epn(x_i, Y)$, $i = 1, 2, 3$. Note that each x'_i is joined by red edges to all vertices in $Y - \{x_i\}$. Since $|Y| \geq 4$, there is a vertex $w \in Y - X$ such that $pn(w, Y) = \{w\}$; hence w is joined by red edges to the vertices in $X \cup X'$. Furthermore, by Remark 3 there is also a vertex $v \in V(B) - N[Y]$; so v is joined by red edges to all vertices in Y. But $\beta(B) < 3$ and so, to avoid a red K_4, the above-mentioned red edges force all edges between vertices in $X' \cup \{v\}$ to be blue. But this is a blue K_4, contradicting $\beta(R) \leq 3$. The case $|X| \leq 2$ is easy and omitted. ■

Now we are ready to prove the following theorem.

Theorem 6. $u(4, 4) \leq 15$.

Proof. Let (R, B) be a 2-edge coloring of K_{15} and suppose that $\Gamma(R) \leq 3$ and $\Gamma(B) \leq 3$. By Theorem 1, $\beta(R) \leq 3$ and $\beta(B) \leq 3$. By Theorem 2, $s(4, 4) = 13$ and therefore, without loss of generality, we may assume that $IR(B) \geq 4$. We only consider the case $IR(B) = 4$; the case $IR(B) \geq 5$ is similar but simpler, and thus omitted. Then, by Lemma 5, there exists an IR-set X of B in which $epn(x, X) \neq \emptyset$ for each $x \in X$. Let $V(K_{15}) = \{0, 1, \ldots, 9, x, y, z, w, t\}$, $X = \{0, 2, 4, 6\}$ and $Y = \{1, 3, 5, 7\}$, where for each $i \in Y$, $i \in epn_B(i - 1, X)$. Thus there is a blue matching consisting of the edges $\{0, 1\}, \{2, 3\}, \{4, 5\}$ and $\{6, 7\}$, and each vertex $i \in X$ is joined to all vertices $j \in Y - \{i + 1\}$ by red edges, according to the private neighbor property. Since $\Gamma(B) < IR(B)$, Remark 3 applied to the irredundant sets X and Y implies that there are vertices u and v joined by red edges to the vertices in X and Y, respectively. If $u = v$, then $X \cup \{u\}$ is irredundant in B and $IR(B) \geq 5$, which is not the case. Hence we may assume that $u \neq v$; say $u = 9$ and $v = 8$. Similarly, we may assume that $\{8, 9\}$ is red, otherwise $X' = X \cup \{8\}$ is irredundant in B (where $9 \in epn(8, X')$).

We now make a few observations about the effects that a red edge between two vertices in X (or Y) has on the colors of the other edges between vertices in $X \cup Y \cup \{8, 9\}$. For simplicity, we consider the edge $\{1, 3\}$; similar remarks hold for the other edges. Suppose therefore that $\{1, 3\}$ is red. Then

Observation 1. $\{i, 8\}$ is blue for $i \in \{4, 6\}$, otherwise $\{1, 3, i, 8\}$ induces a red K_4, thus contradicting $\beta(B) \leq 3$.

Observation 2. $\{4, 6\}$ is blue, otherwise $\{1, 3, 4, 6\}$ induces a red K_4.

Observation 3. $\{1,9\}$ and $\{3,9\}$ are blue, otherwise, if (say) $\{1,9\}$ is red, then $\{2,8\}$ (respectively $\{2,4\}$,$\{2,6\}$) is blue to avoid the red K_4 induced by $\{1,2,8,9\}$ (respectively $\{1,2,4,9\}$, $\{1,2,6,9\}$), thus forming the blue K_4 on $\{2,4,6,8\}$, by Observation 1 and Observation 2. This contradicts $\beta(R) \leq 3$.

Now, if (say) $\{1,3\},\{1,5\}$ and $\{1,7\}$ are all red, then by Observation 2, $\{2,4,6\}$ induces a blue triangle and thus by Observation 1, $\{2,4,6,8\}$ induces a blue K_4, a contradiction. Therefore

Observation 4. No vertex in X (or Y) is adjacent in R to all other vertices of X (or Y).

Observation 5. The red subgraph induced by X is triangle-free, otherwise any such red triangle forms a red K_4 with vertex 9; similarly, the red subgraph induced by Y is triangle-free.

The remaining part of the proof is divided into two parts.

- Part 1: there is a vertex $v \in Y$ such that v is joined by exactly two red edges to the remaining vertices of Y.
- Part 2: there is no vertex $v \in Y$ such that v is joined by two red edges to the remaining vertices of Y.

Part 1

Without loss of generality, let us suppose that edges $\{1,3\}$, $\{1,5\}$ are red. By Observations 1–5 we have $\{1,7\}$, $\{1,9\}$, $\{2,6\}$, $\{2,8\}$, $\{3,5\}$, $\{3,9\}$, $\{4,6\}$, $\{4,8\}$, $\{5,9\}$, $\{6,8\}$ and $\{7,9\}$ are blue, the edge $\{2,4\}$ is red. To avoid a blue triangle $(3,5,7)$ we have that at least one of the edges $\{3,7\}$, $\{5,7\}$ must be red. This forces $\{0,8\}$ to be blue. Now, we have to consider three cases:

- Case 1: $\{3,7\}$ and $\{5,7\}$ are red.
- Case 2: $\{3,7\}$ is blue, $\{5,7\}$ is red.
- Case 3: $\{3,7\}$ is red, $\{5,7\}$ is blue.

Case 1. In this case, we have that $\{3,7\}$ and $\{5,7\}$ are red. By an observation similar to Observation 1, the edges $\{0,2\}$ and $\{0,4\}$ are blue. Similarly, $\{0,6\}$ is red.

Suppose $\{8,x\}$ is blue. If x is joined by red edges to $\{2,4\}$, then, to avoid a red K_4, the edges $\{1,x\}$, $\{7,x\}$ and $\{9,x\}$ are blue, and we obtain a blue K_4 on $\{1,7,9,x\}$.

Similarly, if x is joined by red edges to $\{0,6\}$, then to avoid a red K_4, the edges $\{3,x\}$, $\{5,x\}$ and $\{9,x\}$ are blue, and we obtain a blue K_4 on $\{3,5,9,x\}$.

Suppose $\{2,x\}$ is blue. Then $\{6,x\}$ is red, since otherwise a blue K_4 on $\{2,6,8,x\}$ results. Since $\{6,x\}$ is red, $\{0,x\}$ is blue. But then we have a blue K_4 on $\{0,2,8,x\}$. Thus $\{2,x\}$ is red, and so $\{4,x\}$ is blue. To avoid a blue K_4 on $\{4,6,8,x\}$, $\{6,x\}$ is red. Since $\{6,x\}$ is red, $\{0,x\}$ is blue. But then we have a blue K_4 on $\{0,4,8,x\}$.

Thus vertex 8 is joined by a red edge to every vertex in $\{x,y,z,w,t\}$ and so the red degree of 8 is at least 10. As $r(3,4)=9$, we immediately have a red K_4 containing 8 or a blue K_4 amongst the neighbors of 8.

Case 2. In this case, we have that $\{3,7\}$ is blue and $\{5,7\}$ is red. Similarly to Observation 1, the edge $\{0,2\}$ is blue. To avoid a blue K_4 on $\{0,2,6,8\}$, $\{0,6\}$ is red. If $\{0,4\}$ is blue, then by using similar methods to those in Case 1, we immediately obtain a contradiction. Thus, edge $\{0,4\}$ is red.

Next, suppose that vertex 8 has three blue edges incident to vertices $\{x,y,z,w,t\}$. Without loss of generality, let us suppose that edges $\{8,x\}$, $\{8,y\}$ and $\{8,z\}$ are blue.

Now suppose $\{6,x\}$ is blue. Then $\{2,x\}$ and $\{4,x\}$ are red, since otherwise there are two blue K_4's on $\{2,6,8,x\}$ and $\{4,6,8,x\}$. But then we have a blue K_4 on $\{1,7,9,x\}$. Thus, $\{6,x\}$ is red.

If $\{0,x\}$ is red, we have a blue K_4 on $\{3,5,9,x\}$.

Thus there is only one possible method of coloring the edges joining vertices $\{x,y,z\}$ to vertices $\{0,2,4,6\}$: $\{0,x\}$, $\{0,y\}$, $\{0,z\}$, $\{4,x\}$, $\{4,y\}$, $\{4,z\}$ are blue, and $\{2,x\}$, $\{2,y\}$, $\{2,z\}$, $\{6,x\}$, $\{6,y\}$, $\{6,z\}$ are red. But this coloring forces a red K_4 on the set $\{x,y,z,2\}$, a contradiction.

Thus our assumption that vertex 8 has three blue edges incident to vertices $\{x,y,z,w,t\}$ is incorrect. Similarly, vertex 9 has at most two blue edges to vertices $\{x,y,z,w,t\}$. It is easy to see that there are exactly two blue edges joining vertex 8 (9) to vertices $\{x,y,z,w,t\}$, for otherwise $deg_R(8)\geq 9$ or $deg_R(9)\geq 9$, and by the fact $r(3,4)=9$ we shall to obtain a contradiction. Now, we have to consider three subcases.

Subcase 2.1. In this subcase two blue edges joining vertices 8 and 9 to vertices $\{x,y,z,w,t\}$ have the same end-vertices. Without loss of generality, let us suppose that the end-vertices of these blue edges are x and y.

Suppose $\{6,x\}$ is blue. Then $\{2,x\}$ and $\{4,x\}$ are red, since otherwise there are two blue K_4's on $\{2,6,8,x\}$ and $\{4,6,8,x\}$. But then we have a blue K_4 on $\{1,7,9,x\}$. Thus $\{6,x\}$ is red.

Suppose $\{0,x\}$ is also colored red. Then $\{3,x\}$ and $\{5,x\}$ are blue, since otherwise two red K_4's on $\{0,3,6,x\}$ and $\{0,5,6,x\}$. But then we have a blue K_4 on $\{3,5,9,x\}$. Thus $\{0,x\}$ is blue.

Now, suppose $\{1,x\}$ is red. Then $\{3,x\}$ and $\{5,x\}$ are blue, since otherwise there are two red K_4's on $\{1,3,6,x\}$ and $\{1,5,6,x\}$. But then we have a blue K_4 on $\{3,5,9,x\}$. Thus $\{1,x\}$ is blue.

Consequently, to avoid a blue K_4 on $\{1,7,9,x\}$, $\{7,x\}$ is red, and to avoid a blue K_4 on $\{0,2,8,x\}$, $\{2,x\}$ is red. Then $\{4,x\}$ and $\{5,x\}$ are blue and $\{3,x\}$ is red. Thus there is only one possible method of coloring the edges joining vertices x and y to the vertices of sets X and Y: $\{0,x(y)\}$, $\{1,x(y)\}$, $\{4,x(y)\}$, $\{5,x(y)\}$ are blue, and $\{2,x(y)\}$, $\{3,x(y)\}$, $\{6,x(y)\}$, $\{7,x(y)\}$ are red. But this forces $\{x,y\}$ to be red, and we obtain a red K_4 on vertices $\{3,6,x,y\}$, a contradiction.

Subcase 2.2. In this subcase vertices 8 and 9 are joined by two blue edges to different vertices among $\{x,y,z,w,t\}$. Assume $\{8,z\}$, $\{8,t\}$, $\{9,x\}$ and $\{9,y\}$ are blue.

To avoid the blue K_4 on $\{3,5,9,x\}$, one of the edges $\{3,x\}$ or $\{5,x\}$ is red. Then $\{1,x\}$ is blue, since otherwise there is a red K_4 on either $\{1,3,8,x\}$ or $\{1,5,8,x\}$. Similarly $\{1,y\}$ is blue.

Next, to avoid the blue K_4 on $\{1,7,9,x\}$, edge $\{7,x\}$ is red, and similarly, $\{7,y\}$ is red.

To avoid the blue K_4 on $\{1,9,x,y\}$ edge $\{x,y\}$ is red. But then $\langle\{7,8,x,y\}\rangle$ is a red K_4, a contradiction.

Subcase 2.3. We have to consider the subcase when vertices 8 and 9 are joined by blue edges to exactly one common vertex among $\{x,y,z,w,t\}$. Without loss of generality, assume that $\{8,y\}$, $\{8,z\}$, $\{9,x\}$, $\{9,y\}$ are blue and the remaining edges which join vertices 8 and 9 to $\{x,y,z,w,t\}$ are red. Then we immediately have that $\{w,t\}$ is blue.

Suppose $\{1,x\}$ is red. Then, to avoid two red K_4's on $\{1,3,8,x\}$ and $\{1,5,8,x\}$, we obtain that the edges $\{3,x\}$ and $\{5,x\}$ are blue. But then $\langle\{3,5,9,x\}\rangle$ is a blue K_4, a contradiction. We conclude that $\{1,x\}$ is blue, $\{7,x\}$ is red and $\{5,x\}$ is blue.

Suppose $\{2,y\}$ is blue. Then, to avoid a blue K_4 on $\{0,2,8,y\}$, $\{0,y\}$ is red. Similarly, to avoid a blue K_4 on $\{2,6,8,y\}$, $\{6,y\}$ is red. To avoid a blue

K_4 on $\{3,5,9,y\}$, $\{5,y\}$ or $\{3,y\}$ is red. If $\{5,y\}$ is red, then $\{0,5,6,y\}$ is a red K_4. Thus $\{5,y\}$ is blue, and so $\{3,y\}$ is red. But then $\langle\{0,3,6,y\}\rangle$ is a red K_4. Thus $\{2,y\}$ is red.

Suppose $\{4,y\}$ is red. To avoid a red K_4 on $\{1,2,4,y\}$ it follows that $\{1,y\}$ is blue. To avoid a red K_4 on $\{2,4,7,y\}$, $\{7,y\}$ is blue. But then $\langle\{1,7,9,y\}\rangle$ is a blue K_4, a contradiction. Thus $\{4,y\}$ is blue, and so $\{6,y\}$ is red.

Suppose $\{0,y\}$ is red. To avoid a red K_4 on $\{0,3,6,y\}$, $\{3,y\}$ is blue. To avoid a red K_4 on $\{0,5,6,y\}$, it follows that $\{5,y\}$ is blue. But then $\langle\{3,5,9,y\}\rangle$ is a blue K_4, a contradiction. Thus $\{0,y\}$ is blue.

Suppose $\{1,y\}$ is red. Then, to avoid a red K_4 on $\{1,3,6,y\}$, the edge $\{3,y\}$ is blue. To avoid a red K_4 on $\{1,5,6,y\}$, the edge $\{5,y\}$ is blue. But then $\{3,5,9,y\}$ is a blue K_4, which is a contradiction. Thus $\{1,y\}$ is blue.

Suppose $\{5,y\}$ is red. Then, to avoid a red K_4 on $\{2,5,7,y\}$, it follows that $\{7,y\}$ is blue. But then we obtain a blue K_4 on $\{1,7,9,y\}$. Thus $\{5,y\}$ is blue.

Suppose $\{6,z\}$ is blue. To avoid a blue K_4 on $\{4,6,8,z\}$, $\{4,z\}$ is red. To avoid a blue K_4 on $\{2,6,8,z\}$, the edge $\{2,z\}$ is red. But then $\langle\{2,4,9,z\}\rangle$ is a red K_4, a contradiction.

Thus $\{6,z\}$ is red. Finally:

- to avoid a red K_4 on $\{0,6,9,z\}$, the edge $\{0,z\}$ is blue;
- to avoid a blue K_4 on $\{0,2,8,z\}$, the edge $\{2,z\}$ is red;
- to avoid a red K_4 on $\{2,4,9,z\}$, the edge $\{4,z\}$ is blue;
- to avoid a blue K_4 on $\{3,5,9,y\}$, the edge $\{3,y\}$ is red;
- to avoid a blue K_4 on $\{1,7,9,y\}$, the edge $\{7,y\}$ is red;
- to avoid a blue K_4 on $\{3,5,9,x\}$, the edge $\{3,x\}$ is red;
- to avoid a blue K_4 on $\{5,9,x,y\}$, the edge $\{x,y\}$ is red;
- to avoid a blue K_4 on $\{0,8,y,z\}$, the edge $\{y,z\}$ is red;
- to avoid a red K_4 on $\{2,7,x,y\}$, the edge $\{2,x\}$ is blue;
- to avoid a red K_4 on $\{2,7,y,z\}$, the edge $\{7,z\}$ is blue;
- to avoid a red K_4 on $\{3,6,x,y\}$, the edge $\{6,x\}$ is blue;
- to avoid a red K_4 on $\{3,6,y,z\}$, the edge $\{3,z\}$ is blue.

Suppose, to the contrary, that $\{w,x\}$ is red. Then $\{3,w\}$ and $\{7,w\}$ are blue, since otherwise $\langle\{3,8,w,x\}\rangle$ and $\langle\{7,8,w,x\}\rangle$ are blue K_4's.

Suppose $\{w,z\}$ is red. If $\{2,w\}$ is red, then $\{2,9,z,w\}$ is a red K_4. If $\{6,w\}$ is red, then $\{6,9,w,z\}$ is a red K_4. Thus $\{2,w\}$ and $\{6,w\}$ are blue.

If t sends a blue edge to $\{2,7\}$ and t sends a blue edge to $\{3,6\}$, we obtain a blue K_4, and we are done.

Suppose t is joined by red edges to 2 and 7. Then $\{4,t\}$, $\{5,t\}$ and $\{y,t\}$ are blue, since otherwise there are three red K_4's on $\{2,4,7,t\}$, $\{2,5,7,t\}$ and $\{2,7,y,t\}$. But then we obtain a blue K_4 on $\{4,5,y,t\}$, a contradiction.

Suppose t is joined by red edges to 3 and 6. Then $\{y,t\}$, $\{1,t\}$ and $\{0,t\}$ are blue, since otherwise there are three red K_4's: $\{0,3,6,t\}$, $\{1,3,6,t\}$ and $\{3,6,y,t\}$. But then $\langle\{0,1,y,t\}\rangle$ is a blue K_4. Thus $\{w,z\}$ is blue. But then, in both cases, $\{3,7,w,z\}$ forms a blue K_4, a contradiction. Consequently, $\{w,x\}$ is blue.

Now, by using the same methods to those for the edge $\{w,x\}$, we prove that $\{x,t\}$ is blue. Suppose $\{x,t\}$ is red. Then $\{3,t\}$ and $\{7,t\}$ are blue, since otherwise, $\langle\{3,8,x,t\}\rangle$ and $\langle\{7,8,x,t\}\rangle$ are blue K_4's.

Suppose $\{z,t\}$ is red. If $\{2,t\}$ is red, then $\langle\{2,9,z,t\}\rangle$ is a red K_4. If $\{6,t\}$ is red, then $\langle\{6,9,z,t\}\rangle$ is a red K_4. Thus $\{2,t\}$ and $\{6,t\}$ are blue.

If $\{2,w\}$, $\{3,w\}$, $\{6,w\}$ and $\{7,w\}$ are blue, then $\{3,7,w,t\}$, $\{2,6,w,t\}$ or $\{2,6,w,x\}$ are a blue K_4.

Suppose w is joined by red edges to 2 and 7. Then $\{4,w\}$, $\{5,w\}$ and $\{y,w\}$ are blue, since otherwise there are three red K_4's on $\{2,4,7,w\}$, $\{2,5,7,w\}$ and $\{2,7,y,w\}$. But then we obtain a blue K_4 on $\{4,5,y,w\}$, a contradiction.

Suppose w is joined by red edges to 3 and 6. Then $\{y,w\}$, $\{1,w\}$ and $\{0,w\}$ are blue, since otherwise there are three red K_4's: $\{0,3,6,w\}$, $\{1,3,6,w\}$ and $\{3,6,y,w\}$. But then $\langle\{0,1,y,w\}\rangle$ is a blue K_4. Thus $\{z,t\}$ is blue. But then, in both cases, $\{3,7,t,z\}$ forms a blue K_4, a contradiction. Hence, $\{x,t\}$ is blue.

Suppose now that $\{z,t\}$ is red. Then $\{2,t\}$ and $\{6,t\}$ are blue, since otherwise $\langle\{2,9,z,t\}\rangle$ and $\langle\{7,8,z,t\}\rangle$ are red K_4's. But then we obtain a blue K_4 on $\{2,6,t,x\}$. Thus $\{z,t\}$ is blue. Similarly, $\{w,z\}$ is also colored blue. Then, to avoid a blue K_4 on $\{w,t,x,z\}$, it follows $\{x,z\}$ is red.

Now, let us consider a vertex x and all blue edges incident to it. Since $r(3,4)=9$, we obtain that x is joined by at most one blue edge to one of vertices 0 and 4.

If $\{0,x\}$ and $\{4,x\}$ are red, then we have a red K_4 on $\{0,4,7,x\}$.

First, suppose $\{0, x\}$ is blue. To avoid a red K_4 on vertices $\{1, 5, 8, w\}$ or $\{1, 5, 8, t\}$ we may assume without loss of generality that $\{1, w\}$ is blue. Then $\{5, w\}$ and $\{1, t\}$ are red, and $\{5, t\}$ is blue. To avoid a blue K_4 on vertices $\{0, 1, x, w\}$, $\{0, w\}$ is red. Then $\{6, w\}$ is blue, since otherwise there is a red K_4 on $\{0, 6, 9, w\}$. Similarly, $\{6, t\}$ is red and $\{0, t\}$ is blue. It is easy to see that $\{2, w\}$ and $\{2, t\}$ are red.

Now, consider a vertex z and all blue edges incident to it. Similarly to x, vertex z is joined by exactly one blue edge either to vertex 1 or to 5.

If $\{1, z\}$ is blue and $\{5, z\}$ is red, then, since $\{0, w\}$ is red, we obtain that $\{4, w\}$ is blue and $\{4, t\}$ is red. But then we have a red K_4 on $\{2, 4, 9, t\}$.

If $\{1, z\}$ is red and $\{5, z\}$ is blue, we also easily obtain a contradiction, so $\{0, x\}$ is red. If $\{4, x\}$ is blue, then by using similar arguments we obtain a contradiction, and the proof of this subcase is complete.

Case 3. In this case we have that $\{3, 7\}$ is red and $\{5, 7\}$ is blue. To avoid a red K_4 on $\{0, 3, 4, 7\}$, it follows that $\{0, 4\}$ is blue. To avoid a blue K_4 on $\{0, 4, 6, 8\}$, $\{0, 6\}$ is red. If $\{0, 2\}$ is blue, then by using similar methods to those in Case 1, we obtain a contradiction. Thus edge $\{0, 2\}$ is red and we obtain a coloring isomorphic to that considered in Case 2.

Part 2

Without loss of generality we can assume that $\{1, 3\}$ is red. By Observations 1–5, we obtain that vertices 1 and 3 are joined by blue edges to vertex 9. Edge $\{5, 9\}$ is blue, otherwise a blue K_4 on $\{0, 2, 4, 8\}$ results. Similarly $\{7, 9\}$ is blue, otherwise there is a blue K_4 on $\{0, 2, 6, 8\}$. To avoid a blue K_4 on $\{3, 5, 7, 9\}$, $\{5, 7\}$ is red. So we obtain two red edges, and all the remaining edges of K_5 on $\{0, 2\}$, $\{0, 8\}$, $\{2, 8\}$, $\{4, 6\}$, $\{4, 8\}$, $\{6, 8\}$. When we color the remaining edges of K_5 on $X \cup \{8\}$, we must consider sixteen cases. When $\{0, 4\}$, $\{0, 6\}$, $\{2, 4\}$, $\{2, 6\}$ are red, we obtain a coloring which is isomorphic to that considered in Part 1, Case 1 above. In the nine following cases:

1. $\{0, 4\}$, $\{0, 6\}$ are blue and $\{2, 4\}$, $\{2, 6\}$ are red,
2. $\{0, 4\}$, $\{2, 4\}$ are blue and $\{0, 6\}$, $\{2, 6\}$ are red,
3. $\{0, 6\}$, $\{2, 6\}$ are blue and $\{0, 4\}$, $\{2, 4\}$ are red,
4. $\{2, 4\}$, $\{2, 6\}$ are blue and $\{0, 4\}$, $\{0, 6\}$ are red,
5. $\{0, 4\}$, $\{0, 6\}$, $\{2, 4\}$ are blue and $\{2, 6\}$ is red,
6. $\{0, 4\}$, $\{0, 6\}$, $\{2, 6\}$ are blue and $\{2, 4\}$ is red,

7. $\{0,4\}$, $\{2,4\}$, $\{2,6\}$ are blue and $\{0,6\}$ is red,
8. $\{0,6\}$, $\{2,4\}$, $\{2,6\}$ are blue and $\{0,4\}$ is red,
9. $\{0,4\}$, $\{0,6\}$, $\{2,4\}$ and $\{2,6\}$ are blue,

we immediately obtain a contradiction. In the remaining six cases:

1. $\{0,4\}$, $\{0,6\}$, $\{2,4\}$ are red and $\{2,6\}$ is blue,
2. $\{0,4\}$, $\{0,6\}$, $\{2,6\}$ are red and $\{2,4\}$ is blue,
3. $\{0,4\}$, $\{2,4\}$, $\{2,6\}$ are red and $\{0,6\}$ is blue,
4. $\{0,6\}$, $\{2,4\}$, $\{2,6\}$ are red and $\{0,4\}$ is blue,
5. $\{0,6\}$, $\{2,4\}$ are red and $\{0,4\}$, $\{2,6\}$ are blue,
6. $\{0,4\}$, $\{2,6\}$ are red and $\{0,6\}$, $\{2,4\}$ are blue,

similarily to Case 1, we obtain that vertex 9 is joined by a red edge to every vertex in $\{x,y,z,w,t\}$, so the red degree of 9 is at least 10. This observation completes the proof of Theorem 6. ■

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