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## ARBITRARILY VERTEX DECOMPOSABLE CATERPILLARS WITH FOUR OR FIVE LEAVES

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### Abstract

A graph  $G$  of order  $n$  is called arbitrarily vertex decomposable if for each sequence  $(a_1, \dots, a_k)$  of positive integers such that  $a_1 + \dots + a_k = n$  there exists a partition  $(V_1, \dots, V_k)$  of the vertex set of  $G$  such that for each  $i \in \{1, \dots, k\}$ ,  $V_i$  induces a connected subgraph of  $G$  on  $a_i$  vertices.

D. Barth and H. Fournier showed that if a tree  $T$  is arbitrarily vertex decomposable, then  $T$  has maximum degree at most 4. In this paper we give a complete characterization of arbitrarily vertex decomposable caterpillars with four leaves. We also describe two families of

arbitrarily vertex decomposable trees with maximum degree three or four.

**Keywords:** arbitrarily vertex decomposable graphs, trees, caterpillars, star-like trees.

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## 1. Introduction

Let  $G = (V, E)$  be a graph of order  $n$ . A sequence  $\tau = (a_1, \dots, a_k)$  of positive integers is called *admissible for  $G$*  if it adds up to  $n$ . If  $\tau = (a_1, \dots, a_k)$  is an admissible sequence for  $G$  and there exists a partition  $(V_1, \dots, V_k)$  of the vertex set  $V$  such that for each  $i \in \{1, \dots, k\}$ ,  $|V_i| = a_i$  and a subgraph induced by  $V_i$  is connected then  $\tau$  is called *realizable in  $G$*  and the sequence  $(V_1, \dots, V_k)$  is said to be a  *$G$ -realization of  $\tau$*  or a *realization of  $\tau$  in  $G$* . A graph  $G$  is *arbitrarily vertex decomposable* (avd for short) if for each admissible sequence  $\tau$  for  $G$  there exists a  $G$ -realization of  $\tau$ .

The problem of deciding whether a given graph is arbitrarily vertex decomposable has been considered in several papers (see for example [1]–[4]). Generally, this problem is NP-complete [1] but we do not know if this problem is NP-complete when restricted to trees.

However, it is obvious that each path and each traceable graph is avd. The investigation of avd trees is motivated by the fact that a connected graph is avd if its spanning tree is avd. In [4] M. Horňák and M. Woźniak conjectured that if  $T$  is a tree with maximum degree  $\Delta(T)$  at least five, then  $T$  is not avd. This conjecture was proved by D. Barth and H. Fournier [2].

**Theorem 1.** *If a tree  $T$  is arbitrarily vertex decomposable, then  $\Delta(T) \leq 4$ . Moreover, every vertex of degree four of  $T$  is adjacent to a leaf.*

In [1] D. Barth, O. Baudon and J. Puech studied a family of trees each of them being homeomorphic to  $K_{1,3}$  (they call them tripodes) and showed that determining if such a tree is avd can be done using a polynomial algorithm.

There is an interesting motivation for investigation of avd graphs. Consider a network connecting different computing resources; such a network is modeled by a graph. Suppose there are  $k$  different users, where  $i$ -th one needs  $n_i$  resources in our graph. The subgraph induced by the set of resources attributed to each user should be connected and a resource should

be attributed to at most one user. So we have the problem of seeking a realization of the sequence  $(n_1, \dots, n_k)$  in this graph. Note also that one can find in [4] some references concerning arbitrarily edge decomposable graphs. The aim of this article is a characterization of avd trees with maximum degree at most four that have a very simple structure. Namely, we consider caterpillars or trees which are homeomorphic to a star  $K_{1,q}$ , where  $q$  is three or four.

## 2. Terminology and Results

In this paper, we deal with finite, simple and undirected graphs.

Let  $T = (V, E)$  be a tree. A vertex  $x \in V$  is called *primary* if  $d(x) \geq 3$ . A *leaf* is a vertex of degree one. A path  $P$  of  $T$  is an *arm* if one of its endvertices is a leaf in  $T$ , the other one is primary and all internal vertices of  $P$  have degree two in  $T$ . A tree  $T$  is called *primary* if it contains a primary vertex.

A graph  $T$  is a *star-like tree* if it is a tree homeomorphic to a star  $K_{1,q}$  for some  $q \geq 3$ . Such a tree has one primary vertex (let us denote it by  $c$ ) and  $q$  arms (let us denote them by  $A_i, i \in \{1, \dots, q\}$ ). For each  $A_i$  let  $\alpha_i$  be the order of  $A_i$ . The structure of a star-like tree is (up to a isomorphism) determined by this sequence  $(\alpha_1, \dots, \alpha_q)$  of orders of its arms. Since the ordering of this sequence is not important, we will always assume that  $2 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_q$  and will denote the above defined star-like tree by  $S(\alpha_1, \dots, \alpha_q)$ . Notice that an order of this star-like tree is equal to  $1 + \sum_{i=1}^q (\alpha_i - 1)$ .

A tree  $T$  is a *caterpillar* if the set of vertices of degree at least two induces a path. Let  $T$  be a caterpillar such that  $\Delta(T) \leq 4$ . Let us note that if there are two or more vertices of degree four in  $T$ , then the sequences  $(2, 2, \dots, 2)$  if  $n$  is even or  $(1, 2, 2, \dots, 2)$  if  $n$  is odd are not realizable in  $T$ , hence  $T$  is not avd. Clearly, these particular sequences are realizable in  $T$  if there is a perfect matching or a quasi-perfect matching in  $T$ . According to the above remark we will consider only caterpillars of maximum degree at most four having at most one vertex of degree four.

Let  $T$  be a caterpillar with  $\Delta(T) = 3$  and let  $\{y_1, \dots, y_s\}$  be the set of primary vertices of  $T$ . We call  $T$  a *caterpillar with  $s$  single legs attached at  $y_1, \dots, y_s$* .

Similarly, if  $T$  is a caterpillar and  $\{x, y_1, \dots, y_s\}$  the set of primary vertices of  $T$  such that  $d(x) = 4$  and  $d(y_i) = 3$  for all  $i \in \{1, \dots, s\}$ , then

$T$  is called a *caterpillar with one double leg attached at  $x$  and  $s$  single legs attached at  $y_1, \dots, y_s$* . For simplicity of notation we say sometimes that we have a caterpillar with  $s$  single legs or a caterpillar with one double leg and  $s$  single legs. We present two examples of such caterpillars in Figure 1 and Figure 4.

Here and subsequently, we assume that every admissible sequence for a graph  $G$  is non-decreasing and we write  $d^\lambda$  for the sequence  $\underbrace{(d, \dots, d)}_\lambda$  and  $d^\lambda \cdot g^\mu$  for the sequence  $\underbrace{(d, d, \dots, d)}_\lambda \underbrace{(g, g, \dots, g)}_\mu$ , the concatenation of  $\lambda$  times  $d$  and  $\mu$  times  $g$ . We will note  $d \cdot g^\mu$  and  $d^\lambda \cdot g$  instead of  $d^1 \cdot g^\mu$  and  $d^\lambda \cdot g^1$ .

We denote by  $(a, b)$  the greatest common divisor of two positive integers  $a$  and  $b$  and we write  $t(i, j)$  for the transposition of the elements  $i$  and  $j$  of the set  $\{1, 2, \dots, k\}$ . Note that if  $i = j$ , then by transposition  $t(i, j)$  we mean the identity.

Let  $T$  be a tree, and let  $(V_1, V_2)$  and  $(V'_1, V'_2)$  be two partitions of  $V(T)$  such that each  $V_i$  and each  $V'_i$  induces a tree in  $T$ . We say that we can *transpose*  $V_1$  and  $V_2$  (into  $V'_1$  and  $V'_2$ ) if  $|V'_i| = |V_i|$  ( $i = 1, 2$ ).

Let  $P = y_1, \dots, y_q$  be a subpath of a tree  $T$  and  $U, W$  two disjoint subsets of  $V(T)$ . We shall say that  $U$  and  $W$  are *neighbouring* in  $P$  if for some  $j \in \{1, \dots, q - 1\}$ ,  $y_j \in U$  and  $y_{j+1} \in W$  or  $y_j \in W$  and  $y_{j+1} \in U$ .

The first result characterizing avd star-like trees (i.e., caterpillars with one single leg) was found by D. Barth, O. Baudon and J. Puech [1] and, independently, by M. Hornák and M. Woźniak [3].

**Proposition 2.** *The star-like tree  $S(2, a, b)$ , with  $2 \leq a \leq b$  is avd if and only if  $(a, b) = 1$ . Moreover, each admissible and non-realizable sequence in  $S(2, a, b)$  is of the form  $d^\lambda$ , where  $a \equiv b \equiv 0 \pmod{d}$  and  $d > 1$ .*

In [1] D. Barth, O. Baudon and J. Puech proved the following proposition. In the statement of this result the sequence  $(3, a, b)$  is not assumed to be non-decreasing.

**Proposition 3.** *Each star-like tree  $S(2, 2, a, b)$ , with  $2 \leq a \leq b$  is avd if and only if*

- 1<sup>0</sup> *the star-like tree  $S(3, a, b)$  is avd;*
- 2<sup>0</sup>  *$a, b$  are odd;*
- 3<sup>0</sup>  *$a \not\equiv 2 \pmod{3}$  or  $b \not\equiv 2 \pmod{3}$ .*

The next result due to D. Barth and H. Fournier [2] shows that the structure of avd caterpillars is not obvious.

**Theorem 4.** *For every  $s \geq 1$  there exists an avd caterpillar with  $s$  single legs.*

The main results of this paper are Theorems 5 and 6 of Sections 3 and 4 which give a complete characterization of avd caterpillars with two single legs and avd star-like trees  $S(3, a, b)$ . In Section 4 we also give a necessary and sufficient condition for a star-like tree  $S(2, 2, a, b)$  to be avd. Thus, we describe the family of avd caterpillars with four leaves. In Section 5 we describe an infinite family of avd caterpillars with one double and one single leg (Proposition 9).

### 3. Arbitrarily Vertex Decomposable Caterpillars with Two Single Legs

Every caterpillar  $T$  of order  $n$  with two single legs attached at  $x$  and  $y$  can be obtained by taking a path  $P = x_1, \dots, x_{n-2}$ , where  $x = x_i$  and  $y = x_j$  ( $i < j$ ) are two internal vertices of  $P$ , adding two vertices  $u$  and  $v$ , and joining  $u$  to  $x$  and  $v$  to  $y$  (see Figure 1). For such a graph let us define  $l_x(T) := i$ ,  $r_x(T) := n - i$  and, analogously,  $l_y(T) := j + 1$  and  $r_y(T) := n - j - 1$ .

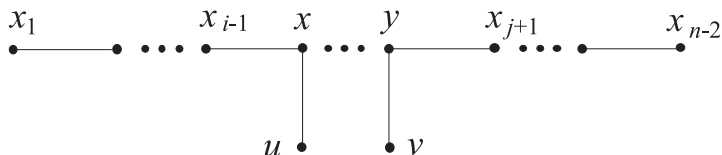


Figure 1. A caterpillar with two single legs.

**Theorem 5.** *Let  $T = (V, E)$  be a caterpillar of order  $n$  with two single legs attached at  $x$  and  $y$ . Then  $T$  is avd if and only if the following conditions hold:*

- 1<sup>0</sup>  $(l_x(T), r_x(T)) = 1$ ;
- 2<sup>0</sup>  $(l_y(T), r_y(T)) = 1$ ;
- 3<sup>0</sup>  $(l_x(T), r_y(T)) = 1$ ;

- 4<sup>0</sup>  $(l_y(T), r_x(T)) < l_y(T) - l_x(T)$  or  $n \equiv 1 \pmod{(l_y(T), r_x(T))}$ ;
- 5<sup>0</sup>  $n \neq \alpha l_x(T) + \beta l_y(T)$  for any  $\alpha, \beta \in \mathbf{N}$ ;
- 6<sup>0</sup>  $n \neq \alpha r_x(T) + \beta r_y(T)$  for any  $\alpha, \beta \in \mathbf{N}$ .

**Proof.** For abbreviation we write  $l_x = l_x(T)$ ,  $r_x = r_x(T)$ ,  $l_y = l_y(T)$  and  $r_y = r_y(T)$ . Observe first that  $n = l_x + r_x = l_y + r_y$  and there is no loss of generality in assuming  $l_x \leq r_y$ .

**Necessity.** Suppose that  $(l_x, r_x) = d > 1$  ( $(l_y, r_y) = d' > 1$ , resp.). Then  $n = \lambda \cdot d$  ( $n = \lambda' \cdot d'$ , resp.) for some  $\lambda \in \mathbf{N}$  ( $\lambda' \in \mathbf{N}$ , resp.). It can be easily seen that the sequence  $d^\lambda$  ( $d^{\lambda'}$ , resp.) is not realizable in  $T$ , so the conditions 1<sup>0</sup> and 2<sup>0</sup> are necessary for  $T$  to be avd.

Suppose now  $l_x = \alpha \cdot d$ ,  $r_y = \beta \cdot d$  for some integers  $\alpha, \beta \geq 1$  and  $d > 1$ . Hence  $n = (\alpha + \beta) \cdot d + r$  and, by 1<sup>0</sup>,  $d$  does not divide  $r$ . Let us consider the sequence  $r \cdot d^\lambda$  if  $r \leq d$  or  $d^\lambda \cdot r$  otherwise. Let  $S$  be a subtree of  $T$  of order  $r$ . It can be easily seen that the graph  $T - S$  has a connected component  $C$  being a star-like tree  $S(2, a, b)$  with  $(a, b) = \mu d$  for some integer  $\mu \geq 1$  or a path of length which is not divisible by  $d$  or else a caterpillar  $T'$  with two single legs attached at  $x$  and  $y$  such that  $d$  divides  $(l_y(T'), r_y(T'))$  or  $(l_x(T'), r_x(T'))$ . Thus, using the previous argument or Proposition 2 we may deduce that such a sequence is not realizable in  $C$  and this implies the necessity of the condition 3<sup>0</sup>.

Assume then  $(l_y, r_x) = d \geq l_y - l_x \geq 2$  and  $n$  is not congruent to 1 modulo  $d$ . If  $d = l_y - l_x$ , then  $l_x \equiv 0 \pmod{d}$  and we can show as above that  $T$  is not avd. Assume  $d > l_y - l_x$  and let  $\lambda$  and  $r \in \{1, \dots, d - 1\}$  be two integers such that  $l_x = \lambda d + r$ . Thus,  $r_x = \alpha d$ ,  $l_y = \beta d$  for some integers  $\alpha, \beta$  and  $n = \lambda d + \alpha d + r$ . Hence  $r \geq 2$  and, because  $l_y - l_x < d$ ,  $\beta = \lambda + 1$ . Consider now the sequence  $\tau = r \cdot d^{\alpha + \lambda}$ . Taking the graph  $T - S$ , where  $S$  is a subtree of  $T$  on  $r$  vertices and using a similar argument as in the previous situation we deduce that  $\tau$  is not realizable in  $T$ , so the condition 4<sup>0</sup> is necessary for  $T$  to be avd.

Finally, if  $n = \alpha l_x + \beta l_y$  for some  $\alpha, \beta \in \mathbf{N}$  (or  $n = \alpha r_x + \beta r_y$ ), then the sequence  $l_x^\alpha \cdot l_y^\beta$  (or  $r_y^\beta \cdot r_x^\alpha$ , resp.) is not realizable in  $T$  and this implies the necessity of the conditions 5<sup>0</sup> and 6<sup>0</sup>.

**Sufficiency.** Suppose the conditions 1<sup>0</sup>–6<sup>0</sup> hold and let  $\tau = (a_1, \dots, a_k)$  be an admissible sequence for  $T$ . We first show that if  $a_1 = 1$ , then there exists a  $T$ -realization of  $\tau$ . Indeed, consider a caterpillar  $T' = T - u$  i.e., a caterpillar with one leg attached at  $y$  and an admissible sequence  $\tau' =$

$(a_2, a_3, \dots, a_k)$  for  $T'$ . Obviously, if  $\tau'$  is a realizable sequence for  $T'$ , then  $\tau$  is realizable for  $T$ . Suppose then, that  $\tau'$  is not realizable for  $T'$ . It follows from Proposition 2 that  $(l_y - 1, r_y) = d$  for some integer  $d > 1$  and  $\tau' = (d, \dots, d)$ . Thus  $d$  divides  $r_y$  and, by  $3^0$ ,  $l_x$  is not divisible by  $d$ , so  $\tau'$  is realizable in the tree  $T'' = T - v$ . It follows that  $\tau = (1, d, \dots, d)$  is realizable in  $T$  as claimed.

From now on we will assume that  $a_1 \geq 2$ , i.e., for every  $i = 1, \dots, k$ ,  $a_i \geq 2$ .

Observe that  $T$  is avd if and only if for any admissible sequence  $\tau = (a_1, \dots, a_k)$  for  $T$  there exists a permutation  $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$  such that for all  $s \in \{1, \dots, k\}$

$$(*) \quad \sum_{i=1}^s a_{\sigma(i)} \notin \{l_x, l_y\}.$$

Let  $m$  be the minimum number  $j \in \{1, \dots, k\}$  such that  $a_1 + \dots + a_j \geq l_x$ . Thus, for  $m > 1$  we get  $a_1 + \dots + a_{m-1} < l_x$ .

*Case 1.*  $a_1 + \dots + a_m = l_x$ . If  $a_j = a_1$  for all  $j \in \{1, \dots, k\}$ , then we have a contradiction with condition  $1^0$ . Therefore, there exists  $j_0 \geq m + 1$  such that  $a_{j_0} > a_1$ . We may assume that  $j_0$  is minimal with this property. Let  $\sigma$  be the product of three transpositions:  $t(1, m)$ ,  $t(m + 1, j_0)$  and  $t(m, m + 1)$  taken in this order. It can be easily seen that  $a_{\sigma(1)} + \dots + a_{\sigma(m)} > l_x$  and  $a_{\sigma(1)} + \dots + a_{\sigma(m-1)} = a_2 + \dots + a_m < l_x$  for  $m > 1$ .

Assume that there exists  $m' \geq m$  such that  $a_{\sigma(1)} + \dots + a_{\sigma(m')} = l_y$ . Now, if  $a_{\sigma(j)} = a_1$  for each  $j \in \{m' + 1, \dots, k\}$  then  $r_y \geq 2a_1$  ( $k - 1 > m'$ ), because  $l_x \leq r_y$  and  $(l_x, r_y) = 1$ . So  $j_0 = k$  and  $a_i = a_1$  for each  $i < k$ . It follows that  $l_x = ma_1$  and  $l_y = (m' - 1)a_1 + a_k$ ; consequently  $r_y = n - l_y = \alpha a_1$  for some  $\alpha$  which contradicts  $3^0$ . Hence, we can also assume there exists  $s \in \{m' + 1, \dots, k\}$  such that  $a_{\sigma(s)} > a_1$ .

*Case 1.1.*  $m = m'$ . Hence  $a_{\sigma(m)} \geq l_y - l_x + 1$ . If  $a_{\sigma(j)} > a_{\sigma(m)}$  for some  $j > m$  then we can take the permutation  $t(m, m + 1) \circ t(m + 1, j) \circ \sigma$  satisfying  $(*)$ . Thus we may assume that if  $j > m$  then  $a_{\sigma(j)}$  can take only two values:  $a_1$  and  $a_{\sigma(m)}$ . Moreover, by  $5^0$ , we have  $m \geq 2$ . Set

$$\begin{aligned} d &= a_{\sigma(m)}, \\ r &= \sum_{i=2}^{m-1} a_i \text{ for } m > 2 \text{ and} \\ r &= 0 \text{ for } m = 2. \end{aligned}$$

Hence  $l_x = a_1 + r + a_m$  and  $l_y = r + a_m + d$ .

*Case 1.1.1.*  $d > a_m$ . Suppose first  $a_m > a_1$  and take the permutation  $\sigma' = t(1, m+1) \circ \sigma$  (recall that  $a_{\sigma(1)} = a_m$  and  $a_{\sigma(m+1)} = a_1$ ). We have now  $a_{\sigma'(1)} + \dots + a_{\sigma'(m-1)} = a_1 + r < a_1 + r + a_m = l_x$ ,  $l_y = r + a_m + d > a_{\sigma'(1)} + \dots + a_{\sigma'(m)} = a_1 + r + d > l_x$  (because  $a_m > a_1$  and  $d > a_m$ ),  $a_{\sigma'(1)} + \dots + a_{\sigma'(m+1)} = a_1 + r + d + a_m = l_y + a_1 > l_y$ , therefore  $\sigma'$  verifies (\*). Suppose then  $a_1 = a_m$ , i.e.,  $a_j = a_1$  for all  $j \in \{1, \dots, m\}$  and  $l_x = \lambda a_1$  for some integer  $\lambda \geq 2$ . Therefore, by  $3^0$ , there exists  $i_0 \geq m+1$ ,  $i_0 \neq j_0$ , such that  $a_{i_0} = d$ . Consider now the permutation  $\sigma' = t(m-1, i_0) \circ \sigma$ . We have  $a_{\sigma'(1)} + \dots + a_{\sigma'(m)} = (\lambda - 2)a_1 + 2d > l_y = (\lambda - 1)a_1 + d$ . Thus, if  $(\lambda - 2)a_1 + d \neq l_x = \lambda a_1$ , i.e.,  $d \neq 2a_1$ , then  $\sigma'$  satisfies (\*). But if  $d = 2a_1$ , then  $r_y$  is divisible by  $a_1$  and we get a contradiction with  $3^0$ .

*Case 1.1.2.*  $d = a_m$ . By construction of our permutation  $\sigma$ , we get  $a_j = d$ , for all  $j \geq m$ , so  $r_x = (k - m)d$  and  $a_1 < d$ . Instead of our permutation  $\sigma$  take another permutation  $\rho$  given by the following formula:  $\rho(i) = k - i + 1$ ,  $i = 1, 2, \dots, k$ . Clearly,  $a_{\rho(i)} = a_m = d$  for  $i = 1, \dots, k - m$  and, since  $l_y < r_x$ , we obtain  $\sum_{i=1}^{k-m} a_{\rho(i)} > l_y$ . From  $1^0$ ,  $l_x$  is not divisible by  $d$ , therefore the condition (\*) does not hold for  $\rho$  if  $l_y = \gamma d$  for some integer  $\gamma$ . But in this case there are three positive integers  $w, \alpha', \beta'$  such that  $(l_y, r_x) = wd \geq d > d - a_1 = l_y - l_x$  and  $n = r_x + l_x = r_x + l_y - d + a_1 = \alpha'wd + \beta'wd - d + a_1 = (\alpha' + \beta' - 1)wd + (w - 1)d + a_1 \neq 1 \pmod{wd}$  (because  $d > a_1 \geq 2$ ) and we obtain a contradiction with  $4^0$ .

*Case 1.2.*  $m < m'$ . Suppose that there exists  $s_0 \in \{m' + 1, \dots, k\}$  such that  $a_{\sigma(s_0)} \neq a_{\sigma(m')}$ . Without loss of generality we can assume that  $s_0 = m' + 1$  (if necessary, we can perform an appropriate transposition). Now taking the transposition  $t(m', m' + 1)$  we get a permutation that satisfies (\*). Assume then  $a_{\sigma(s)} = a_{\sigma(m')}$  for all  $s \in \{m' + 1, \dots, k\}$ .

Now, if  $m + 1 < m'$  and for some  $i \in \{m + 1, m' - 1\}$  we have  $a_{\sigma(i)} \neq a_{\sigma(m')}$ , then we can take the permutation  $t(m', m' + 1) \circ t(i, m') \circ \sigma$  that verifies (\*). Therefore, we can assume that  $a_{\sigma(s)} = a_{\sigma(m')}$  for all  $s \in \{m + 1, \dots, m'\}$ , so  $a_{\sigma(s)} = a_1$  for  $s \in \{m + 1, \dots, k\}$  and  $l_x = ma_1$ , which is impossible by  $3^0$ .

*Case 2.*  $a_1 + \dots + a_m > l_x$ . We may assume that there exists  $m' \geq m$  such that  $a_1 + \dots + a_{m'} = l_y$ , because otherwise the identity permutation satisfies (\*). Now, since  $a_i \geq a_{m'}$  for  $i > m'$ , it is enough to consider only the case where  $a_i = a_{m'}$  for  $i > m'$ , i.e.,  $r_y = \alpha a_{m'}$  for some integer  $\alpha$ . Using the same method as in Case 1.2 we see that if there is no permutation



verifying (\*), then  $a_i = a_{m'}$  for all  $i > m$ . Notice that if  $a_{m+1} > a_m$  then the transposition  $\sigma = t(m, m' + 1)$  satisfies (\*). So assume  $a_i = a_{m'}$  for all  $i \geq m$ . Hence  $l_y < r_x < (k - m + 1)a_{m'}$ . Now take the permutation  $\rho$  defined as follows:  $\rho(i) = k - i + 1, i = 1, 2, \dots, k$ . Since  $r_y = \alpha a_{m'}$ , for some integer  $\alpha$ , it follows by  $3^0$  and  $2^0$  that the condition (\*) holds for  $\rho$  and we are done. This finishes the proof of the theorem. ■

#### 4. Arbitrarily Vertex Decomposable $S(3, a, b)$ and $S(2, 2, a, b)$

**Theorem 6.** *Let  $a, b, 3 \leq a \leq b$ , be two integers and  $T = S(3, a, b)$  a star-like tree with three arms. Then  $T$  is avd if and only if the following conditions hold:*

- $1^0 (a, b) \leq 2;$
- $2^0 (a + 1, b) \leq 2;$
- $3^0 (a, b + 1) \leq 2;$
- $4^0 (a + 1, b + 1) \leq 3;$
- $5^0 n \neq \alpha \cdot a + \beta \cdot (a + 1)$  for  $\alpha, \beta \in \mathbf{N}$ .

**Proof.** Let  $c$  be the primary vertex of degree three of  $T$  and  $A_1, A_2, A_3$  its arms. The vertices of three arms will be denoted as follows:

$$\begin{aligned} V(A_1) &= \{c, x, y\}, \\ V(A_2) &= \{x_1, \dots, x_a = c\}, \\ V(A_3) &= \{x_a = c, x_{a+1}, \dots, x_{a+b-1}\}, \end{aligned}$$

(see Figure 2).

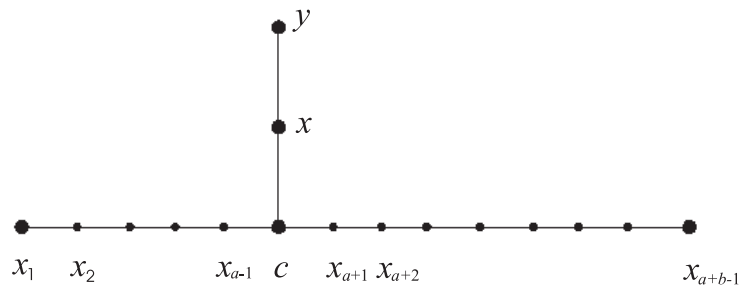


Figure 2.  $S(3, a, b)$

**Necessity.** Suppose that  $(a, b) = d > 2$ . Then  $n = \lambda \cdot d + 1$  for some integer  $\lambda \geq 2$ , and it can be easily seen that the sequence  $d^{\lambda-1} \cdot (d + 1)$  is not realizable in  $T$ .

Let  $(a + 1, b) = d \geq 3$  ( $(a, b + 1) = d' \geq 3$ ). We have  $n = \lambda \cdot d$ ,  $\lambda \in \mathbf{N}, \lambda \geq 2$  ( $n = \lambda' \cdot d', \lambda' \in \mathbf{N}, \lambda' \geq 2$ , resp.) and it is easy to check that the sequence  $d^\lambda$  ( $(d')^{\lambda'}$ , resp.) is not realizable in  $T$ .

Similarly, if  $(a + 1, b + 1) = d > 3$ , then  $n = \lambda \cdot d - 1$ ,  $\lambda \in \mathbf{N}$ , so the sequence  $(d - 1) \cdot d^{\lambda-1}$  is not realizable in  $T$ .

We now turn to the case  $n = \alpha \cdot a + \beta \cdot (a + 1)$ ,  $\alpha, \beta \in \mathbf{N}$ . This implies that the sequence  $a^\alpha \cdot (a + 1)^\beta$  is not realizable in  $T$ .

**Sufficiency.** Suppose that conditions  $1^0-5^0$  hold and let  $\tau = (m_1, \dots, m_k)$  be an admissible sequence for the tree  $T$ . Such a sequence is realizable in  $T$  if  $m_k = 1$  (because it is ordered in a non-decreasing way), so we will assume  $m_k > 1$ . Let  $\hat{\tau} = (n_1, \dots, n_k)$  be a non-decreasing ordering of the sequence  $(m_1, \dots, m_{k-1}, m_k - 1)$ , with  $n_s = m_k - 1$ . Consider the tree  $\hat{T} = T - y$  which is isomorphic to the star-like tree  $S(2, a, b)$ . Clearly, the sequence  $\hat{\tau}$  is admissible for the tree  $\hat{T}$ . Suppose  $\hat{\tau}$  is not realizable in  $\hat{T}$ . Then, by  $1^0$  and Proposition 2,  $(a, b) = 2$  and  $\hat{\tau} = 2^k$ . Hence  $\tau = 2^{k-1} \cdot 3$  is obviously realizable in  $T$ . From now on we will assume that  $\hat{\tau}$  is realizable in  $\hat{T}$ .

Furthermore, since  $\tau$  is realizable in  $T$  if  $m_i \in \{1, 2\}$  for some  $i$ , we can assume  $n_j \geq 3$  for all  $j \neq s$  and  $n_s \geq 2$ . Let  $\hat{M} = (V_1, \dots, V_s, \dots, V_k)$  be a  $\hat{T}$ -realization of  $\hat{\tau}$  such that  $|V_i| = n_i$  for  $i = 1, \dots, k$ , and  $V_p$  induces the primary tree of  $\hat{T}$ . Observe that if

$$(*) \quad |V_p| = m_k - 1,$$

then the sequence  $M$ , obtained from  $\hat{M}$  by replacing  $V_p$  by  $V_p \cup \{y\}$ , is a  $T$ -realization of  $\tau$ . Therefore, we will assume that the condition  $(*)$  does not hold (so  $V_p \neq V_s$ ).

*Case 1.*  $V_s \subset V(A_2)$ . Suppose  $x_{a-1} \in V_p$ . Because  $A_2 - V_p$  is a path in  $\hat{T}$ , we can arrange the sets  $V_i$ 's covering this path in such a way that  $V_p$  and  $V_s$  are neighboring in  $A_2$ . Therefore, the subtree of  $T$  induced by  $V_p \cup V_s \cup \{y\}$  can be covered by  $(V_s \cup \{z\}, V_p \setminus \{z\} \cup \{y\})$ , where  $z$  is the first vertex of  $V_p$  on  $A_2$ . Adding the remaining sets  $V_i$  we get a  $T$ -realization of  $\tau$ . Thus, let us assume that  $V_p$  induces a path in  $\hat{T}$  such that  $V_p \setminus \{x\} \subset V(A_3)$  (see Figure 3).

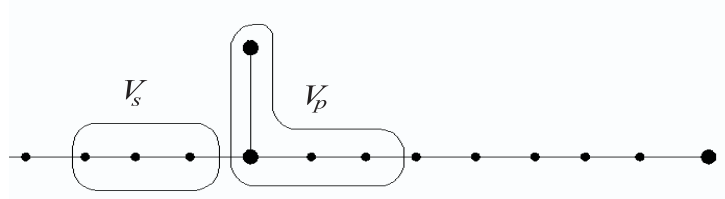


Figure 3.  $V_p$  and  $V_s$  are neighboring in  $A_2$ .

Suppose now  $n_s > n_p$ . Since  $A_2 - V_p$  is a path in  $\hat{T}$ , we can assume without loss of generality that  $V_p$  and  $V_s$  are neighboring in  $A_2$  (see Figure 3). Now the subtree of  $\hat{T}$  induced by the set  $V_s \cup V_p$  can be covered by  $(V'_s, V'_p)$ , where  $V'_s$  induces a subpath of  $A_2$  on  $n_p$  vertices, and  $V'_p$  a star-like tree on  $n_s = m_k - 1$  vertices that contains  $c$ . Put  $V'_i = V_i$  for  $i \neq p, s$ . It is easy to see that  $(V'_1, \dots, V'_k)$  is a  $\hat{T}$ -realization of  $\hat{\tau}$  satisfying  $(*)$  and we can easily obtain a  $T$ -realization of  $\tau$ . Hence, by the choice of  $n_s$ , we can assume that  $n_s = n_p - 1 = m_k - 1$ . Then  $n_i \leq n_p$  for all  $i$ 's. If for some  $i \neq s, p$ ,  $V_i \subset A_2$  and  $|V_i| \leq n_p - 2$ , then, assuming that  $V_i$  and  $V_p$  are neighboring in  $A_2$ , we can cover  $V_i \cup V_p$  by the pair  $(V'_i, V'_p)$ , where  $V'_i$  induces a subpath of  $A_3 - c$  on  $n_i$  vertices and  $V'_p$  induces a tree containing  $c$ . Applying the same argument as above we get a  $T$ -realization of  $\tau$ . Hence,  $n_p - 1 \leq |V_i| \leq n_p$  for all  $i$ 's such that  $V_i \subset V(A_2)$ . Suppose that for some  $j$ ,  $V_j \subset V(A_3)$  and  $|V_j| < n_p$ . Now, because  $V_p$  induces a path in  $\hat{T}$ , we can place this  $V_j$  at the beginning of the path  $xcx_{a+1} \dots x_{a+b-1}$  and find a  $T$ -realization of  $\tau$  as in the previous cases. Thus,  $|V_i| = n_p$  for all  $i$ 's such that  $V_i \subset V(A_3)$ .

Let  $q := n_p$ . We have now  $a = \lambda q + \mu(q - 1)$  and  $b + 1 = \nu q$ , for some integers  $\lambda > 0$ ,  $\mu \geq 0$  and  $\nu > 0$ . Moreover, the sequence  $\tau$  is of the form

$$(q - 1)^\mu \cdot q^{\lambda + \nu}.$$

If  $\mu = 0$ , then, by  $3^0$ ,  $q \leq 2$ , a contradiction with our assumption on  $n_p$ . Suppose  $\mu = 1$ . Then  $a + 1 = (\lambda + 1)q$ , hence, by  $4^0$ ,  $q = 3$ , so  $\tau = 2 \cdot 3^k$  and this sequence is clearly  $T$ -realizable. So consider the case  $\mu \geq 2$ . Because  $a < b$ , it follows that  $\nu \geq 2$ , so the sequence  $(q - 1)^2 \cdot q^{\nu - 2}$  is realizable in  $A_3 - c$  and the sequence  $(q - 1)^{\mu - 2} \cdot q^{\lambda + 2}$  is realizable in the tree induced by  $A_2 \cup \{x, y\}$ , hence  $\tau$  is realizable in  $T$ .

*Case 2.*  $V_s \subset V(A_3)$ . As in Case 1 we assume that  $x_{a+1} \notin V_p$ ,  $q - 1 \leq |V_i| \leq q$  for  $V_i \subset V(A_3)$  and  $|V_j| = q$  for  $V_j \subset V(A_2)$ , where  $q = n_p$ . Now we

can write  $b = \lambda q + \mu(q - 1)$  and  $a + 1 = \nu q$ , for some integers  $\lambda > 0$ ,  $\mu \geq 0$  and  $\nu > 0$ . If  $\mu = 0$ , then, by  $2^0$ ,  $q \leq 2$ , and we get a contradiction with our assumption on  $n_p$ . For  $\mu = 1$  we proceed as in Case 1 and show that  $\tau$  is realizable in  $T$ . Suppose then  $\mu \geq 2$ . If  $\nu \geq 2$  we proceed as in Case 1 and we show that  $\tau$  is realizable in  $T$ . If  $\nu = 1$  (the essential difference between Case 1 and Case 2), then  $q = a + 1$  and  $n = (\lambda + 1)(a + 1) + \mu a$ , a contradiction. This finishes the proof of the theorem. ■

**Corollary 7.** *Let  $a, b, 3 \leq a \leq b$  be two integers and  $T = S(2, 2, a, b)$  a star-like tree on  $n$  vertices. Then  $T$  is avd if and only if the following conditions hold:*

- 1'  $(a, b) = 1$ ;
- 2'  $(a + 1, b) = 1$ ;
- 3'  $(a, b + 1) = 1$ ;
- 4'  $(a + 1, b + 1) = 2$ ;
- 5'  $n \neq \alpha \cdot a + \beta \cdot (a + 1)$  for  $\alpha, \beta \in \mathbf{N}$ .

**Proof. Necessity.** Assume that  $T$  is avd. Hence, from Proposition 3,  $S(3, a, b)$  is avd,  $a, b$  are odd, and  $a \not\equiv 2 \pmod{3}$  or  $b \not\equiv 2 \pmod{3}$ .

Therefore, the odd numbers  $a$  and  $b$  satisfy the conditions  $1^0$ – $5^0$  of Theorem 6, hence also the conditions 1' and 5' of our theorem. Since  $a$  and  $b$  are odd, it follows by  $1^0$ ,  $2^0$  and  $3^0$  that  $(a, b) = 1$ ,  $(a + 1, b) = 1$  and  $(a, b + 1) = 1$ . So  $a$  and  $b$  satisfy 1', 2' and 3'. By  $4^0$ ,  $(a + 1, b + 1) \in \{2, 3\}$  and since  $a \not\equiv 2 \pmod{3}$  or  $b \not\equiv 2 \pmod{3}$ , we have  $(a + 1, b + 1) \neq 3$  and the condition 4' holds.

**Sufficiency.** If  $a$  and  $b$  verify the conditions 1'–5' then the conditions  $1^0$ – $5^0$  of Theorem 6 are satisfied. Thus  $S(3, a, b)$  is avd and, by 1'–3',  $a$  and  $b$  are odd.

Suppose that  $a \equiv 2 \pmod{3}$  and  $b \equiv 2 \pmod{3}$ . Then  $a + 1 \equiv 0 \pmod{3}$  and  $b + 1 \equiv 0 \pmod{3}$ , so  $(a + 1, b + 1) \geq 3$ , a contradiction. This implies that  $a \not\equiv 2 \pmod{3}$  or  $b \not\equiv 2 \pmod{3}$ , and, by Proposition 3,  $T$  is avd. This finishes the proof. ■

**Corollary 8.** *There are infinitely many arbitrarily vertex decomposable star-like trees  $S(3, a, b)$  and  $S(2, 2, a, b)$ .*

**Proof.** Let  $a \geq 5$  be a prime and  $b = a + 2$ . It can be easily seen that  $a$  and  $b$  satisfy the conditions  $1'-5'$  (and also  $1^0-5^0$ ) for  $n = 2a + 3$ . ■

### 5. Caterpillars with One Double and One Single Leg

Every caterpillar with one double and one single leg attached at  $x$  and  $y$  can be constructed in the following way. Take a path  $P = x_1, \dots, x_{n-3}$  where  $x = x_a$  and  $y = x_j$  ( $a < j$ ) are two internal vertices of  $P$ , add three vertices  $u, v$  and  $z$  and join  $u$  and  $v$  to  $x$  and  $v$  to  $y$  (see Figure 4).

Let  $L_x = \{x_1, x_2, \dots, x\}$ ,  $R_x = \{x, x_{a+1}, \dots, x_{n-3}\} \cup \{z\}$ ,  $L_y = \{x_1, x_2, \dots, y\} \cup \{u, v\}$ ,  $R_y = \{y, x_{j+1}, \dots, x_{n-3}\}$  and let  $l_x = |L_x|$ ,  $r_x = |R_x|$ ,  $l_y = |L_y|$  and  $r_y = |R_y|$ .

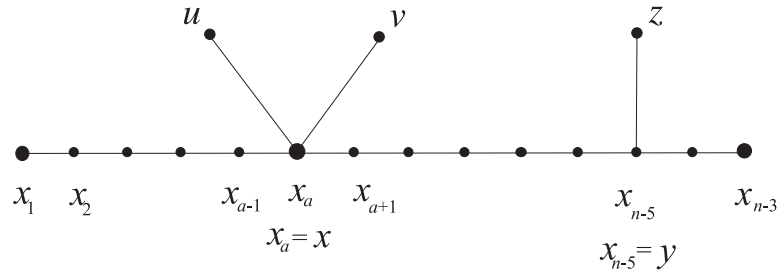


Figure 4. A caterpillar with one double and one single leg.

**Proposition 9.** Let  $T$  be a caterpillar of order  $n$  with one double and one single leg attached at  $x$  and  $y$  resp. Let  $a = l_x$  and  $b = r_x$ . If  $a \equiv 1 \pmod{6}$ ,  $b \equiv 0 \pmod{3}$ ,  $7 \leq a < b$ ,  $(a - 3, b) = 1$ ,  $n - 1 \neq \alpha a$  ( $\alpha \in \mathbf{N}$ ),  $r_y = 3$  and  $a$  and  $b$  satisfy the conditions  $1'-5'$  of Corollary 7, then  $T$  is avd.

**Proof.** Let  $u$  and  $v$  denote two vertices of degree one adjacent to  $x$  and let  $z$  be the vertex of degree one adjacent to  $y$  (see Figure 4). It follows from our assumptions that  $n = a + b + 1 \equiv 2 \pmod{3}$ . Let  $\tau = (a_1, \dots, a_k)$  be an admissible sequence for the tree  $T$ . We will show that it suffices to consider the case where  $a_t \geq 2$  for all  $t$ . Indeed, the caterpillar  $T' = T - v$  with two single legs satisfies  $l'_x = a$ ,  $r'_x = b$ ,  $l'_y = a + b - 3 = n - 4 \equiv 1 \pmod{3}$ ,  $r'_y = 3$ , so the conditions  $1^0-3^0$  of Theorem 5 are satisfied. We also have  $(l'_y, r'_x) = (a + b - 3, b) = (a - 3, b) = 1 < l'_y - l'_x = b - 3$ , so the condition

$4^0$  holds. Furthermore, if  $\alpha l'_x + \beta l'_y = \alpha a + (n - 4)\beta = n - 1$ , for some  $\alpha, \beta \in \mathbf{N}$ , then, since  $a \geq 7$ , we have  $\beta = 0$ , which is a contradiction. Assume  $n - 1 = \alpha r'_x + \beta r'_y = \alpha b + 3\beta \equiv 0 \pmod{3}$  ( $\alpha, \beta \in \mathbf{N}$ ). But  $n - 1 = a + b \equiv 1 \pmod{3}$ , and we get a contradiction. So also  $5^0-6^0$  of Theorem 5 hold. Now, if  $a_1 = 1$ , we can put  $V_1 = \{v\}$  and the existence of  $T$ -realization of  $\tau$  is obvious. Therefore, we may assume  $a_t \geq 2$  for all  $t$ .

Notice that, by Corollary 7, the star-like tree  $\hat{T} = S(2, 2, a, b)$  obtained by deleting the edge  $zy$  and adding  $zx_{n-3}$  is avd. Let  $\hat{M} = (V_1, \dots, V_k)$  be a  $\hat{T}$ -realization of  $\tau$  such that  $V_p$  ( $V_s$ , resp.) induces a primary tree (a primary tree or a subpath, resp.) of  $\hat{T}$  containing  $x$  ( $y$ , resp.). Observe that if  $V_s$  contains  $x_{n-4}$  (the vertex that follows  $y$  in the path  $x_1, \dots, x_{n-3}$ ) then  $\tau$  is  $T$ -realizable. Indeed, if  $z \in V_s$ , then  $\hat{M}$  is also a  $T$ -realization of  $\tau$  and if  $z \in V_{s'}$  for some  $s' \neq s$ , then  $V_{s'} = \{x_{n-3}, z\}$  and replacing in  $\hat{M}$   $V_s$  and  $V_{s'}$  by the sets  $(V_s \setminus \{x_{n-4}\}) \cup \{z\}$  and  $(V_{s'} \setminus \{z\}) \cup \{x_{n-4}\}$ , we get a  $T$ -realization of  $\tau$ .

Therefore, we shall assume that  $V_s$  does not contain  $x_{n-4}$ . Hence, because  $a_r \geq 2$  for all  $r$ , there is  $g$  with  $V_g = \{x_{n-4}, x_{n-3}, z\}$  (see Figure 5).

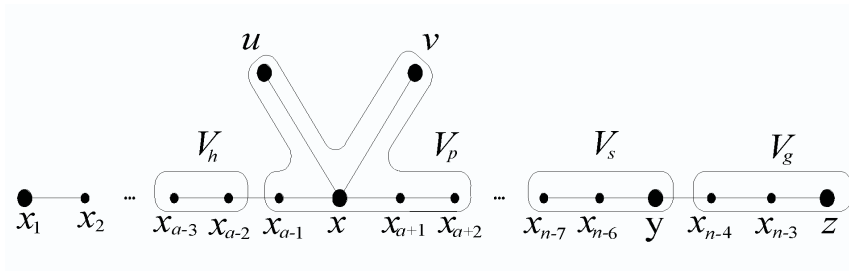


Figure 5. Case 1.1

Notice that for every  $r$  such that  $V_r \subset R_x$  we have  $|V_r| = 3$ , for otherwise  $g \neq r$  and assuming  $V_r$  and  $V_g$  are neighboring in  $R_x$  we could transpose  $V_r$  and  $V_g$  into  $V'_r$  and  $V'_g$ , in such a way that  $V'_r$  or  $V'_g$  contains the set  $\{y, x_{n-4}\}$ .

Now, since  $|R_x| = b \equiv 0 \pmod{3}$ , we have  $|V_p \cap (R_x \setminus \{x\})| \equiv 2 \pmod{3}$ , hence  $|V_p \cap (R_x \setminus \{x\})| \geq 2$ . Furthermore, since  $a_r \geq 2$  for all  $r$ , we have  $u, v \in V_p$  and  $|V_p| \geq 5$ .

*Case 1.* There is  $h$  such that  $V_h \subset L_x$  and  $|V_h| \neq 3$ . Obviously, we may suppose that  $V_h$  and  $V_p$  are neighboring in  $L_x$ .

*Case 1.1.*  $|V_h| \leq |V_p \cap (R_x \setminus \{x\})|$  (see Figure 5). Now we can transpose  $V_p$  and  $V_h$  into  $V'_p$  and  $V'_h$  with  $V'_h \subset R_x$ . Using the same argument as above, we easily find a  $T$ -realization  $(V'_1, \dots, V'_k)$  of  $\tau$ .

*Case 1.2.*  $|V_h| > |V_p \cap (R_x \setminus \{x\})|$ . Let  $b = 3q$  and  $|V_h| = 3w + r$ , where  $q, w, r$  are three integers such that  $3 \leq q$ ,  $1 \leq w$  and  $r \in \{0, 1, 2\}$ . We have by assumption  $3 < |V_h| = 3w + r < a < b = 3q$ , so setting:

$$V'_h = \{x_{n-3w-r-1}, x_{n-3w-r}, \dots, x_{n-3}, z\},$$

$$V'_p = \{x_t, x_{t+1}, \dots, x_a, \dots, x_{a+2-r}\} \cup \{u, v\},$$

where  $|V'_p| = |V_p| = a_p$ , we can cover the remaining vertices of  $R_x$  by  $q - w - 1 \geq 0$  sets of cardinality 3 and the existence of a  $T$ -realization  $(V'_1, \dots, V'_k)$  of  $\tau$  is obvious.

*Case 2.*  $\tau = (3, 3, \dots, 3, |V_p|)$ . Because  $a - 1 \equiv 0 \pmod{3}$  and  $|V_p| > 3$ , we can place the set of cardinality  $|V_p|$  at the end of the path  $x_1, x_2, \dots, x_{n-3}, z$  and easily construct a realization of  $\tau$  in  $T$ . ■

**Theorem 10.** *The number of avd caterpillars with one double and one single leg is infinite.*

**Proof.** Take  $a$  such that  $b = a + 2 = 3p$ , where  $p$  is a prime number greater than five. Therefore,  $a \equiv 1 \pmod{6}$ ,  $(b, a - 3) = 1$ ,  $n = 2a + 3$ ,  $n - 1 = 2a + 2$  and it is easy to check that the assumptions 1'-5' of Corollary 7 hold. By Proposition 9 our caterpillar is avd. ■

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