

13th WORKSHOP  
'3in1' GRAPHS 2004  
Krynica, November 11-13, 2004



International  
Visegrad Fund

## CHVÁTAL-ERDŐS CONDITION AND PANCYCLISM

EVELYNE FLANDRIN,<sup>\*</sup> HAO LI,<sup>\*</sup> ANTONI MARCZYK<sup>†</sup>

INGO SCHIERMEYER<sup>‡</sup> AND MARIUSZ WOŹNIAK<sup>†</sup>

<sup>\*</sup>*LRI, UMR 8623, Bât. 490, Université de Paris-Sud  
91405 Orsay, France*

<sup>†</sup>*AGH University of Science and Technology  
Faculty of Applied Mathematics  
Al. Mickiewicza 30, 30-059 Kraków, Poland*

<sup>‡</sup>*Fakultät für Mathematik und Informatik  
Technische Universität Bergakademie Freiberg  
D-09596 Freiberg, Germany*

### Abstract

The well-known Chvátal-Erdős theorem states that if the stability number  $\alpha$  of a graph  $G$  is not greater than its connectivity then  $G$  is hamiltonian. In 1974 Erdős showed that if, additionally, the order of the graph is sufficiently large with respect to  $\alpha$ , then  $G$  is pancyclic. His proof is based on the properties of cycle-complete graph Ramsey numbers. In this paper we show that a similar result can be easily proved by applying only classical Ramsey numbers.

**Keywords:** hamiltonian graphs, pancyclic graphs, cycles, connectivity, stability number.

**2000 Mathematics Subject Classification:** 05C38, 05C45.

## 1. Introduction

We use Bondy and Murty's book [5] for terminology and notation not defined here and consider finite, undirected and simple graphs only. For a graph  $G$  we denote by  $V = V(G)$  its vertex-set and by  $E = E(G)$  its set of edges. The symbols  $\alpha = \alpha(G)$  and  $\kappa = \kappa(G)$  stand for the stability number and the connectivity of  $G$ . By  $C_p$  we denote a  $p$ -cycle of  $G$ , i.e., a cycle of length  $p$ . The order of  $G$  will be denoted by  $n$ . A graph of order  $n$  is said to be pancyclic if it contains cycles of every length  $p$  with  $3 \leq p \leq n$ .

In 1971 Bondy [2] suggested the famous "metaconjecture":  
*almost all nontrivial sufficient conditions for a graph to be hamiltonian also imply that it is pancyclic except for maybe a simple family of exceptional graphs.*

There are various conditions for hamiltonicity that were examined in light of this conjecture, see [16]. Recall now the well-known Chvátal-Erdős [8] theorem.

**Theorem 1.** *Every  $k$ -connected graph on  $n \geq 3$  vertices with stability number  $\alpha \leq k$  is hamiltonian.*

There is a large family of triangle-free graphs (see for example the survey [7]) that satisfy the Chvátal-Erdős condition ( $\alpha(G) \leq \kappa(G)$ ), thus they are not pancyclic. This family contains the complete bipartite graphs as well as the Andrásfai graphs  $G_i = \overline{C_{3i+2}^i}$ ,  $i \geq 1$ , i.e., each  $G_i$  is the complement of the  $i$ -th power of the cycle  $C_{3i+2}$ . For example  $G_1 = C_5$  and  $G_8$  is a cycle on 8 vertices with the longest chords. The lexicographic product  $G_i[\vec{K}_s]$  ( $s \geq 1$ ) is a triangle-free  $r = s(i+1)$ -regular graph with stability number  $\alpha = r$ , connectivity  $r$  and order  $3\alpha - s \leq 3\alpha - 1$ , so it also satisfies the Chvátal-Erdős condition and is not pancyclic.

There are several articles that investigate the set of cycle lengths in graphs satisfying this condition. We cite below some results of importance for us. Amar, Fournier and Germa [1]) proved the following.

**Theorem 2** (Amar, Fournier and Germa [1]). *Let  $G$  be a  $k$ -connected graph of stability  $\alpha \leq k$  and of order  $n$ . If  $G \neq K_{k,k}$  and  $G \neq C_5$ , then  $G$  has a  $C_{n-1}$ .*

The next result due to Lou [13] was conjectured by Amar, Germa and Fournier [1].

**Theorem 3.** *If a triangle-free graph  $G$  satisfies  $\alpha(G) \leq \kappa(G)$ , then  $G$  has cycles of all length between four and the order of  $G$ , unless  $G = K_{r,r}$  or  $G = C_5$ .*

But if  $\alpha(G) < \kappa(G)$  then  $G$  has to contain a  $C_3$ . Taking into account this observation Jackson and Ordaz [12] formulated the following conjecture.

**Conjecture 1.** Let  $G$  be a  $k$ -connected graph with stability number  $\alpha$ . If  $\alpha < k$ , then  $G$  is pancyclic.

There are few results about this conjecture. By results due to Amar, Fournier and Germa [1] and Chakroun, Sotteau [9] the conjecture is valid for every graph  $G$  with  $\alpha(G) \leq 3$  while Marczyk and Saclé [14] proved it for any graph  $G$  satisfying  $\alpha(G) \leq 4$ .

The most beautiful result related to both Bondy's "metaconjecture" and the Jackson-Ordaz conjecture is due to Erdős [11]. Applying the properties of cycle-complete graph Ramsey numbers [4] he proved the following result which had been conjectured by Zarins.

**Theorem 4.** *Every hamiltonian graph with the stability number less than  $p$  and the order greater than  $4p^4$  is pancyclic.*

Note that the Erdős' proof is not complete and has a small gap which is filled in Section 2.

From the last result and the Chvátal-Erdős theorem we get at once the following corollary.

**Corollary 1.** *If the stability number  $\alpha$  of a graph  $G$  does not exceed its connectivity and the order of  $G$  is greater than  $4(\alpha+1)^4$ , then  $G$  is pancyclic.*

The purpose of this paper is to present a simple proof of a similar result which use only the classical Ramsey numbers  $R(l, m)$ , i.e., to show the pancyclicity of every graph satisfying the Chvátal-Erdős condition and having sufficiently large order in relation to  $\alpha$ . Our proof is quite different and simpler than that of Erdős though our bound is not as good as that of Corollary 1. Let us recall the simplest version of the Ramsey theorem [15].

**Theorem 5.** *For every pair  $l, m \geq 2$  of integers there exists an integer  $r(l, m)$  such that each graph of order  $n \geq r(l, m)$  contains a clique on  $l$  vertices or a stable set of cardinality  $m$ .*

The Ramsey number  $R(l, m)$  is defined to be the smallest number  $r(l, m)$  with this property. Our main result reads as follows:

**Theorem A.** *Let  $G$  be  $k$ -connected graph with stability number  $\alpha$ . If  $\alpha \leq k$  and the order of  $G$  is at least  $2R(4\alpha, \alpha + 1)$ , then  $G$  is pancyclic.*

In the last theorem the order of the graph  $G$  satisfying the hypothesis is very large and our Theorem is weaker than Corollary 1, however we feel that the bound  $2R(4\alpha, \alpha + 1)$  can be considerably lowered (see Section 2). The proof of Theorem A is given in Section 4.

## 2. Some Remarks on the Theorem by Erdős

It is surprising, but true, that the beautiful theorem by Erdős (Theorem 4) was forgotten for a long period. For example, it was not mentioned in the survey [12]. Consequently, we obtained our main result of the present paper without any knowledge of this theorem.

In his proof Erdős used the Ramsey number  $R(C_m, K_p)$  i.e., the smallest number such that each graph of order  $n \geq R(C_m, K_p)$  contains a cycle of length  $m$  or a stable set of cardinality  $p$ . A theorem of Bondy and Erdős [4] states that  $R(C_m, K_p) = (m - 1)(p - 1) + 1$  for  $m \geq p^2 - 2$ . Thus, if  $p^2 - 2 \leq m \leq \frac{n}{p}$  then  $n \geq (m - 1)(p - 1) + 1$  and any graph of order  $n$  contains a cycle  $C_m$  for  $p^2 - 2 \leq m \leq \frac{n}{p}$  (provided  $p^2 - 2 \leq \frac{n}{p}$ ). For  $\frac{n}{p} < m < n$  Erdős gave an original proof. However, he forgot to write down the case  $3 \leq m \leq p^2 - 3$ . The existence of  $C_m$  belonging to this interval follows easily from another result of Bondy and Erdős published in the same paper:  $R(C_m, K_p) \leq mp^2$  for all  $m$  and  $p$ . Indeed, if  $m \leq p^2 - 3$ , then  $mp^2 \leq p^4 - 3p^2 < 4p^4 < n$ , so if the stability number is at most  $p - 1$ , a  $C_m$  exists in  $G$ .

In his paper Erdős conjectured that the same conclusion holds if we replace the bound  $4p^4$  by  $Cp^2$ , where  $C$  is a constant (sufficiently large). He also wrote that a simple example shows that it certainly fails for  $n < \frac{p^2}{4}$ , but did not present it in the article. Consider now another example. Take  $p - 1$  disjoint copies  $A_1, \dots, A_{p-1}$  of the complete graph  $K_{2p-4}$ , where  $p \geq 3$ . Choose two vertices  $x_i, y_i$  in each copy  $A_i$  and add  $p - 1$  independent edges  $x_i y_{i+1}$  (indices are taken modulo  $p - 1$ ). It can be easily seen that the stability number of this hamiltonian graph is  $p - 1$  and there exist cycles

$C_m$  for every  $m$  except  $m = 2p - 3$ , therefore, we cannot lower the bound of the theorem of Erdős below the number  $(p - 1)(2p - 4) = 2p^2 - 6p + 4$ .

However, for graphs satisfying the Chvátal-Erdős condition perhaps the following is true: there exist two constants  $c$  and  $C$ ,  $c < 2$ , such that every graph  $G$  with  $\alpha(G) = \alpha = \kappa(G)$  and  $|V(G)| > C\alpha^c$  is pancyclic ([6, 10]). The graphs  $G_i[\vec{K}_s]$  show that such the constant  $c$  must be at least one.

### 3. Notation

Let  $C$  be a cycle of  $G$  and  $a$  a vertex of  $C$ . We shall denote by  $\vec{C}$  the cycle  $C$  with a given orientation, by  $a^+ = a^{+1}$  the successor of  $a$  on  $\vec{C}$  and by  $a^- = a^{-1}$  its predecessor. We write  $a^{++}$  for  $(a^+)^+$ ,  $a^{+k}$  for  $(a^{+(k-1)})^+$  and  $a^{-k}$  for  $(a^{-(k-1)})^-$ .

Let  $a$  and  $b$  be two vertices of  $C$ . By  $a \vec{C} b$  we denote the set of consecutive vertices of  $C$  from  $a$  to  $b$  ( $a$  and  $b$  included) in the direction specified by the orientation of  $C$ . It will be called the *segment of  $\vec{C}$  from  $a$  to  $b$* . The orientation of  $C$  defines the natural relation of order in  $a \vec{C} b$  (denoted by  $\prec$ ). When  $a = b$  the symbol  $a \vec{C} b$  means the one-vertex subset  $\{a\}$  of  $V(C)$ .

Throughout the paper the indices of a cycle  $C = x_1, x_2, \dots, x_p$  are to be taken modulo  $p$ .

Suppose  $A$  is a subset of  $V(G - C)$ . The symbol  $N_C(A)$  stands for the set  $\{y \in V(C) \mid \text{there is a vertex } x \in A \text{ such that } xy \in E(G)\}$ . We write  $N_C(x)$  for  $A = \{x\}$  and we denote by  $d_C(x)$  the number  $|N_C(x)|$ .

Let  $P = x_1, x_2, \dots, x_r$  be an oriented path in  $G$ . The symbol  $\overleftarrow{P}$  stands for the path obtained by reversing the order of  $P$ , i.e.,  $\overleftarrow{P} = x_r, x_{r-1}, \dots, x_1$ . Consider another path  $Q = y_1, y_2, \dots, y_p$  of  $G$  such that  $y_1 = x_r$ . If  $Q$  is vertex-disjoint (except for  $x_r$ ) from  $P$ , then by  $P, Q$  we mean the path  $x_1, x_2, \dots, x_r, y_2, \dots, y_p$ .

### 4. Proof of Theorem A

Suppose that  $G$  is a  $k$ -connected graph of stability number  $\alpha \leq k$  such that  $n \geq 2R(4\alpha, \alpha + 1)$ , where  $n$  is the order of  $G$ . Obviously we may assume  $\alpha \geq 2$ ,  $n \geq 2 \cdot 28 = 56$  and, since  $n > 2\alpha$ ,  $G$  is not bipartite.

1. First we shall show that  $G$  contains a  $C_p$  for each  $p > \frac{n}{2} - 2$ . Observe that, by Chvátal-Erdős theorem and Theorem 2, this statement is evident

for  $p = n$  and  $p = n - 1$ . Suppose  $G$  contains a cycle  $C_p$  with  $p > \frac{n}{2}$ . We shall prove that it contains also a  $C_{p-2}$ . Indeed, since  $p > \frac{n}{2} \geq R(4\alpha, \alpha + 1)$ , and the graph  $\langle C_p \rangle$  induced by  $C_p$  has no stable set of cardinality  $\alpha + 1$ , it follows from Ramsey theorem that it contains a clique, say  $K$ , having  $4\alpha$  vertices. Let  $\vec{C}_p$  denote the cycle  $C_p$  with a given orientation and let  $x_1, x_2, \dots, x_{4\alpha}$  be the vertices of  $K$  appearing on  $\vec{C}_p$  in order of their indices. Clearly, for every  $l = 1, 2, \dots, 2\alpha$  (indices are taken modulo  $4\alpha$ ) the vertices  $x_{2l}$  and  $x_{2l+2}$  are separated by at least one vertex on  $C_p$ . Consider now the set  $S = \{x_2^{++}, x_4^{++}, x_6^{++}, \dots, x_{2l}^{++}, \dots, x_{4\alpha}^{++}\}$  of  $2\alpha > \alpha + 1$  vertices. Since the stability number of  $\langle C_p \rangle$  is at most  $\alpha$ , there is in  $\langle C_p \rangle$  an edge of the form  $x_{2i}^{++}x_{2j}^{++}$  ( $i \neq j$ ). Therefore, the following cycle  $x_{2i}^{++}, x_{2i}^{+3}, \dots, x_{2j}, x_{2i}, x_{2i}^-, \dots, x_{2j}^{+3}, x_{2j}^{++}, x_{2i}^{++}$  (we allow that the paths  $x_{2i}^{++}, x_{2i}^{+3}, \dots, x_{2j}$  and  $x_{2i}, x_{2i}^-, \dots, x_{2j}^{+3}, x_{2j}^{++}$  are trivial) has  $p-2$  vertices and our claim is proved.

Now, using the fact that  $C_n$  and  $C_{n-1}$  exist, it is a simple matter to prove recursively that  $G$  contains a  $C_p$  for  $p > \frac{n}{2} - 2$ .

2. Now we shall show that  $G$  contains a cycle on  $p$  vertices for every  $p$  such that  $3 \leq p \leq \frac{n}{2} - 2$ . It is evident for  $3 \leq p \leq 4\alpha$  because  $n > R(4\alpha, \alpha + 1)$  and  $G$  has no stable set of cardinality  $\alpha + 1$ , so it follows from Ramsey's theorem that it contains a clique on  $4\alpha$  vertices.

Suppose  $G$  has a  $C_p$  for some  $p$  satisfying  $p \leq \frac{n}{2} - 4\alpha$ . We claim that it contains also a cycle on  $p + 4\alpha - 2$  vertices. Indeed, the order of the graph  $G - C_p$  is equal to  $n - p > n/2$ . By Ramsey's theorem it contains a clique, say  $K$ , on  $4\alpha$  vertices. Since  $\alpha \leq k$ , it follows by Menger's theorem that we can choose  $r = \min(\alpha, p)$  vertex-disjoint paths, say  $P_1, P_2, \dots, P_r$ , that join  $C_p$  with  $K$ . Denote by  $x_i \in V(C_p)$  and  $y_i \in V(K)$  the end-vertices of  $P_i$  ( $i = 1, 2, \dots, r$ ). The vertex  $x_i$  will be called starting vertex of  $P_i$  ( $i = 1, \dots, r$ ). Since the stability number of  $G$  is equal to  $\alpha$  we may assume that the length of every path  $P_i$  is less than or equal to  $2\alpha - 1$ . Let  $\vec{C}_p$  denote the cycle  $C_p$  with a given orientation and suppose there is some  $i$  such that  $x_i$  and  $x_{i+1}$  are consecutive on  $\vec{C}_p$ . Then the length of the cycle  $x_i^-, x_i, P_i, Q_i, P_{i+1}^-, x_{i+1}, x_{i+1}^+, \dots, x_i^-$  is  $p + 4\alpha - 2$ , where  $Q_i$  is a path from  $y_i$  to  $y_{i+1}$  in  $K$  of  $4\alpha - |V(P_i)| - |V(P_{i+1})| + 2 \geq 2$  vertices. Suppose then that any two vertices  $x_i$  and  $x_{i+1}$  are separated by at least one vertex on  $C_p$ . Hence  $r = \alpha$  and the set  $\{x_1^+, x_2^+, \dots, x_\alpha^+\}$  has  $\alpha$  elements. If there are two indices, say  $i$  and  $j$ , such that  $x_i^+x_j^+$  belongs to  $E(G)$ , then the

cycle  $x_i^-, x_i, P_i, Q_{ij}, P_j, x_j, x_j^-, \dots, x_i^+, x_j^+, x_j^{++}, \dots, x_i^-$ , where  $Q_{ij}$  is a  $y_i - y_j$  path of  $4\alpha - |V(P_i)| - |V(P_j)| + 2 \geq 2$  vertices which is contained in  $K$  (see Figure 1). Obviously, the length of this cycle is  $p + 4\alpha - 2$ .

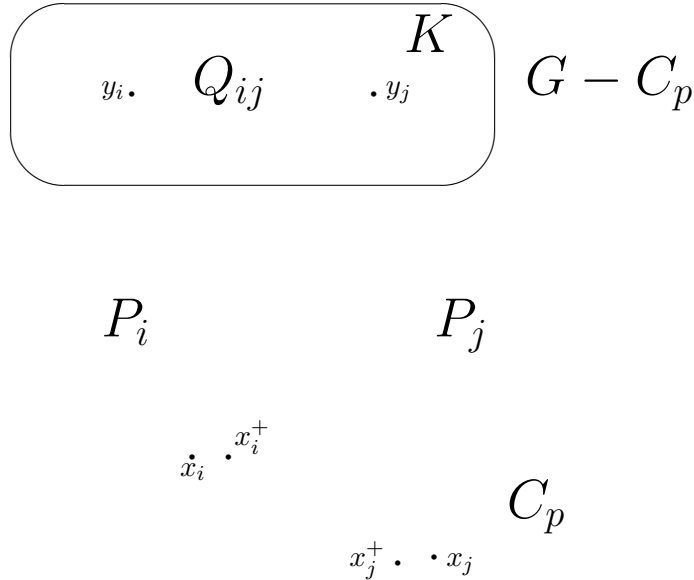


Figure 1

Thus suppose the vertices  $x_1^+, x_2^+, \dots, x_\alpha^+$  are independent and let  $u$  be the second vertex on  $P_1$  (starting at  $x_1$ ). If  $ux_i^+ \in E(G)$  for some  $i$ ,  $2 \leq i \leq \alpha$ , then the starting vertices of  $P_i$  and of the path obtained by replacing in  $P_1$  the vertex  $x_1$  by  $x_i^+$  and the edge  $ux_1$  by  $ux_i^+$  are consecutive on  $C$ . So we can construct a cycle of length  $p + 4\alpha - 2$  as above. So assume  $ux_i^+ \notin E(G)$  for  $i = 2, \dots, \alpha$ . Thus, because the stability number of  $G$  is  $\alpha$ ,  $ux_1^+ \in E(G)$  and we can replace the path  $P_1$  by another one starting at  $x_1^+$ . Repeating this reasoning (if necessary) we obtain  $ux_2^- \in E(G)$ . Therefore, there are two disjoint paths whose starting vertices are consecutive on the cycle and we can construct a  $C_{p+4\alpha-2}$ . So our claim is proved.

Because  $G$  contains a cycle  $C_p$  for every  $p$  between 3 and  $4\alpha$ , the existence of a cycle of length  $p$  for  $3 \leq p \leq n/2 - 2$  follows by induction from our now-proved claim. This completes the proof of the theorem.

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Received 17 February 2005

Revised 21 November 2005