ON A PERFECT PROBLEM

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Abstract

We solve Open Problem (xvi) from Perfect Problems of Chvátal [1] available at ftp://dimacs.rutgers.edu/pub/perfect/problems.tex:

(a) \( \mathcal{C} \) does not include all perfect graphs and
(b) every perfect graph contains a vertex whose neighbors induce a subgraph that belongs to \( \mathcal{C} \).

A class \( \mathcal{P} \) is called locally reducible if there exists a proper subclass \( \mathcal{C} \) of \( \mathcal{P} \) such that every graph in \( \mathcal{P} \) contains a local subgraph belonging to \( \mathcal{C} \). We characterize locally reducible hereditary classes. It implies that there are infinitely many solutions to Open Problem (xvi). However, it is impossible to find a hereditary class \( \mathcal{C} \) of perfect graphs satisfying both (a) and (b).

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1. Locally Reducible Classes

A class of graphs \( \mathcal{P} \) is hereditary if \( H \in \mathcal{P} \) for each induced subgraph \( H \) of every graph \( G \in \mathcal{P} \). As usual, \( N(u) = N_G(u) \) is the neighborhood of a vertex \( u \) in a graph \( G \). A local subgraph in a graph \( G \) is a subgraph induced by \( N(u) \), where \( u \) is a vertex of \( G \). If \( u \) is an isolated vertex [i.e., \( N(u) = \emptyset \)], then the corresponding local subgraph is \( K_0 \), the vertexless graph. Let \( \mathcal{P} \) be a hereditary class of graphs. If there is a proper subclass \( \mathcal{C} \) of \( \mathcal{P} \) such
that every graph in $\mathcal{P}$ with at least one vertex contains a local subgraph belonging to $\mathcal{C}$, then $\mathcal{P}$ is called a locally reducible class.

**Problem 1.** Characterize locally reducible hereditary classes.

Not all hereditary classes are locally reducible. For example, let us consider the class $\mathcal{K} = \{K_n : n \geq 0\}$, of all complete graphs. Let $\mathcal{C}$ be an arbitrary proper subclass of $\mathcal{K}$. Since $\mathcal{C} \neq \mathcal{K}$, there exists $m$ such that $K_m \notin \mathcal{C}$. The graph $K_{m+1}$ belongs to $\mathcal{K}$. However, all local subgraphs in $K_{m+1}$ are $K_m$, and therefore they are not in $\mathcal{C}$. By definition, $\mathcal{K}$ is not locally reducible.

**Theorem 1.** A non-empty hereditary class $\mathcal{P}$ is locally reducible if and only if $\mathcal{P} \neq \mathcal{K}$.

**Proof.** Necessity was shown above.

**Sufficiency.** As usual, the star $K_{1,n}$ has $n + 1$ vertices $v_0, v_1, \ldots, v_n$ and $n$ edges $v_0v_1, v_0v_2, \ldots, v_0v_n$, the vertex $v_0$ being the center of the star.

**Claim 1.** For a fixed $n \geq 2$, there is no graph $G$ such that the neighborhood of each vertex of $G$ induces $K_{1,n}$.

**Proof.** Suppose that there exists a graph $G$ such that the neighborhood of each vertex induces $K_{1,n}$. We consider an arbitrary vertex $u$ of $G$. Its neighborhood induces the subgraph $H$ isomorphic to $K_{1,n}$. We denote $V(H) = \{v_0, v_1, \ldots, v_n\}$, where $v_0$ is the center, see Figure 1.

![Figure 1. An illustration](image_url)
The set $N_G(v_0) = \{u, v_1, v_2, \ldots, v_n\}$ induces $K_{1,n}$ centered at $u$. The vertex $v_1$ is adjacent to both $u$ and $v_0$, and $v_1$ is non-adjacent to all the vertices $v_2, v_3, \ldots, v_n$. It follows that $\{u, v_0\}$ is a connected component of the induced subgraph $G(N(v_1))$. Since $n \geq 2$, $N(v_1)$ cannot induce $K_{1,n}$, a contradiction.

First suppose that the path $P_3$ belongs to $\mathcal{P}$. Then $C = \mathcal{P} \setminus \{P_3\}$ is a proper subclass of $\mathcal{P}$. We consider an arbitrary graph $G \in \mathcal{P}$. Claim 1 implies that there exists a vertex $x \in V(G)$ such that $N_G(x)$ does not induce $P_3 \cong K_{1,2}$. By the definition of $C$, $G(N(x)) \in C$, as required.

It remains to consider the case, where $P_3 \notin \mathcal{P}$. Since $P_3$ is a forbidden induced subgraph, each graph $G \in \mathcal{P}$ is a disjoint union of complete subgraphs. Clearly, all local subgraphs of $G$ are complete graphs.

Suppose that $\mathcal{P}$ contains $O_2$, the graph with two non-adjacent vertices. Clearly, we can define $C = \mathcal{P} \setminus \{O_2\}$. If $\mathcal{P}$ does not contain $O_2$, then $\mathcal{P}$ consists of complete graphs only. According to the condition, $\mathcal{P} \neq \mathcal{K}$, i.e., there exists $m$ such that $K_m \notin \mathcal{P}$. Note that the class $\mathcal{P}$ is not empty implying that $m \geq 1$. We may assume that $K_{m-1} \in \mathcal{P}$. Since $\mathcal{P}$ is a hereditary class, $\mathcal{P} = \{K_0, K_1, \ldots, K_{m-1}\}$. We may set $C = \mathcal{P} \setminus \{K_{m-1}\}$, thus completing the proof.

Recall that a graph $G$ is called perfect if $\omega(H) = \chi(H)$ for each induced subgraph $H$ of $G$, where $\omega(H)$ is the clique number of $H$ – the size of the largest complete subgraph in $H$, and $\chi(H)$ is the chromatic number of $H$ – the minimum number of colors in proper vertex colorings of $H$; see \cite{3}. If $\mathcal{P} = \mathcal{PERF}$ is the class of all perfect graphs, Problem 1 coincides with Open Problem (xvi) in Chvátal's list \cite{1}. Theorem 1 gives a solution to this problem. Since all stars are perfect graphs, Claim 1 implies a more general fact.

**Corollary 1.** There are infinitely many proper subclasses $C$ of $\mathcal{PERF}$ such that every perfect graph contains a local subgraph belonging to $C$.

**Proof.** We define $C_n = \mathcal{PERF} \setminus \{K_{1,n}\}$ for each $n \geq 2$ and apply Claim 1.

A Zykov graph $H$ is defined by the property that there exists a graph $G$ such that neighborhood of each vertex $u \in V(G)$ induces $H$, see the Neighborhood Problem in Zykov \cite{4}. In our proof we used the fact that all stars $K_{1,n}$ with $n \geq 2$ are not Zykov graphs.
Corollary 2. Let $\mathcal{P}$ be a class of graphs closed under taking local subgraphs. If $\mathcal{P}$ contains a graph $H$ which is not a Zykov graph, then $\mathcal{P}$ is locally reducible.

Proof. We define $\mathcal{C} = \mathcal{P} \setminus \{H\}$. Since $H$ is not a Zykov graph, an arbitrary graph $G \in \mathcal{P}$ has a local subgraph $L \not\sim H$. According to the condition, $L \in \mathcal{P}$. Thus, $L \in \mathcal{P} \setminus \{H\} = \mathcal{C}$.

2. Hereditary Subclasses

Now we consider a more complicated problem. A hereditary class $\mathcal{P}$ of graphs is called locally $h$-reducible if there exists a proper hereditary subclass $\mathcal{C}$ of $\mathcal{P}$ such that every graph in $\mathcal{P}$ with at least one vertex contains a local subgraph belonging to $\mathcal{C}$.

Problem 2. Characterize locally $h$-reducible hereditary classes.

Join of graphs $G$ and $H$, denoted by $G + H$, is obtained from vertex-disjoint copies of $G$ and $H$ by adding all edges between $V(G)$ and $V(H)$. A class $\mathcal{P}$ of graphs is called join-closed if $G + H \in \mathcal{P}$ whenever $G, H \in \mathcal{P}$.

Claim 2. Each join-closed hereditary class $\mathcal{P}$ having a graph $H$ with at least one vertex is not locally $h$-reducible.

Proof. Suppose that $\mathcal{P}$ is a locally $h$-reducible class, i.e., there exists a proper hereditary subclass $\mathcal{C}$ of $\mathcal{P}$ such that every graph in $\mathcal{P}$ with at least one vertex contains a local subgraph belonging to $\mathcal{C}$. There exists a graph $H \in \mathcal{P} \setminus \mathcal{C}$. Since the class $\mathcal{C}$ is hereditary, each graph in $\mathcal{C}$ is $H$-free. We consider the graph $G = H + H \in \mathcal{P}$. We see that each local subgraph $L$ in $G$ contains $H$ as an induced subgraph. It implies that $L \not\subseteq \mathcal{C}$, a contradiction to the assumption that $\mathcal{P}$ is a locally $h$-reducible class.

Claim 2 shows that the class $\mathcal{P}^{ERF}$ is not locally $h$-reducible. Indeed, join of perfect graphs $G$ and $H$ always produces a perfect graph: $\omega(G + H) = \omega(G) + \omega(H)$ and $\chi(G + H) = \chi(G) + \chi(H)$. Thus, it is impossible to strengthen Corollary 1 requiring that $\mathcal{C}$ is a hereditary class.

A graph is chordal if it does not contain the cycles $C_n$ with $n \geq 4$ as induced subgraphs. Claim 2 does not hold for the class $\mathcal{P} = \text{CHORD}$ of all chordal graphs. Indeed, according to Dirac [2] each chordal graph
$G \neq K_0$ has a simplicial vertex — a vertex whose neighborhood induces a complete subgraph. It shows that we can choose $\mathcal{C} = \mathcal{K}$ as a hereditary proper subclass of all chordal graphs. The reason is that the class CHORD is not join-closed: $C_4 = O_2 + O_2$ is not a chordal graph, while $O_2$ is. Thus, Problem 2 remains open for all hereditary classes which are not join-closed.

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References


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