

## SELF-COMPLEMENTARY HYPERGRAPHS

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### Abstract

A  $k$ -uniform hypergraph  $H = (V; E)$  is called *self-complementary* if there is a permutation  $\sigma : V \rightarrow V$ , called *self-complementing*, such that for every  $k$ -subset  $e$  of  $V$ ,  $e \in E$  if and only if  $\sigma(e) \notin E$ . In other words,  $H$  is isomorphic with  $H' = (V; \binom{V}{k} - E)$ .

In the present paper, for every  $k$ , ( $1 \leq k \leq n$ ), we give a characterization of self-complementing permutations of  $k$ -uniform self-complementary hypergraphs of the order  $n$ . This characterization implies the well known results for self-complementing permutations of graphs, given independently in the years 1962–1963 by Sachs and Ringel, and those obtained for 3-uniform hypergraphs by Kocay, for 4-uniform hypergraphs by Szymański, and for general (not uniform) hypergraphs by Zwonek.

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## 1. Introduction

Let  $V$  be a finite set with  $n$  vertices. By  $\binom{V}{k}$  we denote the set of all  $k$ -subsets of  $V$ . A  $k$ -uniform hypergraph is a pair  $H = (V; E)$ , where  $E \subset \binom{V}{k}$ .  $V$  is called a *vertex set*, and  $E$  an *edge set* of  $H$ . Two  $k$ -uniform

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hypergraphs  $H = (V; E)$  and  $H' = (V'; E')$  are *isomorphic* if there is a bijection  $\tau : V \rightarrow V'$  such that  $\tau$  induces a bijection of  $E$  onto  $E'$ . If  $H = (V; E)$  is isomorphic to  $H' = (V; \binom{V}{k} - E)$ , then  $H$  is called a *self-complementary  $k$ -uniform hypergraph*. Every permutation  $\sigma : V \rightarrow V$  which induces a bijection  $\sigma^* : E \rightarrow \binom{V}{k} - E$  is called a *self-complementing permutation*. It is clear that if  $H$  is a self-complementary  $k$ -uniform hypergraph and  $\sigma$  is its self-complementing permutation, then every odd power  $\sigma^{2p+1}$  is a self-complementing permutation of  $H$ , while every even power  $\sigma^{2p}$  is an automorphism of  $H$ . The self-complementary 2-uniform hypergraphs, so just self-complementary graphs, have been introduced and studied first by Ringel [3] and Sachs [4]. In particular, Ringel and Sachs have independently characterised the self-complementing permutations of self-complementary graphs. The self-complementing permutations of 3-uniform self-complementary hypergraphs have been characterised by Kocay in [2], and the self-complementing permutations of 4-uniform self-complementary hypergraphs are described in [5] by Szymański. All these results follow from Theorem 2 and we give them in the last section of the paper.

In [5] Szymański proposed to study the complementation of general hypergraphs. Since there is exactly one subset of  $V$  which has  $|V|$  elements, and exactly one subset of  $V$  with 0 elements, we define a general self-complementary hypergraph as follows.

A hypergraph  $H = (V; E)$  is said to be a *general self-complementary hypergraph* if  $\emptyset, V \notin E$  and  $H$  is isomorphic with  $H' = (V; 2^V - E - \{\emptyset\} - \{V\})$ . A corresponding permutation  $\sigma : V \rightarrow V$  is called a *general self-complementing permutation*. The general self-complementing permutations have been characterised in [6] by Zwonek. Also this result follows from Theorem 2 and we give it in Section 3.

## 2. Results

**Proposition 1.** *Let  $H = (V; E)$  be a  $k$ -uniform self-complementary hypergraph, and let  $\sigma$  be its self-complementing permutation. Then  $H' = (V; E')$ , where  $E' = \{V - e : e \in E\}$  is an  $(n - k)$ -uniform self-complementary hypergraph and  $\sigma$  is its self-complementing permutation. ■*

**Theorem 2.** *Let  $k, m$  and  $n$  be natural numbers and let  $\sigma : V \rightarrow V$  be a permutation of a set  $V$ ,  $|V| = n$ , with the orbits  $O_1, \dots, O_m$ . Let  $2^{q_i}(2s_i + 1)$  denote the cardinality of the orbit  $O_i$ ,  $i = 1, \dots, m$ .  $\sigma$  is a self-complementing*

permutation of a self-complementary  $k$ -uniform hypergraph, if and only if, for every  $l \in \{1, \dots, k\}$  and for every decomposition

$$k = k_1 + k_2 + \dots + k_l$$

of  $k$ , where  $k_j = 2^{p_j}(2r_j + 1)$ ;  $p_j, r_j$  are non negative integers, and for every subsequence of orbits

$$O_{i_1}, O_{i_2}, \dots, O_{i_l}$$

such that  $k_j \leq |O_{i_j}|$  for  $j = 1, 2, \dots, l$ , there is a subscript  $j_0 \in \{1, \dots, l\}$  such that  $p_{j_0} < q_{i_{j_0}}$ .

Theorem 2 follows easily from the lemmas given below. For a hypergraph  $H = (V; E)$  and a subset  $A$  of  $V$ , we denote by  $H\langle A \rangle$  the subhypergraph induced in  $H$  by  $A$ , i.e., the hypergraph  $H\langle A \rangle = (A; E \cap 2^A)$ .

**Lemma 3.** *Let  $k$  and  $l$  be positive integers,  $k < l$ . If  $c = (a_1, \dots, a_l)$  is a cycle of a self-complementing permutation of a  $k$ -uniform self-complementary hypergraph  $H$ , then  $H\langle \{a_1, \dots, a_l\} \rangle$  is a  $k$ -uniform self-complementary hypergraph with self-complementing permutation  $c$ . ■*

The next lemma is also formulated in [5], its version for graphs is given in [1].

**Lemma 4.** *A permutation  $\sigma : V \rightarrow V$  is self-complementing of a  $k$ -uniform self-complementary hypergraph, if and only if, for every  $k$ -subset  $e$  of  $V$   $\sigma^{2t+1}(e) \neq e$  for every integer  $t$ .*

**Proof.** Suppose that  $\sigma : V \rightarrow V$  is a self-complementing permutation of a  $k$ -uniform self-complementary hypergraph  $H = (V; E)$ . Set  $E' = \binom{V}{k} - E$ . Note that  $H' = (V; E')$  is a  $k$ -uniform self-complementary hypergraph too, and  $\sigma$  is its self-complementing permutation. Moreover, every odd power of  $\sigma$  is a self-complementing permutation of  $H$  and of  $H'$  as well. Hence, for every integer  $t$ , we have  $\sigma^{2t+1}(e) \in E'$  for  $e \in E$  and  $\sigma^{2t+1}(e) \in E$  for  $e \in E'$ . Therefore  $\sigma^{2t+1}(e) \neq e$  for every  $e \in \binom{V}{k}$ .

Now, let us suppose that  $\sigma$  is such a permutation of the set  $V$  that for every  $k$ -subset  $e$  of  $V$ , and for every integer  $t$ ,  $\sigma^{2t+1}(e) \neq e$ . We color all the  $k$ -subsets of  $V$  with two colors: red or blue. If  $e \subset V$ ,  $|e| = k$  and  $e$  is yet uncolored, then color all the edges  $\sigma^{2t+1}(e)$  red, and all the edges  $\sigma^{2t}(e)$  blue ( $t \in \mathbf{N}$ ). Every edge of order  $k$  is colored with exactly one color.

It is clear that the hypergraph  $H_r$  on the vertex set  $V$  with the edge set consisting of all the red edges is  $k$ -uniform self-complementary hypergraph and its self-complementing permutation is  $\sigma$ . The complement of  $H_r$  in the complete  $k$ -uniform hypergraph  $(V; \binom{V}{k})$  is the hypergraph on the vertex set  $V$  with all the blue edges. ■

**Lemma 5.** *Let  $p, q, r$  and  $s$  be non negative integers, and let  $k = 2^p(2r + 1)$ ,  $n = 2^q(2s + 1)$  and  $k < n$ . The cyclic permutation  $\sigma = (1, \dots, n)$  is a self-complementing permutation of a  $k$ -uniform self-complementary hypergraph on the vertex set  $[1, n] = \{1, \dots, n\}$  if and only if  $p < q$ .*

**Proof.** Let us suppose first that  $k < n$  and  $p < q$ . Assume, to derive a contradiction, that  $\sigma = (1, \dots, n)$  is not a self-complementing permutation for any  $k$ -uniform self-complementary hypergraph with vertex set  $\{1, \dots, n\}$ . Then, by Lemma 4, there is a non negative integer  $t$  and a  $k$ -subset  $\{a_1, \dots, a_k\} \subset \{1, \dots, n\}$ , such that  $\sigma^{2t+1}(\{a_1, \dots, a_k\}) = \{a_1, \dots, a_k\}$ . Write  $\tau = \sigma^{2t+1}$ . Every orbit of  $\tau$  has the same cardinality, say  $\beta$ . So we have  $k = \beta\gamma$ , where  $\gamma$  is the number of orbits of  $\tau$  contained in  $\{a_1, \dots, a_k\}$ . We have

$$\tau^\beta = id_{[1, n]}$$

and therefore  $\sigma^{(2t+1)k} = \tau^{\beta\gamma} = id_{[1, n]}^\gamma = id_{[1, n]}$ . Hence  $(2t + 1)k$  is a multiple of  $n$  and we have  $(2t + 1) \cdot 2^p(2r + 1) = 2^q(2s + 1) \cdot \alpha$ , where  $\alpha \in \mathbf{Z}$ , a contradiction with  $p < q$ .

Now, let us suppose that  $\sigma$  is a self-complementing permutation of a  $k$ -uniform, self-complementary hypergraph  $H$ . We shall prove that  $p < q$ . Conversely, suppose that  $q \leq p$ . Observe that since  $n > k$ , we have  $n - k = 2^q(2s + 1) - 2^p(2r + 1) = 2^q(2s + 1 - 2^{p-q}(2r + 1)) > 0$  and therefore  $2s + 1 > 2^{p-q}(2r + 1)$ . Consider the set

$$\begin{aligned} e = \{ & 1, 2, \dots, 2^{p-q}(2r + 1), \\ & 2s + 2, 2s + 3, \dots, 2s + 1 + 2^{p-q}(2r + 1), \\ & \dots \\ & \alpha(2s + 1) + 1, \alpha(2s + 1) + 2, \dots, \alpha(2s + 1) + 2^{p-q}(2r + 1), \\ & \dots \\ & (2^q - 1)(2s + 1) + 1, \dots, (2^q - 1)(2s + 1) + 2^{p-q}(2r + 1) \}. \end{aligned}$$

Clearly,  $|e| = 2^q(2^{p-q}(2r + 1)) = k$  and  $\sigma^{2s+1}(e) = e$ , contradicting Lemma 4. ■

**Proof of Theorem 2.** Let  $\sigma$  be a permutation of a set  $V$  of cardinality  $n$ , and let  $O_1, \dots, O_m$  be the orbits of  $\sigma$ ,  $|O_i| = 2^{q_i}(2s_i + 1)$  for  $i = 1, \dots, m$ .

- We first suppose that  $\sigma$  satisfies the conditions of Theorem 2. Let  $O_{i_1}, \dots, O_{i_l}$  be the subsequence of orbits having non empty intersections with  $e$ , say  $e_j = O_{i_j} \cap e$  and  $|e_j| = k_j = 2^{p_j}(2r_j + 1)$ , for  $j = 1, \dots, l$ . So we have  $k_j \leq |O_{i_j}|$  and, by the assumptions of the theorem, there is a  $j_0 \in \{1, \dots, l\}$ , such that  $p_{j_0} < q_{j_0}$ . By Lemmas 3, 4 and 5, for every integer  $t$  we have  $\sigma^{2t+1}(e_{j_0}) \neq e_{j_0}$  and, in consequence,  $\sigma^{2t+1}(e) \neq e$ . Hence, by Lemma 4,  $\sigma$  is a  $k$ -uniform self-complementary hypergraph.
- Now, let us suppose that  $\sigma$  is a self-complementary permutation of a  $k$ -uniform self-complementary hypergraph. By Lemma 4,  $\sigma^{2t+1}(e) \neq e$  for every  $k$ -subset of  $V$  and for every integer  $t$ . Suppose that the conditions of the theorem are not satisfied, i.e., there is such a decomposition

$$k = k_1 + \dots, k_l$$

where  $k_j = 2^{p_j}(2r_j + 1)$  for  $j = 1, \dots, l$ , and such a subsequence of orbits  $O_{i_1}, \dots, O_{i_l}$  that  $k_{i_j} \leq |O_{i_j}|$  and  $q_{i_j} \leq p_j$  for every  $j \in \{1, \dots, l\}$ . Then, by Lemmas 5 and 4, for every  $j \in \{1, \dots, l\}$ , there is such  $e_j$ ,  $|e_j| = k_j$ , and such integer  $t_j$  that  $\sigma^{2t_j+1}(e_j) = e_j$ . Observe that  $u = \prod_{j=1}^l (2t_j + 1)$  is odd and  $\sigma^u(e) = e$ , a contradiction. ■

### 3. Corollaries

The three corollaries given below follow easily from Theorem 2 for  $k = 2, 3$  and 4, respectively. The first one has been proved independently by Ringel [3] and Sachs [4], the second by Kocay [2], and the third by Szymański [5]. The proofs are easy and similar, hence we give only a proof of Corollary 7.

**Corollary 6** ([3, 4]). *A permutation  $\sigma : [1, n] \rightarrow [1, n]$  is a self-complementing permutation of a self-complementary graph if and only if*

1.  $n \equiv 0 \pmod{4}$  or  $n \equiv 1 \pmod{4}$ ,
2. *a length of every cycle of  $\sigma$  is a multiple of 4 unless, in the odd case, one cycle of the length 1 (a fixed point).* ■

**Corollary 7** ([2]). *A permutation  $\sigma$  is a complementing permutation of a 3-uniform self-complementary hypergraph if and only if*

1. every cycle of  $\sigma$  has even length, or
2.  $\sigma$  has 1 or 2 fixed points, and the length of all the other cycles is a multiple of 4.

**Proof.** In the proof we shall use the three decompositions of 3:

- (i)  $3 = 3$ ,
- (ii)  $3 = 1 + 2$ ,
- (iii)  $3 = 1 + 1 + 1$ .

Let us assume first that  $\sigma : V \rightarrow V$  is a self-complementing permutation of a 3-uniform self-complementary hypergraph  $H$ .

By Theorem 2 and the decomposition (i), if an orbit  $O$  of  $\sigma$  has at least three elements then  $|O|$  is even. So we may assume that  $\sigma$  has a fixed point. But then, by the decomposition (ii) and Theorem 2, all the orbits with at least two elements must have cardinalities divisible by 4. It is also clear, by the decomposition (iii) and Theorem 2, that  $\sigma$  may not have more than two fixed points.

Now, suppose that  $\sigma : V \rightarrow V$  satisfies the conditions of Corollary 7.

- If every orbit of  $\sigma$  has an even cardinality then, by Theorem 2,  $\sigma$  is self-complementing, since in every decomposition (i) – (iii) of 3 at least one element is odd.
- If  $\sigma$  has a fixed point then, for every decomposition of  $3 = k_1 + \dots + k_l$ ,  $l \leq 3$ , and for any sequence of mutually different orbits  $O_1, \dots, O_l$ , such that  $k_i \leq |O_i|$  for  $i = 1, \dots, l$ , we have at least one orbit of the order divisible by 4, while all  $k_i$  are either odd or equal to 2. Moreover, if  $k_i \neq 1$  then  $|O_i|$  is a multiple of 4. Hence, by Theorem 2,  $\sigma$  is a self-complementing permutation of a 3-uniform self-complementary hypergraph. ■

**Corollary 8** ([5]). *A permutation  $\sigma$  is a self-complementing permutation of a 4-uniform self-complementary hypergraph if and only if one of the following cases is satisfied.*

1. The length of every cycle of  $\sigma$  is a multiple of 8,
2.  $\sigma$  has 1, 2 or 3 fixed points and all other cycles have length a multiple of 8,
3.  $\sigma$  has 1 cycle of length 2, and all other cycles have length a multiple of 8,

- 4.  $\sigma$  has 1 fixed point, 1 cycle of length 2, and all other cycles have length a multiple of 8,
- 5.  $\sigma$  has 1 cycle of length 3, and all other cycles have length a multiple of 8. ■

The last corollary of this section has been first proved in [6] by Zwonek.

**Corollary 9** ([6]). *Let  $H = (V; E)$  be a general self-complementary hypergraph with  $V = \{1, \dots, n\}$ ,  $n \geq 2$ . Then  $n = 2^m$  for some positive integer  $m$  and the self-complementing permutation  $\sigma$  of  $H$  is cyclic. Moreover, every cyclic permutation  $(1, 2, \dots, 2^m)$  is a self-complementing permutation of a general self-complementary hypergraph of order  $2^m$ .*

**Proof of Corollary 9.** Observe first that a hypergraph  $H = (V; E)$  of order  $n$  is a general self-complementary hypergraph, if and only if for every  $k$  such that  $1 \leq k \leq n - 1$ , the hypergraph  $H^{(k)} = (V; E \cap \binom{V}{k})$  is a  $k$ -uniform self-complementary hypergraph, and all these  $k$ -uniform self-complementary hypergraphs  $H^{(k)}$  have a common self-complementing permutation  $\sigma$ .

If such a permutation  $\sigma$  exists, then it is cyclic — otherwise there is a cycle  $\tau = (a_1, \dots, a_l)$  of  $\sigma$ , with  $l < n$ , and then  $\sigma(e) = \tau(e) = e$  for the hyperedge  $e = \{a_1, \dots, a_l\}$ , what is impossible. Hence, without loss of generality, we may suppose  $\sigma = (1, \dots, n)$ . If  $n = 2$  then — up to the isomorphism — the only self-complementary hypergraph is  $H = (\{1, 2\}; \{\{1\}\})$ . It is also clear that there is no general hypergraph of order  $n = 3$ . Hence we may suppose that  $n \geq 4$ .

Let  $l_0 = \max\{l | 2^l \leq n\}$ . We have  $1 < 2^{l_0-1} < 2^{l_0}$ . Since  $H^{(2^{l_0-1})}$  is a  $2^{l_0-1}$ -uniform self-complementary hypergraph and  $\sigma$  is its self-complementing cyclic permutation we have, by Lemma 5,  $n = 2^{l_0}$ .

Let now  $\sigma = (1, \dots, 2^m)$ . Then for every  $k$  such that  $1 \leq k \leq n - 1$ , we have  $k = 2^p(2r + 1)$  with  $p < m$ . By Lemma 5,  $\sigma$  is a self-complementing permutation of a  $k$ -uniform hypergraph  $H^{(k)}$  on the vertex set  $\{1, \dots, n\}$ . Hence  $\sigma$  is a self-complementing permutation of a general hypergraph on the vertex set  $\{1, \dots, n\}$ . ■

Observe that, for every permutation  $\sigma$  satisfying the conditions of Theorem 2, the algorithm given in the second part of the proof of Lemma 4 produces all the  $k$ -uniform self-complementary hypergraphs having  $\sigma$  as a self-complementing permutation.

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### References

- [1] A. Benhocine and A.P. Wojda, *On self-complementation*, J. Graph Theory **8** (1985) 335–341.
- [2] W. Kocay, *Reconstructing graphs as subsumed graphs of hypergraphs, and some self-complementary triple systems*, Graphs and Combinatorics **8** (1992) 259–276.
- [3] G. Ringel, *Selbstkomplementäre Graphen*, Arch. Math. **14** (1963) 354–358.
- [4] H. Sachs, *Über selbstkomplementäre Graphen*, Publ. Math. Debrecen **9** (1962) 270–288.
- [5] A. Szymański, *A note on self-complementary 4-uniform hypergraphs*, Opuscula Mathematica **25/2** (2005) 319–323.
- [6] M. Zwonek, *A note on self-complementary hypergraphs*, Opuscula Mathematica **25/2** (2005) 351–354.

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