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IN-DEGREE SEQUENCE IN A GENERAL MODEL OF A RANDOM DIGRAPH

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Abstract

A general model of a random digraph $D(n, \mathcal{P})$ is considered. Based on a precise estimate of the asymptotic behaviour of the distribution function of the binomial law, a problem of the distribution of extreme in-degrees of $D(n, \mathcal{P})$ is discussed.

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1. Introduction

We begin with a definition of a general model of a random digraph that was introduced in [6]. Let $\mathcal{P} = (P_0, \ldots, P_{n-1})$ be a probability distribution, i.e., an n-tuple of non-negative real numbers which satisfy $\sum_{i=0}^{n-1} P_i = 1$. Denote by $D(n, \mathcal{P})$ a random digraph on a vertex set $V = \{1, 2, \ldots, n\}$ such that (here, and what follows, $N^+(i)$ denotes the set of images of a vertex i):

(1) each vertex "chooses" its out-degree and then its images independently of all other vertices,

(2) each vertex $i \in V$ chooses its out-degree according to the probability distribution $\mathcal{P}$, i.e.,

$$Pr\{|N^+(i)| = k\} = P_k, \quad k = 0, 1, \ldots, n-1,$$

(3) for every $S \subseteq V \setminus \{i\}$, with $|S| = k$, the probability that S coincides with the set of images of a vertex i equals

$$Pr\{N^+(i) = S\} = P_k / \binom{n-1}{k}$$

i.e., vertex i "chooses" uniformly the set of images.

In particular, if $\mathcal{P}$ is such that $P_d = 1$ for some $d, 1 \leq d \leq n-1$, the model $D(n, \mathcal{P})$ is equivalent to a random d-out regular digraph $D(n, d)$. Such a digraph can also be defined as an element chosen at random from the family of all $\binom{n-1}{d}^n$ digraphs on n labeled vertices each of out-degree d. (Alternatively, $D(n, d)$ can be thought as a representation of a sum of d dependent random mappings as illustrated in [7].)

In a case when $\mathcal{P}$ is a binomial distribution $\mathcal{B}(n-1, p)$, i.e.,

$$\mathcal{P} = \left(q^{n-1}, \ldots, b(k; n-1, p), \ldots, p^{n-1}\right)$$

where

$$b(r; n, p) = \binom{n}{r} p^r q^{n-r}$$

the model $D(n, \mathcal{P})$ is equivalent to a random digraph $D(n, \mathcal{B})$ on n labeled vertices in which each of $n(n-1)$ possible arcs appears independently with a given probability $p = 1 - q$.

2. Preliminaries

Let X^+ be a discrete random variable having a probability distribution $\mathcal{P} = (P_0, P_1, \ldots, P_{n-1})$:

$$Pr\{X^+ = k\} = P_k, \qquad k = 0, 1, \ldots, n-1.$$

Due to the homogeneous structure of the random digraph $D(n, \mathcal{P})$, the random variable $X^+ = X^+(i)$ defines the out-degree of a given vertex $i \in V = \{1, 2, \ldots, n\}$ of $D(n, \mathcal{P})$. Then the probability that a given subset of vertices is contained in the set of images of vertex $i \in V$ can be expressed by appropriate factorial moment of X^+. As a matter of fact the following property is true (see [8]). Here and what follows $(n)_k = n(n-1)\ldots(n-k+1)$ and $E_k(X)$ stands for the k-th factorial moment of a random variable X.

Property 1. *For a given* $i, 1 \leq i \leq n$, *let* $U \subseteq V \setminus \{i\}$ *and* $|U| = t \geq 1$. *Then*

$$Pr\{U \subseteq N^+(i)\} = \frac{1}{(n-1)_t} E_t(X^+) \,.$$

In particular, if $t = 1$ the above property defines an arc occurrence probability in digraph $D(n, \mathcal{P})$. Let

$$E^+ = E^+(\mathcal{P}) = \sum_{k=0}^{n-1} k P_k \,.$$

Then the probability of an arc in $D(n, \mathcal{P})$ is given by

$$p^* = \frac{E^+(\mathcal{P})}{n-1} \,. \tag{1}$$

Now let $X^- = X^-(i)$ be the in-degree of a given vertex $i \in \{1, 2, \ldots, n\}$ of $D(n, \mathcal{P})$. Clearly, the probability distribution of X^- depends on $\mathcal{P}$. We have the following result (see [8]).

Property 2. *For* $i = 1, 2, \ldots, n$ *the random variable* $X^-(i)$ *has binomial distribution* $\mathcal{B}(n-1, p^*)$.

In contrast with out-degrees of vertices of $D(n, \mathcal{P})$, the random variables $X^-(i)$, $i = 1, 2, \ldots, n$, are not, in general, independent. The only case when these variables are independent is when X^+ is binomially distributed (see [8]).

The main aim of our paper is to study the probabilistic properties of extreme in-degrees of the random digraph $D(n, \mathcal{P})$. We show that the in-degree sequence of $D(n, \mathcal{P})$ behaves similarly to the degree sequence of the classical model of a random graph (see [11]). Our results generalize those presented in [10].

Let G_n be an arbitrary random graph model defined on n vertices. If π is a graph property then the assertion "G_n has property π asymptotically almost surely (a.a.s.)" means

$$\lim_{n \to \infty} P(G_n \text{ has property } \pi) = 1 \,.$$

The symbols o, O and $\sim$ are used with respect to $n \to \infty$.

Consider "degree" sequence $d_{(1)} \leq d_{(2)} \leq \cdots \leq d_{(n)}$ of G_n. If G_n is a simple (directed) graph then by the "degree" sequence we mean sequence of degrees (in-degrees or out-degrees) written in non-decreasing order. Denote by X_r, Y_s and Z_t the number of vertices of "degree" $= r, \leq s$ and $\geq t$ in G_n, respectively.

Let $B(s; n, p)$ denote probability of at most s successes in the binomial distribution. Similarly, let $F(t; n, p)$ denote probability of at least t successes in such distribution. In the proofs of our main results we will need a very precise etimate of the asymptotic behaviour of the distribution function of the binomial law with parameters n and p, where $p = p(n) = o(1)$ and $np/\log n \to \infty$ as $n \to \infty$ (see [5] and [12]).

Consider the equation

$$(1+z)\log(1+z) + \frac{1}{a}(1-az)\log(1-az) = u$$

where $0 \leq u < \infty$ and $a \geq 0$. It is known (see e.g. [5]) that this equation has a negative solution $z(u,a)$ and a positive solution $y(u,a)$, which in some neighbourhood of zero are given by the power series

$$z(u,a) = -\left(\frac{2u}{1+a}\right)^{\frac{1}{2}} + \sum_{i=2}^{\infty}(-1)^i f_i(a)\left(\frac{2u}{1+a}\right)^{i/2} \tag{2}$$

and

$$y(u,a) = -\left(\frac{2u}{1+a}\right)^{\frac{1}{2}} + \sum_{i=2}^{\infty} f_i(a)\left(\frac{2u}{1+a}\right)^{i/2} \tag{3}$$

in which

$$f_{i+1}(a) = \frac{(-1)^i}{i+1}\sum \frac{(-1)^k(i+1)(i+3)\dots(i+2k-1)}{k_1!\dots k_i!(2\cdot 3)^{k_1}\dots[(i+1)(i+2)]^{k_i}} \times \frac{(1-a^2)^{k_1}(1+a^3)^{k_2}\dots[1+(-1)^i a^{i+1}]^{k_i}}{(1+a)^k}$$

where $k = k_1 + k_2 + \dots k_i$ and the summation is over all non-negative integers $k_1, \dots, k_i$ such that $k_1 + 2k_2 + \dots + ik_i = i$. In particular,

$$z(u,a) = -\left(\frac{2u}{1+a}\right)^{\frac{1}{2}} + \frac{1-a}{3(1+a)}u + \frac{\sqrt{2}}{36}\frac{1+4a+a^2}{(1+a)^{3/2}}u^{3/2} + \dots \tag{4}$$

and

$$(5)\quad y(u,a) = \left(\frac{2u}{1+a}\right)^{\frac{1}{2}} + \frac{1-a}{3(1+a)}u - \frac{\sqrt{2}}{36}\frac{1+4a+a^2}{(1+a)^{3/2}}u^{3/2} + \dots$$

Now put

$$(6)\qquad u = u(n,p) = \frac{1}{np}\left(\log n - \frac{1}{2}\log\log n\right).$$

In proofs of our main results we will need the following lemma giving a very precise asymptotic behaviour of binomial distribution (see [12]).

Lemma 1. *Let* $m = np = \omega(n)\log n$ *where* $\omega(n) \to \infty$ *as* $n \to \infty$ *in such a way that* $p = p(n) = o(1)$. *Assume that* $x = x(n)$ *satisfies* $x^2 = o(\min\{\omega(n), \log n\})$, *and put*

$$s = m + mz\left(u, \frac{p}{q}\right) - \left(\frac{m}{2\log n}\right)^{1/2}(x - \log\sqrt{4\pi} + o(1))$$

$$t = m + my\left(u, \frac{p}{q}\right) + \left(\frac{m}{2\log n}\right)^{1/2}(x - \log\sqrt{4\pi} + o(1))$$

where u *is given by* (6). *Then*

$$(7)\qquad nB(s;n,p) \sim nF(t;n,p) \sim e^{-x}$$

and

$$(8)\qquad nb(s;n,p) \sim nb(t;n,p) \sim \left(\frac{2\log n}{npq}\right)^{1/2} e^{-x}.$$

■

3. Main Results

Let $X_r^- = X_r^-(\mathcal{P})$ denote the number of vertices of in-degree r in a general model of a random digraph $D(n,\mathcal{P})$. Then by Property 2 we have

Property 3. *The expected value of* X_r^- *equals*

$$E(X_r^-) = nb(r; n-1, p^*)$$

where p^* *is given by* (1).

Now let us put $Y_s^- = Y_s^-(\mathcal{P})$ and $Z_t^- = Z_t^-(\mathcal{P})$ for the number of vertices of in-degree of at most s and at least t in $D(n, \mathcal{P})$, respectively. The following two lemmas, which proofs will be shown in the next section, are the basic tool in proving our main results.

Lemma 2.

$$E(Y_s^-) = nB\left(s; n-1, p^*\right) \tag{9}$$

and

$$E(Z_t^-) = nF\left(t; n-1, p^*\right). \tag{10}$$

Lemma 3. (i) *If* $r = o(n)$ *then*

$$E_2(X_r^-) \leq n^2 b^2(r; n-1, p^*)(1 + o(1)).$$

(ii) *If* $E^+ = E^+(\mathcal{P}) = o(n), s < np^*, t > np^*$ *and* $t = o(n)$ *then*

$$E_2(Y_s^-) \leq n^2 B^2(s; n-1, p^*)(1 + o(1))$$

and

$$E_2(Z_t^-) \leq n^2 F^2(t; n-1, p^*)(1 + o(1)).$$

Let

$$d_{(1)}^- \leq d_{(2)}^- \leq \cdots \leq d_{(n)}^-$$

be the in-degree sequence of vertices in a random digraph $D(n, \mathcal{P})$. The first result shows that for any fixed $i \geq 2$ the first i-th and the last i-th terms of the in-degree sequence of $D(n, \mathcal{P})$ are asymptotically almost surely strictly increasing. For the sake of simplicity let us denote

$$s = s(n, \mathcal{P}) = (1 + z(u, a))E^+ \tag{11}$$

$$t = t(n, \mathcal{P}) = (1 + y(u, a))E^+ \tag{12}$$

and

$$\varphi = \varphi(n, \mathcal{P}) = \left(\frac{E^+}{2\log n}\right)^{1/2} x(n) \tag{13}$$

where power series $z(u, a)$ and $y(u, a)$ are given by (2) and (3), respectively and $x(n)$ is a sequence tending to infinity arbitrary slowly as $n \to \infty$.

Theorem 1. *Let $\mathcal{P} = (P_0, P_1, \ldots, P_{n-1})$ be such that*

$$E^+ = \omega(n)\log(n) = o(n),$$

where $\omega(n) \to \infty$ as $n \to \infty$. Then for any fixed $i \geq 2$

$$s - \varphi < d^-_{(1)} < \cdots < d^-_{(i)} < s + \varphi \quad a.a.s. \tag{14}$$

and

$$t - \varphi < d^-_{(n-i+1)} < \cdots < d^-_{(n)} < t + \varphi \quad a.a.s. \tag{15}$$

where s and t are given by (11) *and* (12) *with*

$$u = u(n, \mathcal{P}) = \frac{1}{E^+}\left(\log n - \frac{1}{2}\log\log n\right) \tag{16}$$

$$a = a(n, \mathcal{P}) = \frac{E^+}{n - 1 - E^+} \tag{17}$$

and φ is given by (13).

Proof. Put $r = s - \varphi$. Then by Lemma 2 we have

$$E(Y_r^-) = nB\left(s - \varphi; n-1, p^*\right).$$

Since

$$p^* = \frac{E^+}{n-1} = \frac{\omega(n)\log n}{n-1}$$

and

$$s - \varphi = \frac{\omega(n)\log n}{n-1}\big(1 + z(u,a)\big) - \left(\frac{\omega(n)}{2(n-1)}\right)^{1/2} x(n)$$

so by Lemma 1

$$\begin{aligned} E(Y_r^-) &\sim e^{-x(n)} \\ &= o(1)\,. \end{aligned} \tag{18}$$

Consequently

$$\begin{aligned} Pr(d_{(1)} \le s - \varphi) &= Pr(Y_r^- \ge 1) \\ &\le E(Y_r^-) \\ &= o(1)\,. \end{aligned}$$

Now let us put $r = s + \varphi$. Then

$$E(Y_r^-) =\sim e^{x(n)} \to \infty \text{ as } n \to \infty\,. \tag{19}$$

Moreover, routine calculations show that by (4), (11), (13), (16) and (17) we have

$$r = s + \varphi < np^*(1 + o(1))\,.$$

So by Lemma 2 and 3

$$E_2(Y_r^-) \le E^2(Y_r^-)(1 + o(1))$$

which implies that

$$Var(Y_r^-) \le E(Y_r^-) + o(E^2(Y_r^-))\,.$$

Thus by Chebyshev's inequality

$$Pr\left(Y_r^- \le \frac{1}{2}E(Y_r^-)\right) \le \frac{4Var(Y_r^-)}{E^2(Y_r^-)} = o(1)\,.$$

Consequently, for any fixed $i \ge 1$

$$\begin{aligned} Pr(d_{(i)}^- \le s + \varphi) &= Pr(Y_r^- \ge i) \\ &\ge 1 - o(1)\,. \end{aligned}$$

To show that the sequence is stricly increasing we have to show that probability that there are at least two vertices of equal in-degree $\le s + \varphi$ tends to zero as $n \to \infty$. We have

$$\sum_{k=0}^{s+\varphi} Pr(X_k^- \ge 2) \le \sum_{k=0}^{s+\varphi} E_2(X_k^-)\,.$$

Since, by Lemma 3,

$$E_2(X_r^-) \le E^2(X_r^-)(1+o(1))$$

so applying Lemma 1 we obtain

$$\begin{aligned}\sum_{k=0}^{s+\varphi} E_2(X_k^-) &\le \sum_{k=0}^{s+\varphi} n^2 b^2(k, n-1, p^*)(1+o(1)) \\ &\le nb(s+\varphi; n-1, p^*) nB(s+\varphi; n-1, p^*)(1+o(1)) \\ &\sim \left(\frac{2\log n}{np^*q^*}\right)^{1/2} e^{-2x(n)} \\ &= o(1)\end{aligned}$$

which completes the proof of (14). The proof of (15) follows analogously. ■

The above theorem gives a very precise estimate of the in-degree distribution of $D(n, \mathcal{P})$ in a case when the out-degree distribution $\mathcal{P} = (P_0, P_1, \ldots, P_{n-1})$ satisfies the condition

$$E^+(\mathcal{P}) = \sum_{k=0}^{n-1} kP_k = \omega(n)\log n\,.$$

The disadvantage of this result is the complicated form for given bounds which are expressed by appropriate power series. It appears that if $E^+(\mathcal{P})$ tends to infinity a bit faster than $\omega(n)\log n$ much more pleasant estimates for in-degree sequence can be given. Now let

$$s = E^+ - (2np^*q^*\log n)^{1/2} + \left(\frac{np^*q^*}{8\log n}\right)^{1/2} \log\log n \tag{20}$$

$$t = E^+ + (2np^*q^*\log n)^{1/2} - \left(\frac{np^*q^*}{8\log n}\right)^{1/2} \log\log n \tag{21}$$

and

$$\varphi(n) = \left(\frac{np^*q^*}{2\log n}\right)^{1/2} x(n) \tag{22}$$

where $x(n) \to \infty$ as $n \to \infty$ but $x(n) = o(\log\log n)$.

Theorem 2. *Let* $E^+ \geq [\gamma(n)(\log n)^3], \gamma(n) \to \infty$ *as* $n \to \infty$. *Then for any fixed* $i \geq 1$

$$s - \varphi \leq d_{(1)} < d_{(2)} < \cdots < d_{(i)} \leq s + \varphi \quad a.a.s. \tag{23}$$

and

$$t - \varphi \leq d_{(n-i+1)} < \cdots < d_{(n-1)} < d_{(n)} \leq t + \varphi \quad a.a.s. \tag{24}$$

where s, t *and* φ *are given by* (20), (21) *and* (22), *respectively.*

Proof. Put

$$r = E^+ - v\sqrt{np^*q^*}$$

where

$$v = v(n) = (2\log n)^{1/2} - \left(\frac{1}{2\log n}\right)^{1/2}\left(\frac{1}{2}\log\log n - x(n)\right).$$

Then the assumption $np^* \geq \gamma(n)(\log n)^3$ implies

$$\frac{v^3}{\sqrt{np^*q^*}} \leq \left(\frac{8}{\gamma(n)}\right)^{1/2} = o(1)$$

so applying the classical DeMoivre-Laplace formula (see Feller [4] Chapter 7) we obtain

$$\begin{aligned} E(Y_r^-) &\sim \frac{n}{\sqrt{2\pi}}\frac{1}{v}e^{-\frac{v^2}{2}} \\ &\sim \frac{1}{\sqrt{2\pi}}e^{-x(n)} \\ &= o(1)\,. \end{aligned}$$

Now putting

$$z = E^+ - w\sqrt{np^*q^*}$$

where

$$w = w(n) = (2\log n)^{1/2} - \left(\frac{1}{2\log n}\right)^{1/2}\left(\frac{1}{2}\log\log n + x(n)\right)$$

we have

$$E(Y_z^-) \to \infty \text{ as } n \to \infty$$

and

$$\sum_{k=0}^{z} Pr(X_k^- \geq 2) = o(1).$$

Therefore the same argument as in the proof of Theorem 1 implies the first part of our result. The second part follows analogously. ■

4. Proofs of Lemmas

***Proof of Lemma* 2.**

$$\begin{aligned} E(Y_t^-) &= n \sum_{k=0}^{t} \binom{n-1}{k} \Bigg\{ \sum_{\substack{a_0,\dots,a_{n-1}\geq 0 \\ a_0+\dots+a_{n-1}=k}} \binom{n-1}{a_0,\dots,a_{n-1}} \prod_{j=0}^{n-1} \left(P_j \frac{j}{n-1}\right)^{a_j} \\ &\times \sum_{\substack{b_0,\dots,b_{n-1}\geq 0 \\ b_0+\dots+b_{n-1}=n-k-1}} \binom{n-1}{b_0,\dots,b_{n-1}} \prod_{j=0}^{n-1} \left[\left(1-\frac{j}{n-1}\right) P_j\right]^{b_j} \Bigg\} \\ &= n \sum_{k=0}^{t} \binom{n-1}{k} \left[\left(\sum_{j=0}^{n-1} P_j \frac{j}{n-1}\right)^k \left(\sum_{j=0}^{n-1} \left(1 - P_j \frac{j}{n-1}\right)\right)^{n-k-1} \right]. \end{aligned}$$

Hence

$$p^* = \sum_{j=0}^{n-1} P_j \frac{j}{n-1}$$

we have

$$\begin{aligned} E(Y_t^-) &= n \sum_{k=0}^{t} \binom{n-1}{k} (p^*)^k (q^*)^{n-k-1} \\ &= nB(t; n-1, p^*)\,. \end{aligned}$$

Proof of (10) is analogous. ■

***Proof of Lemma* 3.** We show part (i). Let $\mathcal{L}$ denotes the set of all arcs in $D(n, \mathcal{P})$. Let A be the event that two given vertices from V, say v_1 and v_2, have the in-degree equal to r in $D(n, \mathcal{P})$. Then

$$E_2(X_r^-) = (n)_2 Pr(A)\,. \tag{25}$$

Let

$$B(v_1) = \{v \in V \setminus \{v_1, v_2\} : (v, v_1) \in \mathcal{L}\}$$

and

$$B(v_2) = \{v \in V \setminus \{v_1, v_2\} : (v, v_2) \in \mathcal{L}\}\,.$$

Then considering the event A_1 that $(v_1, v_2) \notin \mathcal{L}$ and $(v_2, v_1) \notin \mathcal{L}$, we have clearly that

$$|B(v_1)| = |B(v_2)| = r \text{ and } |B(v_1) \cap B(v_2)| = k$$

for $k = f, \ldots, r$, where $f = \max\{0, 2r - (n-2)\}$ and

$$\begin{aligned} Pr(A_1) &= \left[P_1 \frac{\binom{n-2}{1}}{\binom{n-1}{1}} + \cdots + P_{n-1} \frac{\binom{n-2}{n-1}}{\binom{n-1}{n-1}}\right]^2 \\ &= \left[P_1\Big(1 - \frac{1}{n-1}\Big) + \cdots + P_{n-1}\Big(1 - \frac{n-1}{n-1}\Big)\right]^2 \\ &= \left[1 - \frac{1}{n-1}\sum_{i=1}^{n-1} iP_i\right]^2 \\ &= (q^*)^2\,. \end{aligned}$$

Analogously denoting by A_2, A_3 and A_4 the events corresponding to the case

- $(v_1, v_2) \notin \mathcal{L}$ and $(v_2, v_1) \in \mathcal{L}$
- $(v_1, v_2) \in \mathcal{L}$ and $(v_2, v_1) \notin \mathcal{L}$
- $(v_1, v_2) \in \mathcal{L}$ and $(v_2, v_1) \in \mathcal{L}$,

respectively we have

$$Pr(A_2) = Pr(A_3) = p^* q^*$$

and

$$Pr(A_4) = (p^*)^2\,.$$

Furthermore, let B_j stand for the event that a given vertex from the set $V \setminus \{v_1, v_2\}$ emanates j ($j = 0, 1, 2$) arcs to vertices $\{v_1, v_2\}$. Assume that for $j = 1$ it is known to which vertex, v_1 or v_2, an arc is coming to. Then for $j = 0, 1, 2$ we have

$$Pr(B_j) = \sum_{i=j}^{n-j} P_i \frac{\binom{n-3}{i-j}}{\binom{n-1}{i}} \qquad j = 0, 1, 2\,.$$

In particular

$$\begin{aligned} Pr(B_1) &= \sum_{i=1}^{n-2} P_i \frac{\binom{n-3}{i-1}}{\binom{n-1}{i}} \\ &= \sum_{i=1}^{n-2} P_i \frac{i(n-i-1)}{(n-1)(n-2)} \\ &= \sum_{i=1}^{n-2} P_i \frac{i(n-1)}{(n-1)(n-2)} - \sum_{i=1}^{n-2} P_i \frac{i^2}{(n-1)(n-2)} \\ &\le \sum_{i=0}^{n-1} P_i \frac{i}{(n-1)} - \sum_{i=1}^{n-2} P_i \frac{i^2}{(n-1)(n-2)} \\ &\le p^* - (p^*)^2 = p^* q^*\,. \end{aligned}$$

Similarly we get that $Pr(B_0) \le (q^*)^2$ and $Pr(B_2) \le (p^*)^2$. Consequently, with

$H(a, b, c, e)$

$$= \binom{n-2}{a} \sum_{k=b}^{c} \binom{a}{k} \binom{n-2-a}{c-k} Pr(B_2)^k Pr(B_1)^{2(r-k)-e} Pr(B_0)^{n-2-2r+k+e},$$

$f = \max\{0, 2r+2-n\}$, $g = \max\{0, 2r+1-n\}$ and $h = \max\{0, 2r-n\}$ we have

$$\begin{aligned} Pr(A|A_1) &= H(r, f, r, 0) \\ Pr(A|A_2) &= Pr(A|A_3) = H(r, g, r-1, 1) \\ Pr(A|A_4) &= H(r-1, h, r-1, 2)\,. \end{aligned}$$

Applying the well-known relation

$$\sum_{k=0}^{c}\binom{a}{c}\binom{n-2-a}{c-k}=\binom{n-2}{c}$$

we obtain the following estimate

$$\begin{aligned}
&Pr(A|A_1)Pr(A_1)\\
&=\binom{n-2}{r}\sum_{k=f}^{r}\binom{r}{k}\binom{n-2-r}{r-k}Pr(B_2)^k Pr(B_1)^{2(r-k)}Pr(B_0)^{n-2r-2+k}(q^*)^2\\
&\le\binom{n-2}{r}\sum_{k=f}^{r}\binom{r}{k}\binom{n-2-r}{r-k}(p^*)^{2r}(q^*)^{2(n-r-1)}\\
&\le\binom{n-2}{r}\binom{n-2}{r}(p^*)^{2r}(q^*)^{2(n-r-1)}\\
&=\left[\binom{n-1}{r}(p^*)^r(q^*)^{n-r-1}\right]^2\left(1-\frac{r}{n-1}\right)^2\\
&=b^2(n-1;r,p^*)\left(1+O^2(\frac{r}{n})\right).
\end{aligned}$$

Analogously

$$\begin{aligned}
Pr(A|A_2)Pr(A_2)\;&=\;Pr(A|A_3)Pr(A_3)\\
&=\;b^2(n-1;r,p^*)\frac{r}{n-1}\left(1-\frac{r}{n-1}\right)
\end{aligned}$$

and

$$Pr(A|A_4)Pr(A_4)=b^2(n-1;r,p^*)\frac{r^2}{(n-1)^2}\,.$$

Thus by the assumption that $r=o(n)$ we get

$$\begin{aligned}
Pr(A)\;&=\;\sum_{i=1}^{4}Pr(A|A_i)Pr(A_i)\\
&\le\;b^2(n-1;r,p^*)(1+o(1))
\end{aligned}$$

and consequently by (25)

$$E_2(X_r^-) \leq n^2 b^2(n-1; r, p^*)(1 + o(1)).$$

Proof of part (ii) is analogous. ∎

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