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EXTREMAL BIPARTITE GRAPHS WITH A UNIQUE k-FACTOR

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Abstract

Given integers $p > k > 0$, we consider the following problem of extremal graph theory: How many edges can a bipartite graph of order $2p$ have, if it contains a unique k-factor? We show that a labeling of the vertices in each part exists, such that at each vertex the indices of its neighbours in the factor are either all greater or all smaller than those of its neighbours in the graph without the factor. This enables us to prove that every bipartite graph with a unique k-factor and maximal size has exactly $2k$ vertices of degree k and $2k$ vertices of degree $\frac{|V(G)|}{2}$. As our main result we show that for $k \geq 1$, $p \equiv t \pmod{k}$, $0 \leq t < k,$

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a bipartite graph G of order $2p$ with a unique k-factor meets $2|E(G)| \leq p(p+k) - t(k-t)$. Furthermore, we present the structure of extremal graphs.

Keywords: unique k-factor, bipartite graphs, extremal graphs.

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1. Introduction

All graphs considered are finite and simple. We use standard graph terminology as can be found in [1]. A graph G has vertex set $V(G)$, edge set $E(G)$, order $n(G) = |V(G)|$ and size $e(G) = |E(G)|$. A graph is called *bipartite* if the vertex set can be partitioned into two sets A, B such that A and B constitute independent sets. With $K_{a,b}$ we denote the *complete bipartite graph* with partition A, B such that $|A| = a$ and $|B| = b$. The *neighbourhood* $N_G(v)$ of a vertex v is the set of all vertices of the graph G adjacent to v. With $d_G(v) = |N_G(v)|$ we denote the *degree* of v in G. A spanning subgraph F is called a *k-factor*, if $d_F(v) = k$ for all $v \in V(F) = V(G)$. If a graph G has a factor F, we colour the edges belonging to F red and all other ones blue and denote with $N_r(v) = N_F(v)$ and $N_b(v) = N_G(v) \backslash N_F(v)$ the *red* and *blue neighbourhood* of a vertex v, respectively. Then $d_r(v) = |N_r(v)|$ and $d_b(v) = |N_b(v)|$ denote the red and blue degree of v. The red neighbourhood of a set $X \subset V(G)$ is simply the union $N_r(X) = \cup_{x \in X} N_r(x)$. The blue neighbourhood of sets of vertices is defined analogously. We call a *path* or a *circuit alternating*, if its edges are coloured red–blue or blue–red in an alternating way. Note that the graph G has a second k-factor if and only if it has an alternating circuit. Throughout the paper red edges will be symbolized by a thick line $x \,\textbf{—}\, y$ and blue edges will be symbolized by a thin line $x — y$.

Following a result of J. Sheehan [6] on extremal graphs with a unique hamiltonian cycle, G.R.T. Hendry [2] proved sharp results for the maximal size of a graph with a unique 2-factor. P. Johann [4] and L. Volkmann [7] improved Hendry's results in special cases, however, the general case remains unsolved for $k \geq 4$. L. Volkmann further presented graphs of arbitrary order n with a unique k-factor in [7], which he conjectured to have maximal size. Another interesting conjecture in the same paper is that every graph with a unique k-factor, for $k \geq 2$, has exactly k vertices of degree k if its size is maximal.

Two of the authors presented in [3] a method for applying results on general graphs with a unique k-factor to bipartite graphs with a unique k-factor. Through this, sharp upper bounds for the size of a bipartite graph with a unique k-factor if $k \leq 3$ and in some special cases were proven.

The aim of this paper is to present detailed information about the structure of an extremal bipartite graph with a unique k-factor. This will be done in Section 2. In the third section we will use this information to prove our main theorem and present a sharp upper bound for the size of a bipartite graph with a unique k-factor for all $k \geq 1$.

In [3] the following graphs and the observation given for the maximum number of edges in a graph with a unique k-factor have been presented. Let p and k be non-negative integers such that $p = sk + t$ with $s \geq 1$ and $0 \leq t \leq k - 1$. First define a bipartite graph $A(k, t)$ as follows: Let A_1 be a copy of $K_{t,t}$ and A_2 a bipartite $(k - t)$-regular graph on $2k$ vertices (the latter exists as a result of König's Theorem [5]). Let A_{i_j}, with $1 \leq j \leq 2$ denote the two parts of A_i, $1 \leq i \leq 2$. Connect all vertices of A_{1_j} with $A_{2_{(3-j)}}$ for $1 \leq j \leq 2$. The resulting graph $A(k, t)$ is bipartite, has exactly one k-factor, consisting of the edges in A_2 and those connecting A_1 and A_2, and $|E(A(k, t))| = t^2 + k(k + t)$.

Next take $s - 1$ copies of $K_{k,k}$, one copy of $A(k, t)$ and number these graphs $S_1, S_2, \ldots, S_s$, respectively. Let (A, B) be the partition of these graphs. Connect all vertices of $V(S_i) \cap A$ with all vertices in $V(S_j) \cap B$ where $j > i$. The resulting graph $B(p, k)$ is bipartite of order $2p$, has exactly one k-factor, formed by the copies of $K_{k,k}$ and the unique k-factor of $A(k, t)$.

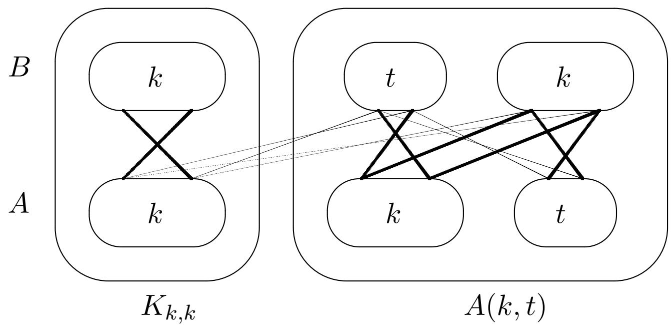

Figure 1. The graph $B(2k + t, k)$ with $t > 0$.

Observation 1.1. Let G be a bipartite graph of order $2p$ with a unique k-factor such that $p \equiv t \pmod{k}$, $0 \leq t < k$. If $|E(G)|$ is maximum, then

$$|E(G)| \geq |E(B(p,k))| = \frac{1}{2}(p^2 + kp - t(k-t)). \tag{1}$$

2. Structural Results on Red and Blue Neighbourhoods

In this section we are going to take a close look at the structure of extremal bipartite graphs with a unique k-factor. So, throughout this section let G always denote a bipartite graph of order $2p$ with a unique k-factor such that $e(G)$ is maximal, if not stated otherwise. With F we will always denote the unique k-factor.

We start out with looking at the red and blue neighbourhoods of vertices and chains of alternating neighbourhoods, defined as follows.

Definition 2.1. Let $x \in V(G)$. For $i > 0$ simultaneously define

for $i = 1$: $R_1(x) := N_r(x)$, $B_1(x) := N_b(x)$,

for $i > 1$: $R_i(x) := N_r(B_{i-1}(x)) \setminus \bigcup_{j=1}^{i-1} R_j(x)$, $B_i(x) := N_b(R_{i-1}(x)) \setminus \bigcup_{j=1}^{i-1} B_j(x)$.

If there is no chance of ambiguity, we simply call the sets R_i and B_i.

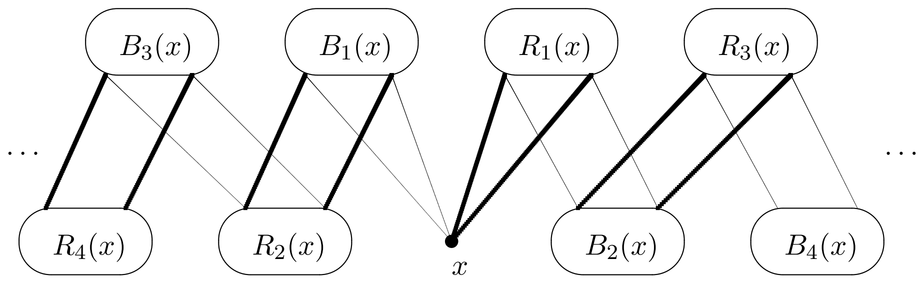

Figure 2. The sets $R_i(x)$ and $B_i(x)$.

Plainly speaking, a set $R_i \neq \emptyset$ contains all red neighbours of the vertices in B_{i-1} which are not in R_j for $j < i$. Similarily for B_i. From the definition it is not clear for which $i > 1$ it holds $R_i(x) = \emptyset$ or $B_i(x) = \emptyset$. The next lemma will show us that in a graph with a unique k-factor such that the size is maximal, the chains of alternating neighbourhoods terminate rather soon.

Lemma 2.2. *For every x and sets $R_i(x)$, $B_i(x)$ as defined in Definition* 2.1 *it holds*

(i) $R_i(x) \cap B_j(x) = \emptyset$ *for all i and j;*

(ii) $B_i(x) = \emptyset$ *for $i \geq 3$ and $R_i(x) = \emptyset$ for $i \geq 4$.*

Proof. By Definition 2.1 $R_i(x)$ and $B_j(x)$ lie in different parts of G if $i \not\equiv j \pmod 2$. Thus $R_i(x) \cap B_j(x) = \emptyset$ in this case. Assume there exist $i \equiv j \pmod 2$ such that $R_i(x) \cap B_j(x) \neq \emptyset$. Choose i, j with that property such that $l := \min\{i, j\}$ is minimal and $|i - j|$ is minimal over all such pairs i, j with $\min\{i, j\} = l$. Without loss of generality let i, j be even, as the proof for i, j odd runs analogously. Choose $y \in R_i(x) \cap B_j(x)$. With Definition 2.1 we get a path $P_1 : x \text{---} x_1 \textbf{---} x_2 \text{---} \ldots \text{---} x_{i-1} \textbf{---} y$ and a path $P_2 : x \textbf{---} y_1 \text{---} y_2 \textbf{---} \ldots \textbf{---} y_{j-1} \text{---} y$. The way the pair i, j was chosen, we have $x_s \neq y_t$ for $1 \leq s \leq i - 1$ and $1 \leq t \leq j - 1$. But then

$$x \text{---} x_1 \textbf{---} x_2 \text{---} \ldots \text{---} x_{i-1} \textbf{---} y \text{---} y_{j-1} \textbf{---} \ldots \textbf{---} y_2 \text{---} y_1 \textbf{---} x$$

forms an alternating circuit. This contradicts the uniqueness of the factor and thus statement (i) of the lemma follows.

For a proof of (ii) we only need to show that $B_3(x) = B_4(x) = \emptyset$. Assume that $B_3(x) \neq \emptyset$. Then there exists a vertex $y \in B_3(x)$. Due to the definition of $B_3(x)$, the vertex y lies in a different part than x and $xy \notin E(G)$. Consider the graph $G' = G \cup xy$ which has F as a k-factor. As G is edge-maximal, there exists an alternating circuit in G' containing the blue edge $x \text{---} y$. Choose such a circuit C with minimum number of edges. Then C contains $y \text{---} x \textbf{---} x_1 \text{---} x_2$ with $x_1 \in R_1(x)$ and $x_2 \in B_2(x)$. A simple counting argument now yields that there either exists an edge $v \textbf{---} w$ with $v \in B_{2j}(x)$ and $w \in B_{2k+1}(x)$ or an edge $v \text{---} w$ with $v \in R_{2j}(x)$ and $w \in R_{2k+1}(x)$. Both cases contradict (i).

Assume that $B_4(x) \neq \emptyset$. Then there exists a vertex $y \in B_4(x)$ and we can find an alternating path $x \textbf{---} v_1 \text{---} v_2 \textbf{---} v_3 \text{---} y$ with $v_1 \in R_1(x)$, $v_2 \in B_2(x)$ and $v_3 \in R_3(x)$. By the definition of the sets B_i, $v_1 y \notin E_b(G)$. Furthermore, $v_1 y \notin E_r(G)$ as otherwise we would have the alternating circuit

$v_1 \text{—} v_2 \boldsymbol{\text{—}} v_3 \text{—} y \boldsymbol{\text{—}} v_1$. Thus $y \in B_3(v_1)$, contradicting $B_3(v_1) = \emptyset$. Hence, $B_4 = \emptyset$, proving statement (ii). ■

We note that statement (i) of the above lemma holds for any graph with a unique k-factor. Statement (ii), however, requires the maximality of $e(G)$.

Lemma 2.3. *If A, B are the parts of G and, without loss of generality, $x \in A$, then it holds:*

(i) $N_b(R_2) \subseteq B_1$ *and* $N_b(R_3) \subseteq B_2$.

(ii) *The subgraph induced by $V(B_1) \cup V(B_2)$ is bipartite complete with every edge coloured blue.*

(iii) *Every $z \in A$ with $d_b(z) \leq d_b(x)$ meets $N_b(z) \subseteq N_b(x)$.*

Proof. Statement (i) is a simple corollary of Lemma 2.2. For a proof of (ii) assume that there exist $v \in B_1(x)$ and $w \in B_2(x)$ such that $vw \notin E_b(G)$. Obviously $w \neq x$ and $vw \notin E(G)$. As G is edge-maximal, the graph $G' = G \cup vw$ contains an alternating circuit C with the path $w \text{—} v \boldsymbol{\text{—}} z$ with $z \in R_2$. However, $N_b(z) \subseteq B_1(x)$ by (i) and $N_r(B_1(x)) = R_2(x)$ by definition and thus the alternating circuit cannot leave $B_1(x) \cup R_2(x)$. This is a contradiction to $w \in B_2(x)$.

Following (ii), every vertex $z \in B_2(x)$ satisfies $d_b(z) > d_b(x)$ as $B_1(x) = N_b(x) \subset N_b(z)$ and as z also has at least one blue neighbour in $R_1(x)$ by definition. As $A = R_2(x) \cup \{x\} \cup B_2(x)$, the only vertices z different from x satisfying $d_b(z) \leq d_b(x)$ are those in $R_2(x)$, proving (iii). ■

With Lemma 2.3 (iv) we are in the position to label our vertices in any of the two parts in such a way, that their blue neighbourhoods form an increasing chain.

Definition 2.4. Let G be a bipartite graph of order 2p, the edges of which are coloured red and blue. Let A, B denote the two parts of G. A labeling (X, Y) of G such that $A = \{x_1, x_2, \ldots, x_p\}$ and $B = \{y_1, y_2, \ldots, y_p\}$ is called a blue labeling if the following conditions hold:

- $N_b(x_1) \subseteq N_b(x_2) \subseteq \ldots \subseteq N_b(x_p)$ and
- $N_b(y_p) \subseteq N_b(y_{p-1}) \subseteq \ldots \subseteq N_b(y_1)$.

Note that $N_b = \emptyset$ is allowed in this definition.

Lemma 2.5. *Let G be a bipartite graph of order $2p$ with a unique k-factor such that $e(G)$ is maximal. Then G has a blue labeling.*

Proof. The statement follows immediately from Lemma 2.3 (iii) and Definition 2.4. ■

Lemma 2.6. *Let A, B be the parts of G and (X, Y) a blue labeling of G. For each $u \in A$, $v \in B$ and $i, j \in \{1, 2, \dots, p\}$ it holds:*

(i) *If $vx_j \in E_b$, then $vx_k \in E_b$ for all $j < k$;*

(ii) *If $uy_i \in E_b$, then $uy_t \in E_b$ for all $t < i$.*

Proof. We give the proof for (i), the proof for (ii) runs analogously. Suppose that there exist $v \in B$ and integers j, k such that $1 \leq j < k \leq p$, $vx_j \in E_b$ and $vx_k \notin E_b$. Then $v \in N_b(x_j)$ and $v \notin N_b(x_k)$, which contradicts $N_b(x_j) \subseteq N_b(x_k)$. ■

We see that an extremal bipartite graph with a unique k-factor and a blue labeling has a fan-shaped structure in its blue edges. This motivates the following general definition.

Definition 2.7. Let G be a bipartite graph of order $2p$, the edges of which are coloured red and blue. Let $A = \{x_1, x_2, \dots, x_p\}$ and $B = \{y_1, y_2, \dots, y_p\}$ denote the parts of G and let $u \in A$ and $v \in B$ be two arbitrary vertices.

- Let i be the last integer such that $uy_i \in E_b$. We say that u has the property $(*)$-right for the sequence $(y_1, y_2, \dots, y_p)$, if $uy_t \notin E_r$ for all $t < i$ (the red edges are on the "right side").
- Let j be the first integer such that $vx_j \in E_b$. We say that v has the property $(*)$-left for the sequence $(x_1, x_2, \dots, x_p)$, if $vx_t \notin E_r$ for all $j < t$ (the red edges are on the "left side").

The definition immediately implies the following lemma.

Lemma 2.8. *Let G be a bipartite graph, the edges of which are coloured red and blue and let A, B denote the parts of G. If one of the following two conditions is met, then G does not have an alternating circuit:*

(i) *There exists a labeling of A such that every vertex of B has the $(*)$-right property.*

(ii) *There exists a labeling of A such that every vertex of B has the $(*)$-left property.*

Looking again at extremal bipartite graphs G with a unique k-factor, Lemma 2.6 and Definition 2.7 give us the next lemma.

Lemma 2.9. *Let A, B denote the parts of G and (X, Y) a blue labeling. Then every vertex of A has the property $(*)$-right and every vertex of B has the property $(*)$-left.*

Lemma 2.10. *Let A, B denote the parts of G and (X, Y) a blue labeling. Then for each $v \in B$ and $u \in A$ it holds:*

(i) *If i is the last integer such that $vx_i \in E_r$ then $vx_j \in E_b$ for $j = i+1, i+2, \ldots, p$.*

(ii) *If i is the first integer such that $uy_i \in E_r$ then $uy_j \in E_b$ for $j = 1, 2, \ldots, i-1$.*

Proof. We only show the proof for (i), the proof for (ii) runs analogously. Suppose that (i) does not hold. Let i be the last integer such that $vx_i \in E_r$ and let j be integer such that $i < j \leq p$ and $vx_j \notin E_b$. Then we add the edge $e = vx_j$ and colour it blue. In the resulting graph $G + e$ each vertex of the set B still has the property $(*)$-left for the sequence $(x_1, x_2, \ldots, x_p)$. Thus by Lemma 2.8 the graph $G + e$ does not contain any alternating circuit. Therefore $G + e$ has a unique k-factor and one more edge than G, contradicting the maximality of $e(G)$. ■

Before turning to our main theorem in the next section, we want to seperately state the following theorem, as it provides us with a nice insight on vertices of minimum degree in an extremal bipartite graph with a unique k-factor.

Theorem 2.11. *Let G be a bipartite graph of order $2p > 2k$ with a unique k-factor such that the size of G is maximal. Let further (X, Y) be a blue labeling of the parts A, B of G. Then G has exactly $2k$ vertices of degree k, namely $x_1, x_2, \ldots, x_k$ in A and $y_{p-k+1}, y_{p-k+2}, \ldots, y_p$ in B.*

Proof. We will only show the proof for $x_1, \ldots x_k$, the proof for $y_{p-k+1}, \ldots, y_p$ runs analogously. Let i_0 denote the smallest index such that $d_b(x_{i_0}) = \min\{d_b(x_i) : d_b(x_i) \geq 1\}$. By the choice of i_0 we either have $i_0 = 1$ or $d_b(x_i) = 0$ for $1 \leq i < i_0$. Let j_0 denote the smallest index such that $x_{i_0} y_{j_0} \in E_r$. As x_{i_0} has the $(*)$-right property and $d_b(x_{i_0}) \geq 1$, Lemma 2.10 gives us $j_0 > 1$. Let us now take a look at y_{j_0-1}. By the choice of j_0 we have $x_{i_0} y_{j_0-1} \in E_b$. On the one hand, the $(*)$-left property holds for y_{j_0-1} and it follows $i < i_0$ for every $x_i \in N_r(y_{j_0-1})$. Hence we get $i_0 > k$. On the other hand, the $(*)$-right property holds for each x_i as well as $d_b(x_i) = 0$ for every

$1 \leq i < i_0$. Again with Lemma 2.10 we get $x_i y_1 \in E_r$ for every $1 \leq i < i_0$. As $d_r(y_1) = k$, we get $i_0 \leq k+1$. Both inequalities together yield $i_0 = k+1$ and $d(x_1) = d(x_2) = \ldots = d(x_k) = k$. ■

3. Extremal Bipartite Graphs with a Unique k-Factor

We start this section with the following theorem.

Theorem 3.1. *Let G be a bipartite graph of order $2p$ with a unique k-factor such that $e(G)$ is maximal. If $p = k + t$ with $0 \leq t \leq k - 1$, then G is isomorphic to one of the graphs $B(p, k)$ defined in the introduction.*

Proof. From Observation 1.1 we know

$$(2) \quad e(G) \geq e(B(p,k)) = \frac{1}{2}((k+t)^2 + k(k+t) - t(k-t)) = k(k+t) + t^2.$$

The statement is obvious for $t = 0$, so let $t > 1$. Colour the edges of the k-factor F red and all other ones blue. With Theorem 2.11 we know that k vertices in each part of G have degree k and only connected to red edges. Thus G can have at most t^2 blue edges, resulting in $e(G) \leq pk + t^2 = k(k+t) + t^2$. As a result we have equality in (2). Thus, the subgraph A_1 induced by the t^2 blue edges of G is isomporphic to $K_{t,t}$. Let A_2 denote the subgraph induced by the vertices of degree k. G has a unique k-factor and thus every vertex of each part of A_1 is connected to every vertex of A_2 in the other part. This leads to $|E(A_2)| = k(k-t)$ and as all edges of A_2 must belong to the k-factor of G, A_2 is a $(k-t)$-regular graph. In consequence, G is isomporphic to one of the graphs $A(k,t)$ and thus to a $B(p,k)$ as defined in the introduction. ■

Let us now present our main theorem.

Theorem 3.2. *Let G be a bipartite graph of order $2p$ with a unique k-factor such that $e(G)$ is maximal. Then $e(G) = \frac{p^2+kp}{2} - \frac{t(k-t)}{2}$, with $0 \leq t < k$ and $p \equiv t \pmod{k}$.*

Proof. We fix k. For $p \geq k$ let $e(p,k)$ be the maximum size of a bipartite graph of order $2p$ with a unique k-factor. Suppose that $e(p,k) > \frac{p^2+kp}{2} - \frac{t(k-t)}{2}$ and choose p minimal in this respect. From Theorem 3.1 it follows $p \geq 2k$. For every graph of order $2p$ and size $e(p,k)$ with a unique k-factor let A, B be parts of G. As $e(p,k)$ denotes the maximal possible size, Lemma 2.5 gives us the existence of a blue labeling (X,Y). By Lemma 2.9 each vertex of B has the property $(*)$-left and each vertex of A has the property $(*)$-right.

Let $A_1 = \{x_1, x_2, \ldots, x_k\} \subset A$. By Theorem 2.11 we have $d_r(u) = k$ and $d_b(u) = 0$ for every $u \in A_1$. It is easy to see that the set $B_1 = \{v \in B : N_r(v) = A_1\}$ meets $0 \leq |B_1| \leq k$. If $|B_1| = k$ then the subgraph induced by $A_1 \cup B_1$ is isomorphic to $K_{k,k}$. Deleting this subgraph leads to a graph G' of order $2(p-k)$ and

$$e(G') \geq e(G) - pk > \frac{(p-k)^2 + k(p-k)}{2} - \frac{t(k-t)}{2}.$$

Thus G' meets the criteria of the assumption and is of smaller order than G, contradicting the choice of G.

Suppose that G is chosen with the maximum number of vertices in B_1 over all bipartite graphs of order $2p$ and size $e(p,k)$ with a unique k-factor. Let us denote $|B_1| = b - 1$. Due to the preceding consideration, suppose that $1 \leq b \leq k$.

Let us first show, that $B_1 = \{y_1, y_2, \ldots, y_{b-1}\}$. For this let i_0 be the smallest index such that $y_{i_0} \notin B_1$. Then y_{i_0} has a red neighbour in $A \setminus A_1$. Suppose that there exists $y_{i_1} \in B_1$ with $i_1 > i_0$. Due to the definition of B_1 and the $(*)$-left property of y_{i_1}, $N_b(y_{i_1}) = A \setminus A_1$. This, however, contradicts the property $N_b(y_{i_1}) \subset N_b(y_{i_0})$ in the definition of the blue labeling. We thus get $i_0 = b$ and $B_1 = \{y_1, y_2, \ldots, y_{b-1}\}$.

Now suppose further that over all such graphs, G is chosen with the maximum number of vertices in $N_r(y_b) \cap A_1$.

Let r be the greatest index such that $y_b x_r \in E_r$. Since $y_b \notin B_1$, we have $x_r \notin A_1$ and there exists an $x_i \in A_1$ such that $x_i y_b \notin E$. As $d_r(x_i) = k$, it follows that there is an index w such that $y_w x_i \in E_r$ and $y_w \notin B_1$. The edge $y_w x_r$ cannot be blue, since the vertex x_r has the property $(*)$-right, $y_b x_r$ is red and $b < w$.

Case 1. $y_w x_r \notin E$.

In this case $y_w x_s \notin E_b$ for all $s \leq r$. We construct a new graph G' by deleting the edges y_b **—** x_r, x_i **—** y_w and adding the edges y_b **—** x_i, y_w **—** x_r, y_b — x_r. The red edges still form a k-factor in G' and each vertex of B still has the property $(*)$-left for the sequence $(x_1, x_2, \ldots, x_p)$. Thus by Lemma 2.8 G' has no alternating circuit. Therefore G' has a unique k-factor and more edges than G, a contradiction.

Case 2. $y_w x_r \in E_r$.
Suppose that there exists a vertex $x_s \notin A_1$, which is not adjacent to y_w. Because of $N_r(x_s) \neq N_r(x_r)$ and as the $(*)$-right property holds for x_r, there exists a vertex $y_j \in N_r(x_s)$, with $j > b$, such that $y_j x_r \notin E$. Then we construct a new graph G' by deleting the edges y_b **—** x_r, x_i **—** y_w, x_s **—** y_j and adding edges y_b **—** x_i, y_w **—** x_s, x_r **—** y_j, y_b — x_r. The red edges still form a k-factor in G' and each vertex of B still has the property $(*)$-left. Thus by Lemma 2.8 G' has no alternating circuit, giving us a contradiction as in Case 1.

Suppose that y_w is adjacent to each vertex of $A \setminus A_1$. Let s be the first integer such that $y_w x_s \in E_b$ $(r < s \leq 2k)$. Again there exists a vertex $y_j \in N_r(x_s)$ such that $y_j x_r \notin E$. Since $y_j \notin B_1$, $y_j \neq y_b$ and x_s has the property $(*)$-right, we have that $b < j < w$. Then we construct a new graph G' by deleting the edges y_b **—** x_r, x_i **—** y_w, x_s **—** y_j, y_w — x_s and adding the edges y_b **—** x_i, y_w **—** x_s, x_r **—** y_j, y_b — x_r. The red edges form a k-factor in G' and each vertex of B still has the property $(*)$-left in G'. Thus by Lemma 2.8 the graph G' has no alternating circuit. By Lemma 2.10 $d_b(y_i) = p - k$ for $1 \leq i \leq b-1$. Then there is the new labeling $B = \{y_{f(1)}, y_{f(2)}, \ldots, y_{f(p)}\}$ of the vertices of B such that $N_b(y_{f(p)}) \subseteq N_b(y_{f(p-1)}) \subseteq \ldots \subseteq N_b(y_{f(1)})$ and $f(b) = b$. However, in G' the set $N_r(y_b) \cap A_1$ has more vertices than in the graph G, contradicting the choice of G.

Thus all cases have been lead to a contradiction and our theorem is proved. ∎

Since in the proof of Theorem 3.2 all cases $|B_1| < k$ lead to a contradiction, we obtain the following

Corollary 3.3. *Let G be a bipartite graphs of order $2p > 4k$ with a unique k-factor such that the size of G is maximal. Let further (X, Y) be a blue labeling of the part A, B of G. Then G has exactly $2k$ vertices of degree p, namely $y_1, y_2, \ldots, y_k$ in B and $x_{p-k+1}, x_{p-k+2}, \ldots, x_p$ in A.*

Note that it has been implicitely shown in Theorem 3.1 that a bipartite graph of order $2p < 4k$ with a unique k-factor such that the size of G is maximal has exactly $2(p-k)$ vertices of degree p. From Corollary 3.3 we get the structure of extremal graphs.

Theorem 3.4. *A bipartite graph of order* $2p$ *with a unique* k*-factor and with the maximum number of edges is isomorphic to one of the graphs* $B(p,k)$*.*

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