

**ON THE BASIS NUMBER AND THE MINIMUM CYCLE  
BASES OF THE WREATH PRODUCT  
OF SOME GRAPHS I**

MOHAMMED M.M. JARADAT

*Department of Mathematics*  
*Yarmouk University*  
*Irbid-Jordan*

**e-mail:** mmjst4@yu.edu.jo

**Abstract**

A construction of a minimum cycle bases for the wreath product of some classes of graphs is presented. Moreover, the basis numbers for the wreath product of the same classes are determined.

**Keywords:** cycle space, basis number, cycle basis, wreath product.

**2000 Mathematics Subject Classification:** 05C38, 05C75.

**1. Introduction**

The basis number of a graph is one of the numbers which give rise to a better understanding and interpretations of a geometric properties of a graph (see [19]). Minimum cycle bases (MCBs) of a cycle spaces have a variety of applications in sciences and engineering, for example, in structural flexibility analysis, electrical networks, and in chemical structure storage and retrieval systems (see [9, 10] and [17]).

In general, required cycle bases, and minimum cycle bases are not very well behaved under graph operations. Neither the basis number  $b(G)$  of a graph  $G$  is monotonic (see [3] and [21]), nor the total length  $l(G)$  and the length of the longest cycle in a minimum cycle basis  $\lambda(G)$  are minor monotone (see [12]). Hence, there does not seem to be a general way of

extending required cycle bases and minimum cycle bases of a certain collection of partial graphs of  $G$  to a required cycle basis and to a minimum cycle basis of  $G$ , respectively. Global upper bounds  $b(G) \leq 2\gamma(G) + 2$  and  $l(G) \leq \dim \mathcal{C}(G) + \kappa(T(G))$  where  $\gamma(G)$  is the genus of  $G$  and  $\kappa(T(G))$  is the connectivity of the tree graph of  $G$  are proven in [21] and [18], respectively.

In this paper, we investigate the basis number for some classes of graphs and we construct minimum cycle bases for same, also, we give their total lengths and the length of longest cycles.

## 2. Definitions and Preliminaries

The graphs considered in this paper are finite, undirected, simple and connected. Most of the notations that follow can be found in [6]. For a given graph  $G$ , we denote the vertex set of  $G$  by  $V(G)$  and the edge set by  $E(G)$ .

### 2.1 Cycle bases

Given a graph  $G$ , let  $e_1, e_2, \dots, e_{|E(G)|}$  be an ordering of its edges. Then a subset  $S$  of  $E(G)$  corresponds to a  $(0, 1)$ -vector  $(b_1, b_2, \dots, b_{|E(G)|})$  in the usual way with  $b_i = 1$  if  $e_i \in S$ , and  $b_i = 0$  if  $e_i \notin S$ . These vectors form an  $|E(G)|$ -dimensional vector space, denoted by  $(\mathbb{Z}_2)^{|E(G)|}$ , over the field of integers modulo 2. The vectors in  $(\mathbb{Z}_2)^{|E(G)|}$  which correspond to the cycles in  $G$  generate a subspace called the *cycle space* of  $G$  and denoted by  $\mathcal{C}(G)$ . We shall say that the cycles themselves, rather than the vectors corresponding to them, generate  $\mathcal{C}(G)$ . It is known that for a connected graph  $G$   $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1$  (see [7]).

A basis  $\mathcal{B}$  for  $\mathcal{C}(G)$  is called a *cycle basis* of  $G$ . A cycle basis  $\mathcal{B}$  of  $G$  is called a  $d$ -fold if each edge of  $G$  occurs in at most  $d$  of the cycles in  $\mathcal{B}$ . The *basis number*,  $b(G)$ , of  $G$  is the least non-negative integer  $d$  such that  $\mathcal{C}(G)$  has a  $d$ -fold basis. The *length*,  $|C|$ , of the element  $C$  of the cycle space  $\mathcal{C}(G)$  is the number of its edges. The *length*  $l(\mathcal{B})$  of a cycle basis  $\mathcal{B}$  is the sum of the lengths of its elements:  $l(\mathcal{B}) = \sum_{C \in \mathcal{B}} |C|$ .  $\lambda(G)$  is defined to be the minimum length of the longest element in an arbitrary cycle basis of  $G$ . A *minimum cycle basis* (MCB) is a cycle basis with minimum length. Since the cycle space  $\mathcal{C}(G)$  is a matroid in which an element  $C$  has weight  $|C|$ , the greedy algorithm can be used to extract a MCB (see [23]). The following results will be used frequently in the sequel.

**Theorem 1.1.1** (MacLane). *The Graph  $G$  is planar if and only if  $b(G) \leq 2$ .*

A cycle is *relevant* if it is contained in some MCB (see [22]).

**Proposition 1.1.2** (Plotkin). *A cycle  $C$  is relevant if and only if it cannot be written as a linear combinations modulo 2 of shorter cycles.*

Chickering, Geiger and Heckerman [8], showed that  $\lambda(G)$  is the length of the longest element in a MCB.

## 2.2 Products

Let  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$  be two graphs.

(1) The *cartesian product*  $G \square H$  has the vertex set  $V(G \square H) = V(G) \times V(H)$  and the edge set  $E(G \square H) = \{(u_1, v_1)(u_2, v_2) | u_1 u_2 \in E(G) \text{ and } v_1 = v_2, \text{ or } v_1 v_2 \in E(H) \text{ and } u_1 = u_2\}$ .

(2) The *direct product*  $G \times H$  is the graph with the vertex set  $V(G \times H) = V(G) \times V(H)$  and the edge set  $E(G \times H) = \{(u_1, u_2)(v_1, v_2) | u_1 v_1 \in E(G) \text{ and } u_2 v_2 \in E(H)\}$ .

(3) The *strong product*  $G \boxtimes H$  is the graph with the vertex set  $V(G \boxtimes H) = V(G) \times V(H)$  and the edge set  $E(G \boxtimes H) = \{(u_1, u_2)(v_1, v_2) | u_1 v_1 \in E(G) \text{ and } u_2 v_2 \in E(H) \text{ or } u_1 = v_1 \text{ and } u_2 v_2 \in E(H) \text{ or } u_1 v_1 \in E(G) \text{ and } u_2 = v_2\}$ .

(4) The *semi-strong product*  $G_1 \bullet G_2$  is the graph with the vertex set  $V(G \bullet H) = V(G) \times V(H)$  and the edge set  $E(G \bullet H) = \{(u_1, u_2)(v_1, v_2) | u_1 v_1 \in E(G) \text{ and } u_2 v_2 \in E(H) \text{ or } u_1 = v_1 \text{ and } u_2 v_2 \in E(H)\}$ .

(5) The *lexicographic product*  $G_1[G_2]$  is the graph with vertex set  $V(G[H]) = V(G) \times V(H)$  and the edge set  $E(G[H]) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1 \text{ and } u_2 v_2 \in E(H) \text{ or } u_1 v_1 \in E(G)\}$ .

(6) The *wreath product*  $G \ltimes H$  has the vertex set  $V(G \ltimes H) = V(G) \times V(H)$  and the edge set  $E(G \ltimes H) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2 \text{ and } v_1 v_2 \in H, \text{ or } u_1 u_2 \in G \text{ and there is } \alpha \in \text{Aut}(H) \text{ such that } \alpha(v_1) = v_2\}$  (see [1] and [11]).

Many authors studied the basis number and the minimum cycle bases of graph products. The cartesian product of any two graphs was studied by Ali and Marougi [4] and Imrich and Stadler [12].

**Theorem 1.2.1** (Ali and Marougi). *If  $G$  and  $H$  are two connected disjoint graphs, then  $b(G \square H) \leq \max\{b(G) + \Delta(T_H), b(H) + \Delta(T_G)\}$  where  $T_H$  and  $T_G$  are spanning trees of  $H$  and  $G$ , respectively, such that the maximum degrees  $\Delta(T_H)$  and  $\Delta(T_G)$  are minimum with respect to all spanning trees of  $H$  and  $G$ .*

**Theorem 1.2.2** (Imrich and Stadler). *If  $G$  and  $H$  are triangle free, then  $l(G \square H) = l(G) + l(H) + 4[|E(G)|(|V(H)| - 1) + |E(H)|(|V(G)| - 1) - (|V(H)| - 1)(|V(G)| - 1)]$  and  $\lambda(G \square H) = \max\{4, \lambda(G), \lambda(H)\}$ .*

Schmeichel [21], Ali [2, 3] and Jaradat [13] gave an upper bound for the basis number of the semi-strong and the direct products of some special graphs. They proved the following results:

**Theorem 1.2.3** (Schmeichel). *For each  $n \geq 7$ ,  $b(K_n \bullet P_2) = 4$ .*

**Theorem 1.2.4** (Ali). *For each integers  $n, m$ ,  $b(K_m \bullet K_n) \leq 9$ .*

**Theorem 1.2.5** (Ali). *For any two cycles  $C_n$  and  $C_m$  with  $n, m \geq 3$ ,  $b(C_n \times C_m) = 3$ .*

**Theorem 1.2.6** (Jaradat). *For each bipartite graphs  $G$  and  $H$ ,  $b(G \times H) \leq 5 + b(G) + b(H)$ .*

**Theorem 1.2.7** (Jaradat). *For each bipartite graph  $G$  and cycle  $C$ ,  $b(G \times C) \leq 3 + b(G)$ .*

The strong product was studied by Imrich and Stadler [12] and Jaradat [15]. They gave the following results:

**Theorem 1.2.8** (Imrich and Stadler). *For any two graphs  $G$  and  $H$ ,  $l(G \boxtimes H) = l(G) + l(H) + 3[\dim C(G \boxtimes H) - \dim C(G) - \dim C(H)]$  and  $\lambda(G \boxtimes H) = \max\{3, \lambda(G), \lambda(H)\}$ .*

**Theorem 1.2.9** (Jaradat). *Let  $G$  be a bipartite graph and  $H$  be a graph. Then  $b(G \boxtimes H) \leq \max\{b(H) + 1, 2\Delta(H) + b(G) - 1, \lfloor \frac{3\Delta(T_G) + 1}{2} \rfloor, b(G) + 2\}$ .*

The results cited above trigger off the following question: Can we construct a minimum cycle basis and find the basis number of the wreath product of graphs? In this paper we will answer this question for a class of graphs. In fact, we construct a minimum cycle basis of the wreath product of two paths, a cycle with a path, a path with a star, a cycle with a star, a path with a wheel and a cycle with a wheel and we give their basis numbers. Moreover, we give the total lengths and lengths of longest cycles of the minimum cycle bases of the same.

In the rest of this paper,  $f_B(e)$  stand for the number of elements of  $B$  containing the edge  $e$  where  $B \subseteq \mathcal{C}(G)$ .

### 3. The Basis Number of the Wreath Product of Graphs

In this section, we investigate the basis number of the wreath product of two paths, a cycle with a path, a path with a star, a cycle with a star, a path with a wheel and a cycle with a wheel. Also, in this section, we shall say  $\mathcal{B}$  is a basis of  $\mathcal{C}(G)$ , rather than saying  $\mathcal{B}$  is a cycle basis of  $G$ . Let  $\{v_1, v_2, \dots, v_m\}$  be a set of vertices and  $ab$  be an edge. Also, let  $P_m = v_1v_2 \dots v_m$ . Then the automorphism group of the path  $P_m$  consists of two elements the identity,  $I$ , and the automorphism  $\alpha$  which is defined as follows:

$$\alpha(v_i) = v_{m-j+1}, j = 1, 2, \dots, m.$$

Therefore,  $ab \times P_m$  is decomposable into  $ab \square P_m \cup M_1$  where  $M_1$  is the graph with the edge set  $\{(a, v_j)(b, v_{m-j+1}), (a, v_{m-j+1})(b, v_j) | j = 1, 2, \dots, \lfloor m/2 \rfloor\}$ . Now, we define the following sets of cycles (see Figure 1):

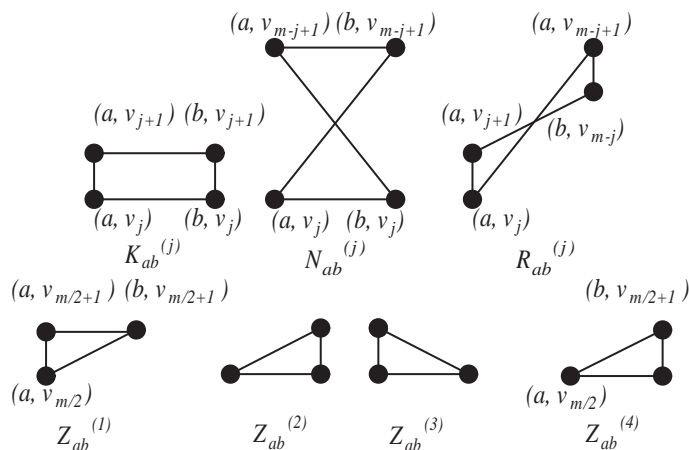


Figure 1. These graphs illustrate the cycles  $\mathcal{K}_{ab}^{(j)}$ ,  $\mathcal{N}_{ab}^{(j)}$ ,  $\mathcal{R}_{ab}^{(j)}$ ,  $\mathcal{Z}_{ab}^{(1)}$ ,  $\mathcal{Z}_{ab}^{(2)}$ ,  $\mathcal{Z}_{ab}^{(3)}$  and  $\mathcal{Z}_{ab}^{(4)}$  for even  $m$ .

$$\begin{aligned}\mathcal{K}_{ab} &= \left\{ \mathcal{K}_{ab}^{(j)} = (a, v_j)(b, v_j)(b, v_{j+1})(a, v_{j+1})(a, v_j) \mid j = 1, 2, \dots, m-1 \right\}, \\ \mathcal{N}_{ab} &= \left\{ \mathcal{N}_{ab}^{(j)} = (a, v_j)(b, v_j)(a, v_{m-j+1})(b, v_{m-j+1})(a, v_j) \mid j = 1, 2, \dots, \lfloor m/2 \rfloor \right\}, \\ \mathcal{R}_{ab} &= \left\{ \mathcal{R}_{ab}^{(j)} = (a, v_j)(a, v_{j+1})(b, v_{m-j})(b, v_{m-j+1})(a, v_j) \mid j = 1, 2, \dots, \lfloor m/2 \rfloor \right\}, \\ \mathcal{Z}_{ab} &= \left\{ \begin{array}{l} \mathcal{Z}_{ab}^{(1)} = (a, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor+1})(a, v_{\lfloor m/2 \rfloor+1})(a, v_{\lfloor m/2 \rfloor}), \\ \mathcal{Z}_{ab}^{(2)} = (a, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor+1})(a, v_{\lfloor m/2 \rfloor}), \\ \mathcal{Z}_{ab}^{(3)} = (a, v_{\lfloor m/2 \rfloor})(a, v_{\lfloor m/2 \rfloor+1})(b, v_{\lfloor m/2 \rfloor})(a, v_{\lfloor m/2 \rfloor}), \\ \mathcal{Z}_{ab}^{(4)} = (a, v_{\lfloor m/2 \rfloor+1})(b, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor+1})(a, v_{\lfloor m/2 \rfloor+1}) \end{array} \right\}.\end{aligned}$$

**Lemma 3.1.** *Let  $m$  be an odd integer. Then  $\mathcal{A}_{ab} = \mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab}$  is a linearly independent subset of  $\mathcal{C}(ab \times P_m)$ .*

**Proof.** We prove that  $\mathcal{K}_{ab}$  is linearly independent using mathematical induction on  $m$ . If  $m = 1$ , then  $\mathcal{K}_{ab}$  consists only of one cycle  $\mathcal{K}_{ab}^{(1)}$ . Thus,  $\mathcal{K}_{ab}$  is linearly independent. Assume that  $m$  is greater than 2 and it is true for less than  $m$ . Note that  $\mathcal{K}_{ab} = (\cup_{j=1}^{m-2} \mathcal{K}_{ab}^{(j)}) \cup \mathcal{K}_{ab}^{(m-1)}$ . Since  $\mathcal{K}_{ab}^{(m-1)}$  contains the edge  $(a, v_m)(b, v_m)$  which is not in any cycle of  $\cup_{j=1}^{m-2} \mathcal{K}_{ab}^{(j)}$ , as a result  $\mathcal{K}_{ab}$  is linearly independent. By a similar way we show that each of  $\mathcal{N}_{ab}$  and  $\mathcal{R}_{ab}$  are linearly independent. Any linear combination of cycles of  $\mathcal{R}_{ab}$  must contain an edge of the form  $(b, v_{m-j+1})(a, v_j)$  for some  $1 \leq j \leq \lfloor m/2 \rfloor$ , which is not in any cycle of  $\mathcal{K}_{ab}$ . Thus,  $\mathcal{K}_{ab} \cup \mathcal{R}_{ab}$  is linearly independent. Similarly, each linear combination of  $\mathcal{N}_{ab}$  contains an edge of the form  $(b, v_j)(a, v_{m-j+1})$  for some  $1 \leq j \leq \lfloor m/2 \rfloor$ , which is not in any cycle of  $\mathcal{K}_{ab} \cup \mathcal{R}_{ab}$ . Therefore,  $\mathcal{A}_{ab}$  is linearly independent. The proof is complete. ■

**Remark 3.1.** For an odd integer  $m$  let  $e \in ab \times P_m$ . Then

- (1) if  $e = (a, v_j)(a, v_{j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{K}_{ab}}(e) = f_{\mathcal{R}_{ab}}(e) = 1$  and  $f_{\mathcal{N}_{ab}}(e) = 0$ .
- (2) If  $e = (a, v_j)(a, v_{j+1})$  such that  $j \geq \lfloor m/2 \rfloor + 1$ , then  $f_{\mathcal{K}_{ab}}(e) = 1$  and  $f_{\mathcal{R}_{ab}}(e) = f_{\mathcal{N}_{ab}}(e) = 0$ .

- (3) If  $e = (b, v_j)(b, v_{j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{K}_{ab}}(e) = 1$  and  $f_{\mathcal{R}_{ab}}(e) = f_{\mathcal{N}_{ab}}(e) = 0$ .
- (4) If  $e = (b, v_j)(b, v_{j+1})$  such that  $j \geq \lfloor m/2 \rfloor + 1$ , then  $f_{\mathcal{K}_{ab}}(e) = f_{\mathcal{R}_{ab}}(e) = 1$  and  $f_{\mathcal{N}_{ab}}(e) = 0$ .
- (5) If  $e = (a, v_j)(b, v_j)$  such that  $j \neq \lfloor m/2 \rfloor + 1$ , then  $f_{\mathcal{K}_{ab}}(e) \leq 2$ ,  $f_{\mathcal{R}_{ab}}(e) = 0$  and  $f_{\mathcal{N}_{ab}}(e) = 1$ .
- (6) If  $e = (a, v_{\lfloor m/2 \rfloor + 1})(b, v_{\lfloor m/2 \rfloor + 1})$ , then  $f_{\mathcal{K}_{ab}}(e) = 2$ ,  $f_{\mathcal{R}_{ab}}(e) = 1$  and  $f_{\mathcal{N}_{ab}}(e) = 0$ .
- (7) If  $e = (a, v_j)(b, v_{m-j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{K}_{ab}}(e) = 0$ ,  $f_{\mathcal{R}_{ab}}(e) \leq 2$  and  $f_{\mathcal{N}_{ab}}(e) = 1$ .
- (8) If  $e = (b, v_j)(a, v_{m-j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{K}_{ab}}(e) = f_{\mathcal{R}_{ab}}(e) = 0$  and  $f_{\mathcal{N}_{ab}}(e) = 1$ .

**Lemma 3.2.** *Let  $m$  be an even integer. Then  $\mathcal{T}_{ab} = \mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}, \mathcal{Z}_{ab}^{(3)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$  is a linearly independent subset of  $\mathcal{C}(ab \times P_m)$ .*

**Proof.** Using the same argument as in Lemma 3.1, we have that  $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$  is linearly independent. Since  $\mathcal{Z}_{ab}^{(1)}$  contains  $(a, v_{\lfloor m/2 \rfloor})(a, v_{\lfloor m/2 \rfloor + 1})$  which is not in any cycle of  $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$ ,  $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$  is linearly independent. Also,  $\mathcal{Z}_{ab}^{(2)}$  contains  $(b, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor + 1})$  which is not in any cycle of  $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$ . Hence,  $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$  is linearly independent. Similarly,  $\mathcal{Z}_{ab}^{(3)}$  contains the edge  $(a, v_{\lfloor m/2 \rfloor + 1})(b, v_{\lfloor m/2 \rfloor})$  which is not in any cycle of  $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$ . Therefore,  $\mathcal{T}_{ab}$  is linearly independent. The proof is complete.  $\blacksquare$

Throughout this paper we consider  $\mathcal{B}_{ab} = \begin{cases} \mathcal{A}_{ab}, & \text{if } m \text{ is odd,} \\ \mathcal{T}_{ab}, & \text{if } m \text{ is even.} \end{cases}$

**Remark 3.2.** For any integer  $m$  let  $e \in ab \times P_m$ . Then from Lemma 3.1 and Lemma 3.2 and as in Remark 3.1 we have that

- (1) if  $e = (a, v_j)(a, v_{j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{B}_{ab}}(e) \leq 2$ .
- (2) If  $e = (a, v_j)(a, v_{j+1})$  such that  $j \geq \lfloor m/2 \rfloor + 1$ , then  $f_{\mathcal{B}_{ab}}(e) = 1$ .

- (3) If  $e = (b, v_j)(b, v_{j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{B}_{ab}}(e) = 1$ .
- (4) If  $e = (b, v_j)(b, v_{j+1})$  such that  $j \geq \lfloor m/2 \rfloor + 1$ , then  $f_{\mathcal{B}_{ab}}(e) \leq 2$ .
- (5) If  $e = (a, v_j)(b, v_j)$  such that  $j \neq 1, m$ , then  $f_{\mathcal{B}_{ab}}(e) \leq 3$ .
- (6) If  $e = (a, v_1)(b, v_1)$  or  $(a, v_m)(b, v_m)$ , then  $f_{\mathcal{B}_{ab}}(e) \leq 2$ .
- (7) If  $e = (a, v_j)(b, v_{m-j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{B}_{ab}}(e) \leq 3$ .
- (8) If  $e = (b, v_j)(a, v_{m-j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{B}_{ab}}(e) = 1$ .

**Lemma 3.3.** *If  $m \geq 3$ , then  $b(ab \times P_m) \geq 3$ .*

**Proof.** *Case 1.*  $m$  is odd. Let  $H_1$  be the subgraph of  $ab \times P_m$  induced by the following set of vertices  $\{(a, v_{\lfloor m/2 \rfloor}), (a, v_{\lfloor m/2 \rfloor + 2}), (b, v_{\lfloor m/2 \rfloor + 1}), (b, v_{\lfloor m/2 \rfloor}), (b, v_{\lfloor m/2 \rfloor + 2}), (a, v_{\lfloor m/2 \rfloor + 1})\}$ . Then  $H_1$  is isomorphic to  $K_{3,3}$  and so by MacLane's Theorem  $b(ab \times P_m) \geq 3$ .

*Case 2.*  $m$  is even. Let  $H_2$  be the subgraph with vertex set  $\{(a, v_{\lfloor m/2 \rfloor - 1}), (a, v_{\lfloor m/2 \rfloor}), (a, v_{\lfloor m/2 \rfloor + 1}), (a, v_{\lfloor m/2 \rfloor + 2}), (b, v_{\lfloor m/2 \rfloor - 1}), (b, v_{\lfloor m/2 \rfloor}), (b, v_{\lfloor m/2 \rfloor + 1}), (b, v_{\lfloor m/2 \rfloor + 2})\}$  and edge set consists of the following nine paths:  $P_1 = (a, v_{\lfloor m/2 \rfloor + 1})(a, v_{\lfloor m/2 \rfloor + 2})(b, v_{\lfloor m/2 \rfloor + 2})$ ,  $P_2 = (a, v_{\lfloor m/2 \rfloor + 1})(a, v_{\lfloor m/2 \rfloor})$ ,  $P_3 = (a, v_{\lfloor m/2 \rfloor + 1})(b, v_{\lfloor m/2 \rfloor})$ ,  $P_4 = (a, v_{\lfloor m/2 \rfloor})(a, v_{\lfloor m/2 \rfloor - 1})$ ,  $P_5 = (a, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor + 1})$ ,  $P_6 = (a, v_{\lfloor m/2 \rfloor - 1})(b, v_{\lfloor m/2 \rfloor - 1})(b, v_{\lfloor m/2 \rfloor})$ ,  $P_7 = (a, v_{\lfloor m/2 \rfloor - 1})(b, v_{\lfloor m/2 \rfloor + 2})$ ,  $P_8 = (b, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor + 1})$ ,  $P_9 = (b, v_{\lfloor m/2 \rfloor + 1})(b, v_{\lfloor m/2 \rfloor + 2})$ . Then  $H_2$  is homeomorphic to  $K_{3,3}$ . Thus, by MacLane's Theorem  $b(ab \times P_m) \geq 3$ . The proof is complete. ■

Let  $P_n = a_1 a_2 \dots a_n$ . Then the graph  $P_n \times P_m$  is decomposable into  $P_n \square P_m \cup M_2$  where  $M_2$  is the graph consisting of the following edge set  $\cup_{i=1}^{n-1} \{(a_i, v_j)(a_{i+1}, v_{n-j+1}), (a_i, v_{n-j+1})(a_{i+1}, v_j) \mid j = 1, 2, \dots, \lfloor m/2 \rfloor\}$ . Hence,  $|E(P_n \times P_m)| = n(m-1) + m(n-1) + 2(n-1)\lfloor m/2 \rfloor$ . Therefore,  $\dim \mathcal{C}(P_n \times P_m) = n(m-1) + m(n-1) + 2(n-1)\lfloor m/2 \rfloor - nm + 1 = mn - n - m + 2(n-1)\lfloor m/2 \rfloor + 1$ . The following lemma will be used frequently in the sequel.

**Lemma 3.4** (Jaradat, Alzoubi and Rawashdeh). *Let  $A$  and  $B$  be two linearly independent sets of cycles such that  $E(A) \cap E(B)$  is an edge set of a forest. Then  $A \cup B$  is linearly independent.*

**Theorem 3.5.** *Let  $P_n$  and  $P_m$  be two paths of order  $n, m \geq 2$ . Then  $b(P_n \times P_m) \leq 3$ . Moreover, the equality holds if  $n \geq 2$  and  $m \geq 3$ .*



**Proof.** By Lemma 3.3 we have that  $b(P_n \times P_m) \geq 3$  for any  $n \geq 2, m \geq 3$ . To prove that  $b(P_n \times P_m) \leq 3$ , it suffices to exhibit a 3-fold cycle basis. Define  $\mathcal{B}(P_n \times P_m) = \cup_{i=1}^{n-1} \mathcal{B}_{a_i a_{i+1}}$ . We now show that  $\mathcal{B}(P_n \times P_m)$  is linearly independent using mathematical induction on  $n$ . If  $n = 2$ , then  $\mathcal{B}(P_n \times P_m) = \mathcal{B}_{a_1 a_2}$  which is linearly independent by Lemma 3.1 and 3.2. Assume  $n \geq 3$  and it is true for less than or equal to  $n-2$ . Note that  $\mathcal{B}(P_n \times P_m) = (\cup_{i=1}^{n-2} \mathcal{B}_{a_i a_{i+1}}) \cup (\mathcal{B}_{a_{n-1} a_n})$  and  $E(\cup_{i=1}^{n-2} \mathcal{B}_{a_i a_{i+1}}) \cap E(\mathcal{B}_{a_{n-1} a_n}) = E(a_{n-1} \square P_m)$  which is an edge set of a path. Thus, by Lemma 3.4,  $\mathcal{B}(P_n \times P_m)$  is linearly independent. Since

$$|\mathcal{B}_{a_i a_{i+1}}| = |\mathcal{A}_{a_i a_{i+1}}| = (m-1) + 2 \lfloor m/2 \rfloor$$

if  $m$  is odd, and

$$\begin{aligned} |\mathcal{B}_{a_i a_{i+1}}| &= |\mathcal{T}_{a_i a_{i+1}}| = (m-2) + 2(\lfloor m/2 \rfloor - 1) + 3 \\ &= (m-1) + 2 \lfloor m/2 \rfloor, \end{aligned}$$

if  $m$  is even, we obtain

$$\begin{aligned} \mathcal{B}(P_n \times P_m) &= \sum_{i=1}^{n-1} |\mathcal{B}_{a_i a_{i+1}}| \\ &= (n-1)((m-1) + 2 \lfloor m/2 \rfloor) \\ &= mn - m - n + 2(n-1) \lfloor m/2 \rfloor + 1 \\ &= \dim \mathcal{C}(P_n \times P_m). \end{aligned}$$

Thus,  $\mathcal{B}(P_n \times P_m)$  is a basis for  $\mathcal{C}(P_n \times P_m)$ . We now show that  $\mathcal{B}(P_n \times P_m)$  is a 3-fold basis. Let  $e \in E(P_n \times P_m)$ . Then

- (1) If  $e = (a_i, v_j)(a_i, v_{j+1})$  such that  $j \leq \lfloor m/2 \rfloor$ , then  $f_{\mathcal{B}(P_n \times P_m)}(e) = f_{\mathcal{B}_{a_{i-1} a_i}}(e) + f_{\mathcal{B}_{a_i a_{i+1}}}(e) = 1 + 2 = 3$ .
- (2) If  $e = (a_i, v_j)(a_i, v_{j+1})$  such that  $j \geq \lfloor m/2 \rfloor + 1$ , then  $f_{\mathcal{B}(P_n \times P_m)}(e) = f_{\mathcal{B}_{a_{i-1} a_i}}(e) + f_{\mathcal{B}_{a_i a_{i+1}}}(e) = 2 + 1 = 3$ .
- (3) If  $e$  is not of the above form, then  $e$  belongs only to cycles of  $\mathcal{B}_{a_i a_{i+1}}$  for some  $1 \leq i \leq n-1$  and so  $f_{\mathcal{B}(P_n \times P_m)}(e) = f_{\mathcal{B}_{a_i a_{i+1}}}(e) \leq 3$ .

The proof is complete. ■

Now we turn our attention to deal with  $C_n \times P_m$ . Let  $C_n = a_1 a_2 \dots a_n a_1$ . Note that  $C_n \times P_m$  is decomposable into  $P_n \times P_m \cup M_3$  where  $M_3$  is the subgraph consists of the following edges:  $\{(a_1, v_j)(a_n, v_j) | j = 1, 2, \dots, m\} \cup$

$\{(a_1, v_j)(a_n, v_{m-j+1}), (a_1, v_{m-j+1})(a_n, v_j) | j = 1, 2, \dots, \lfloor m/2 \rfloor\}$ . Therefore,  $|E(C_n \times P_m)| = |E(P_n \times P_m)| + m + 2 \lfloor m/2 \rfloor$  and so  $\dim \mathcal{C}(C_n \times P_m) = \mathcal{C}(P_n \times P_m) + m + 2 \lfloor m/2 \rfloor = mn - n + 2n \lfloor m/2 \rfloor + 1$ .

Let  $G$  and  $H$  be two graphs and  $e = (a, u)(b, v) \in E(G \times H)$ . Then the *projection* of  $e$  in  $G$ ,  $P_G(e)$ , is defined to be the edge  $ab$  if  $a \neq b$  and to be the vertex  $a$  if  $a = b$  (see [12]).

**Lemma 3.6.**  $C_n \square v_i$  is relevant in  $C_n \times P_m$  for each  $i = 1, 2, \dots, n$ .

**Proof.** For simplicity assume that  $e_i = a_i a_{i+1}$  for each  $i = 1, 2, \dots, n-1$  and  $e_n = a_n a_1$ . Let  $O$  be any cycle of  $C_n \times P_m$  of length less than  $n$ . Since

$$E(e_i \times P_m) \cap E(e_j \times P_m) = \begin{cases} a_j \square P_m, & \text{if } j - i = 1, n - 1, \\ \phi, & \text{if } j - i \neq 1, n - 1, \end{cases}$$

(assuming  $i < j$ ), as a result  $O$  consists of edges of successive graphs of  $\{e_r \times P_m\}_{r=1}^n$ , say  $e_{l+1} \times P_m, e_{l+2} \times P_m, \dots, e_{l+k} \times P_m$  for some  $l, k < n$ . Since  $e_i \times P_m = (a_i \square P_m) \cup (a_{i+1} \square P_m) \cup X_i$  where  $X_i$  is a bipartite graph with independent sets of vertices  $a_i \times V(P_m)$  and  $a_{i+1} \times V(P_m)$ , as a result  $O$  contains even number of edges with projection  $e_{l+s}$  for each  $s = 1, 2, \dots, k$ . Thus, if

$$C_n \square v_1 = \sum_{i=1}^f O_i \pmod{2}.$$

where  $O_i$  is a cycle of length less than  $n$ . Then the number of edge of the ring sum  $O_1 \oplus O_2 \oplus \dots \oplus O_f$  with projection  $e_i$  in  $C_n$  is even for each  $i = 1, 2, \dots, n$ . In contrast, the number of edges of  $C_n \square v_1$  with projection  $e_i$  in  $C_n$  is 1 for each  $i = 1, 2, \dots, n$ . Thus,  $C_n \square v_1$  is relevant. The proof is complete. ■

**Theorem 3.7.** Let  $C_n$  be a cycle and  $P_m$  be a path. Then  $b(C_n \times P_m) = 3$ .

**Proof.** By Lemma 3.3, to prove that  $b(C_n \times P_m) \geq 3$ , it suffices to show that  $b(C_n \times P_2) \geq 3$ . Let  $H$  be the spanning subgraph of  $C_n \times P_2$  with the edge set consists of the following paths:  $(a_1, v_1)(a_1, v_2), (a_2, v_1)(a_2, v_2), (a_1, v_1)(a_2, v_2), (a_1, v_2)(a_2, v_1), (a_1, v_2)(a_n, v_1), (a_1, v_1)(a_n, v_2), (a_n, v_1)(a_n, v_2), (a_2, v_2)(a_3, v_1)(a_4, v_1) \dots (a_n, v_1)$  and  $(a_2, v_1)(a_3, v_2)(a_4, v_2) \dots (a_n, v_2)$ . Then,  $H$  is homeomorphic to  $K_{3,3}$  and so  $b(C_n \times P_2) \geq 3$ . Define  $\mathcal{B}(C_n \times P_m) = \mathcal{B}(P_n \times P_m) \cup \mathcal{B}_{a_n a_1} \cup \{C_n \square v_1\}$  where  $\mathcal{B}(P_n \times P_m)$  is as defined in

Theorem 3.5. Now, since  $E(P_n \times P_m) \cap E(\mathcal{B}_{a_n a_1}) = E(a_1 \square P_m) \cup E(a_n \square P_m)$  which is an edge set of a forest, by Lemma 3.4  $\mathcal{B}(P_n \times P_m) \cup \mathcal{B}_{a_n a_1}$  is linearly independent. Since each cycle of  $\mathcal{B}(C_n \times P_m) - \{C_n \square v_1\}$  is of length less than  $n$  and since  $C_n \square v_1$  is relevant in  $C_n \times P_m$  (Lemma 3.6),  $\mathcal{B}(C_n \times P_m)$  is linearly independent. Since

$$\begin{aligned} |\mathcal{B}(C_n \times P_m)| &= mn - m - n + (n-1)2 \lfloor m/2 \rfloor + 1 + (m-1) + 2 \lfloor m/2 \rfloor + 1 \\ &= mn - n + 2n \lfloor m/2 \rfloor + 1 \\ &= \dim \mathcal{C}(C_n \times P_m), \end{aligned}$$

$\mathcal{B}(C_n \times P_m)$  is a basis of  $\mathcal{C}(C_n \times P_m)$ . It is an easy task to show that  $\mathcal{B}(C_n \times P_m)$  is a 3-fold basis. The proof is complete. ■

Now consider  $S_m$  to be the star with the vertex set  $\{v_1, v_2, \dots, v_m\}$  and  $d_{S_m}(v_1) = m-1$ . Note that the automorphism group of  $S_m$  is isomorphic to the symmetric group on the set  $\{v_2, v_3, \dots, v_m\}$ . Therefore, for any  $\gamma \in \text{Aut}(G)$ ,  $\gamma(v_1) = v_1$ . Moreover, for any two vertices  $v_i, v_j$  such that  $2 \leq i, j \leq m$  there is an automorphism  $\alpha$  such that  $\alpha(v_i) = v_j$ . Hence, the graph  $ab \times S_m$  is decomposable into  $(a \square S_m) \cup (b \square S_m) \cup \{(a, v_1)(b, v_1)\} \cup ab[N_{m-1}]$  where  $N_{m-1}$  is the null graph with the vertex set  $\{v_2, v_3, \dots, v_m\}$ . Let

$$\mathcal{H}_{ab} = \{(a, v_j)(b, v_l)(a, v_{j+1})(b, v_{l+1})(a, v_j) : 2 \leq j, l \leq m\}.$$

Then  $\mathcal{H}_{ab}$  is the Schemichel's 4-fold basis of  $\mathcal{C}(ab[N_{m-1}])$  (see Theorem 2.4 in [21]). Moreover,

- (1) if  $e = (a, v_2)(b, v_m)$  or  $e = (a, v_m)(b, v_2)$  or  $e = (a, v_2)(b, v_2)$  or  $e = (a, v_m)(b, v_m)$ , then  $f_{\mathcal{H}_{ab}}(e) = 1$ .
- (2) If  $e = (a, v_2)(b, v_l)$  or  $(a, v_j)(b, v_2)$  or  $(a, v_m)(b, v_l)$  or  $(a, v_j)(b, v_m)$ , then  $f_{\mathcal{H}_{ab}}(e) \leq 2$ .
- (3) If  $e \in E(e[N_{m-1}])$  and is not of the above form, then  $f_{\mathcal{H}_{ab}}(e) \leq 4$ . Now, define the following sets of cycles (see Figure 2):

$$\mathcal{G}_{ab} = \left\{ \mathcal{G}_{ab}^{(l)} = (a, v_1)(a, v_l)(b, v_2)(a, v_{l+1})(a, v_1) \mid 2 \leq l \leq m-1 \right\},$$

and

$$\mathcal{S}_{ab} = \mathcal{K}_{ab}^{(1)} = \{(a, v_1)(a, v_2)(b, v_2)(b, v_1)(a, v_1)\}.$$

**Lemma 3.8.**  $\mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$  is a linearly independent subset of cycles of  $\mathcal{C}(ab \times S_m)$ .

**Proof.**  $\mathcal{G}_{ab}$  is a basis for the cycle subspace of  $\mathcal{C}(ab \times S_m)$  corresponding to the planar subgraph of  $ab \times S_m$  obtained by pasting all the cycles of  $\mathcal{G}_{ab}$ , which are 4-cycles, at the common edges of the successive cycles. Similarly,  $\mathcal{G}_{ba}$  is a basis for the cycle subspace of  $\mathcal{C}(ab \times S_m)$ . Since  $E(\mathcal{G}_{ab}) \cap E(\mathcal{G}_{ba}) = \{(a, v_1)(a, v_2), (a, v_2)(b, v_2), (b, v_2)(b, v_1)\}$  which is an edge set of a path and since  $\mathcal{S}_{ab}$  contains  $(b, v_1)(a, v_1)$  which occurs in no cycle of  $\mathcal{G}_{ab} \cup \mathcal{G}_{ba}$ ,  $\mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$  is a linearly independent set. The proof is complete. ■

**Lemma 3.9.**  $\mathcal{L}_{ab} = \mathcal{H}_{ab} \cup \mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$  is a linearly independent set of cycles.

**Proof.** The proof of this lemma follows by noting that every linear combination of cycles of  $\mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$  contains at least one edge of the set  $E(a \square S_m) \cup E(b \square S_m)$  which occurs in no cycle of  $\mathcal{H}_{ab}$ . The proof is complete. ■

**Remark 3.3.** Let  $e \in E(P_n \times S_m)$ . Then

- (1) if  $e = (a, v_1)(a, v_l)$  or  $(b, v_1)(b, v_l)$ , then  $f_{\mathcal{L}_{ab}}(e) \leq 2$ .
- (2) If  $e = (a, v_2)(b, v_2)$ , then  $f_{\mathcal{L}_{ab}}(e) = 4$ .
- (3) If  $e = (a, v_m)(b, v_m)$  or  $(a, v_1)(b, v_1)$ , then  $f_{\mathcal{L}_{ab}}(e) = 1$ .
- (4) If  $e = (a, v_2)(b, v_l)$  or  $(a, v_j)(b, v_2)$  such that  $m > j, l \geq 2$ , then  $f_{\mathcal{L}_{ab}}(e) \leq 4$ .
- (6) If  $e = (a, v_m)(b, v_l)$  or  $(a, v_j)(b, v_m)$  such that  $j, l \geq 2$ , then  $f_{\mathcal{L}_{ab}}(e) \leq 2$ .
- (7) If  $e$  is not of the above form, then  $f_{\mathcal{L}_{ab}}(e) = f_{\mathcal{H}_{ab}}(e) \leq 4$ .

The graph  $P_n \times S_m$  is decomposable into  $(\cup_{i=1}^n (a_i \square S_m)) \cup P_n[N_{m-1}] \cup (P_n \square v_1)$ . Thus  $|E(P_n \times S_m)| = m^2(n-1) - m(n-2) + n - 2$ . And so  $\dim \mathcal{C}(P_n \times S_m) = m^2(n-1) - 2m(n-1) + n - 1$ .

**Lemma 3.10.** If  $n \geq 4$  and  $(m^2 + 1)(n - 1) - m(5n - 2) + 3 \leq 0$ , then  $m < 6$ .

**Proof.**  $(m^2 + 1)(n - 1) \leq m(5n - 2) - 3$ . Hence  $(m^2 + 1)(n - 1) \leq 5m(n - 1) + 3(m - 1)$  which implies that  $(m^2 + 1) \leq 5m + 3(m - 1)/(n - 1)$  and so  $m \leq 5 + 3(m - 1)/(m(n - 1) - 1/m)$ . But  $3(m - 1)/(m(n - 1) - 1/m) < 1$ . Therefore,  $m < 6$ . ■

**Theorem 3.11.** For any path  $P_n$  of order  $n \geq 2$  and star  $S_m$ , we have that  $b(P_n \times S_m) \leq 4$ . Moreover, the equality holds if  $n \geq 4$  and  $m \geq 6$ .

**Proof.** Defined  $\mathcal{B}(P_n \times S_m) = \cup_{i=1}^{n-1} \mathcal{L}_{a_i a_{i+1}}$ . Then by Lemma 3.9 and using the same arguments as in Theorem 3.5 we have that  $\mathcal{B}(P_n \times S_m)$  is linearly independent. Now,

$$|\mathcal{L}_{a_i a_{i+1}}| = m^2 - 2m + 1.$$

Thus,

$$\mathcal{B}(P_n \times S_m) = \sum_{i=1}^{n-1} (m^2 - 2m + 1) = \dim \mathcal{C}(P_n \times S_m).$$

Therefore,  $\mathcal{B}(P_n \times S_m)$  is a basis for  $\mathcal{C}(P_n \times S_m)$ . Now we show that  $\mathcal{B}(P_n \times S_m)$  is a 4-fold basis. Let  $e \in E(P_n \times S_m)$ . Then

- (1) if  $e = (a_i, v_1)(a_i, v_l)$ , then  $f_{\mathcal{B}(P_n \times S_m)}(e) = f_{\mathcal{L}_{a_{i-1} a_i}}(e) + f_{\mathcal{L}_{a_i a_{i+1}}}(e) \leq 2 + 2 = 4$ .
- (2) If  $e = (a_i, v_j)(a_{i+1}, v_l)$  such that  $j, l \geq 2$ , then  $f_{\mathcal{B}(P_n \times S_m)}(e) = f_{\mathcal{L}_{a_i a_{i+1}}}(e) \leq 4$ .
- (3) If  $e = (a_i, v_1)(a_{i+1}, v_1)$ , then  $f_{\mathcal{B}(P_n \times S_m)}(e) = f_{\mathcal{L}_{a_i a_{i+1}}}(e) = 1$ .

Now, we show that  $b(P_n \times S_m) \geq 4$  for each  $n \geq 4$  and  $m \geq 6$ . Suppose that  $\mathcal{B}$  is a 3-fold basis of  $\mathcal{C}(P_n \times S_m)$  for each  $n \geq 4$  and  $m \geq 6$ . Since the girth of  $P_n \times S_m$  is 4, as a result

$$4 \dim \mathcal{C}(P_n \times S_m) \leq 3|E(P_n \times S_m)|$$

and so

$$\begin{aligned} 4(nm^2 - 2mn - m^2 + 2m + n - 1) &\leq 3(nm^2 - mn - m^2 + 2m + n - 2), \\ nm^2 - 5mn - m^2 + 2m + n + 2 &\leq 0, \\ m^2(n - 1) - m(5n - 2) + (n - 1) + 3 &\leq 0, \\ (m^2 + 1)(n - 1) - m(5n - 2) + 3 &\leq 0. \end{aligned}$$

By Lemma 3.10, for  $n \geq 4$ , we have that  $m < 6$ . This is a contradiction. The proof is complete.  $\blacksquare$

Now,  $C_n \times S_m$  is decomposable into  $P_n \times S_m \cup a_1 a_m [N_{m-1}] \cup \{(a_1, v_1)(a_m, v_1)\}$  where  $N_{m-1}$  is the null graph with the vertex set  $\{v_2, v_3, \dots, v_m\}$ . Thus,  $|E(C_n \times S_m)| = |E(P_n \times S_m)| + (m - 1)^2 + 1$ . Hence,  $\dim \mathcal{C}(C_n \times S_m) = \dim \mathcal{C}(P_n \times S_m) + (m - 1)^2 + 1 = n(m - 1)^2 + 1$ . By applying the same arguments as in Lemma 3.6 in  $C_n \times S_m$ , we have the following result:

**Lemma 3.12.**  $C_n \square v_i$  is relevant in  $C_n \times S_m$  for each  $i = 1, 2, \dots, n$ .

**Theorem 3.13.** *For any cycle  $C_n$  and star  $S_m$ , we have that  $b(C_n \times S_m) \leq 4$ . Moreover, the equality holds if  $n \geq 4$  and  $m \geq 5$ .*

**Proof.** Define  $\mathcal{B}(C_n \times S_m) = \mathcal{B}(P_n \times S_m) \cup \mathcal{L}_{a_n a_1} \cup \{C_n \square v_1\}$ . By using the same arguments as in Theorem 3.7, we show that  $\mathcal{B}(C_n \times S_m)$  is a 4-fold basis of  $\mathcal{C}(C_n \times S_m)$ . On the other hand to show that  $b(C_n \times S_m) \geq 4$ , we suppose that  $\mathcal{B}$  is a 3-fold basis of the space  $\mathcal{C}(C_n \times S_m)$  for each  $n \geq 4$  and  $m \geq 5$ , then we argue more or less as in Theorem 3.12 by taking into account that if  $n \geq 4$  and  $nm^2 - 5mn + n + 4 \leq 0$ , then  $m < 5$ . The proof is complete. ■

Now, consider  $W_m$  to be the wheel graph with vertex set  $\{v_1, v_2, \dots, v_m\}$  and  $d_{W_m}(v_1) = m - 1$ . Note that for  $m \geq 5$ ,  $\text{Aut}(W_m)$  is isomorphic to  $\text{Aut}(S_m)$ . Hence,  $ab \times W_m$  is decomposable into  $ab \times S_m \cup (a \square C) \cup (b \square C)$  where  $C = v_2 v_3 \dots v_m v_2$ . For each  $k = 2, 3, \dots, m$ , define,

$$\mathcal{P}_{ab}^{(k)} = \{\mathcal{P}_{ab}^{(k,j)} = (b, v_k)(a, v_j)(a, v_{j+1})(b, v_k) | 2 \leq j \leq m - 1\},$$

$$\mathcal{Q}_a = \{(a, v_2)(a, v_3) \dots (a, v_m)(a, v_2)\}.$$

Analogously, we define  $\mathcal{Q}_b$  (see Figure 2).

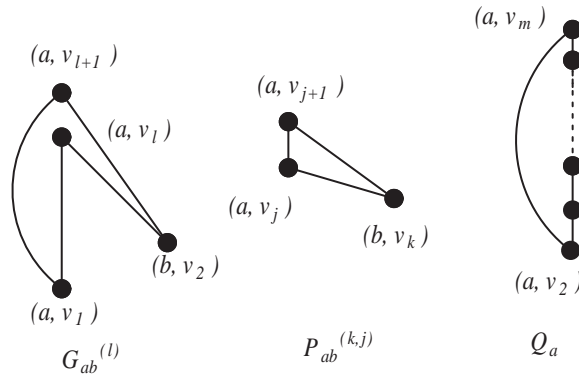


Figure 2. These graphs illustrate the cycles  $\mathcal{G}_{ab}^{(l)}$ ,  $\mathcal{P}_{ab}^{(k,j)}$  and  $\mathcal{Q}_a$ .

**Lemma 3.14.**  $(\cup_{k=2}^m \mathcal{P}_{ab}^{(k)}) \cup \mathcal{P}_{ba}^{(m)}$  is linearly independent.

**Proof.** Since  $\mathcal{P}_{ab}^{(k,j)}$  contains an edge of the form  $(a, v_j)(a, v_{j+1})$  which is not in any other cycle of  $\mathcal{P}_{ab}^{(k)}$ , as a result  $\mathcal{P}_{ab}^{(k)}$  is linearly independent for each

$k = 2, 3, \dots, m$ . Now by the inductive step, we assume that  $\cup_{k=2}^{m-1} \mathcal{P}_{ab}^{(k)}$  is linearly independent. Note that  $E(\cup_{k=2}^{m-1} \mathcal{P}_{ab}^{(k)}) \cap E(\mathcal{P}_{ab}^{(m)}) = E(a \square v_2 v_3 \dots v_m)$  which is an edge set of a path. Thus,  $\cup_{k=2}^m \mathcal{P}_{ab}^{(k)}$  is linearly independent. Now, each cycle  $\mathcal{P}_{ba}^{(m,j)}$  contains an edge of the form  $(b, v_j)(b, v_{j+1})$  which occurs in no other cycles of  $(\cup_{k=2}^m \mathcal{P}_{ab}^{(k)}) \cup \mathcal{P}_{ba}^{(m)}$ . Thus,  $(\cup_{k=2}^m \mathcal{P}_{ab}^{(k)}) \cup \mathcal{P}_{ba}^{(m)}$  is linearly independent. The proof is complete. ■

**Lemma 3.15.** *If  $n \geq 2$  and  $m^2(n-1) - 4m(n-1) - 2m + 3n - 1 \leq 0$ , then  $m < 6$ .*

**Proof.**  $m^2(n-1) \leq 4m(n-1) + 2m - 3n + 1$ . Thus,  $m \leq 4 + 2/(n-1) - 3n/m(n-1) + 1/m(n-1)$  which implies that  $m \leq 4 + 2 - 2/m(n-1)$ . Hence,  $m < 6$ . ■

**Lemma 3.16.** *If  $n \geq 2$  and  $m^2(n-1) - 4m(n-1) - 2m + 2 \leq 0$ , then  $m < 6$ .*

**Proof.** As in Lemma 3.15 we have that  $m \leq 4 + 2/(n-1) - 2/m(n-1)$  which implies that  $m \leq 4 + 2 - 2/m(n-1) < 6$ . ■

**Lemma 3.17.** *If  $n \geq 2$  and  $m^2(n-1) - 7m(n-1) - 5m + 3n + 2 \leq 0$ , then  $m < 12$ .*

**Proof.** As in Lemma 3.15, we have that  $m \leq 7 + 5/(n-1) - 3n/m(n-1) - 2/m(n-1) < 12$ .

Note that  $P_n \times W_m$  is decomposable into  $P_n \times S_m \cup (\cup_{i=1}^n (a_i \square C))$  where  $C = v_2 v_3 \dots v_m v_2$ . Thus,  $|E(P_n \times W_m)| = |E(P_n \times S_m)| + (m-1)n$ . Hence,  $\dim \mathcal{C}(P_n \times W_m) = (n-1)m^2 + 2m - mn - 1$ . ■

**Theorem 3.18.** *For each wheel  $W_m$  of order  $m \geq 5$  and path  $P_n$  of order  $n \geq 2$ , we have that  $b(P_n \times W_m) \leq 4$ . Moreover, the equality holds if  $n \geq 2$  and  $m \geq 12$ .*

**Proof.** Define  $\mathcal{B}(P_n \times W_m) = \mathcal{B}(P_n \times S_m) \cup (\cup_{i=1}^{n-1} \mathcal{P}_{a_i a_{i+1}}^{(m)}) \cup \mathcal{P}_{a_n a_{n-1}}^{(m)} \cup (\cup_{i=1}^n \mathcal{Q}_{a_i})$  where  $\mathcal{B}(P_n \times S_m)$  is defined as in Theorem 3.11. By Lemma 3.14 each one of  $\mathcal{P}_{a_i a_{i+1}}^{(m)}$  and  $\mathcal{P}_{a_n a_{n-1}}^{(m)}$  is linearly independent. Since  $E(\mathcal{P}_{a_i a_{i+1}}^{(m)}) \cap E(\mathcal{P}_{a_l a_{l+1}}^{(m)}) = \emptyset$  whenever  $i \neq l$ ,  $\cup_{i=1}^{n-1} \mathcal{P}_{a_i a_{i+1}}^{(m)}$  is linearly independent. Now, each linear combination of cycles of  $\mathcal{P}_{a_n a_{n-1}}^{(m)}$  contains at least one edge of

$E(a_n \square v_1 v_2 \dots v_m)$  which is not in any cycle of  $\cup_{i=1}^{n-1} \mathcal{P}_{a_i a_{i+1}}^{(m)}$ . Thus  $(\cup_{i=1}^{n-1} \mathcal{P}_{a_i a_{i+1}}^{(m)}) \cup \mathcal{P}_{a_n a_{n-1}}^{(m)}$  is linearly independent.  $E(\mathcal{Q}_{a_i}) \cap E(\mathcal{Q}_{a_j}) = \emptyset$  whenever  $i \neq j$ , also  $\mathcal{Q}_{a_i}$  is the only cycle of  $\mathcal{B}(P_n \times W_m)$  containing  $(a_i, v_m)(a_i, v_2)$  for each  $i$ . Therefore,  $(\cup_{i=1}^{n-1} \mathcal{P}_{a_i a_{i+1}}^{(m)}) \cup \mathcal{P}_{a_n a_{n-1}}^{(m)} \cup (\cup_{i=1}^n \mathcal{Q}_{a_i})$  is linearly independent. Any linear combination of cycles of  $(\cup_{i=1}^{n-1} \mathcal{P}_{a_i a_{i+1}}^{(m)}) \cup \mathcal{P}_{a_n a_{n-1}}^{(m)} \cup (\cup_{i=1}^n \mathcal{Q}_{a_i})$  contains at least one edge of the set  $\cup_{i=1}^n E(a_i \square v_2 v_3, \dots, v_m v_2)$  which is not in any cycle of  $\mathcal{B}(P_n \times S_m)$ . Thus,  $\mathcal{B}(P_n \times W_m)$  is linearly independent. Since

$$\begin{aligned} |\mathcal{B}(P_n \times W_m)| &= |\mathcal{B}(P_n \times S_m)| + \sum_{i=1}^{n-1} |\mathcal{P}_{a_i a_{i+1}}^{(m)}| + |\mathcal{P}_{a_n a_{n-1}}^{(m)}| + \sum_{i=1}^n |\mathcal{Q}_{a_i}| \\ &= m^2 n - 2mn - m^2 + 2m + (n-1) + (n-1)(m-2) \\ &\quad + (m-2) + n \\ &= (n-1)m^2 + 2m - mn - 1 \\ &= \dim \mathcal{C}(P_m \times W_n), \end{aligned}$$

$\mathcal{B}(P_m \times W_n)$  is a basis for  $\mathcal{C}(P_m \times W_n)$ . Now we show that  $\mathcal{B}(P_m \times W_n)$  is a 4-fold basis. Let  $e \in E(P_n \times W_m)$ . Then

- (1) if  $e = (a_i, v_1)(a_i, v_l)$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{L}_{a_{i-1} a_i}}(e) + f_{\mathcal{L}_{a_i a_{i+1}}}(e) \leq 2 + 2 = 4$ .
- (2) If  $e = (a_i, v_j)(a_{i+1}, v_l)$  such that  $m > j, l \geq 2$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{L}_{a_i a_{i+1}}}(e) \leq 4$ .
- (3) If  $e = (a_i, v_m)(a_{i+1}, v_l)$  or  $(a_i, v_l)(a_{i+1}, v_m)$  such that  $m > l \geq 2$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{L}_{a_i a_{i+1}}} + f_{\mathcal{P}_{a_i a_{i+1}}^{(m)} \cup \mathcal{P}_{a_n a_{n-1}}^{(m)}}(e) \leq 2 + 2 = 4$ .
- (4) If  $e = (a_i, v_1)(a_{i+1}, v_1)$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{S}_{a_i a_{i+1}}}(e) = 1$ .
- (5) If  $e = (a_i, v_2)(a_i, v_m)$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{Q}_{a_i}}(e) = 1$ .
- (6) If  $e = (a_i, v_m)(a_{i+1}, v_m)$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{H}_{a_i a_{i+1}}}(e) + f_{\mathcal{P}_{a_i a_{i+1}}^{(m)}}(e) + f_{\mathcal{P}_{a_n a_{n-1}}^{(m)}}(e) \leq 1 + 1 + 1 = 3$ .
- (7) If  $e = (a_i, v_j)(a_i, v_{j+1})$  such that  $j \geq 2$  and  $i \leq n-1$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{P}_{a_i a_{i+1}}^{(m)}}(e) + f_{\mathcal{Q}_{a_i}}(e) \leq 1 + 1 = 2$ .
- (8) If  $e = (a_n, v_j)(a_n, v_{j+1})$  such that  $j \geq 2$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{P}_{a_n a_{n-1}}^{(m)}}(e) + f_{\mathcal{Q}_{a_n}}(e) \leq 1 + 1 = 2$ .



(9) If  $e = (a_i, v_2)(a_{i+1}, v_m)$  or  $(a_i, v_m)(a_{i+1}, v_2)$ , then  $f_{\mathcal{B}(P_n \times W_m)}(e) = f_{\mathcal{L}_{a_n a_{n-1}}}(e) + f_{\mathcal{P}_{a_i a_{i+1}} \cup \mathcal{P}_{a_n a_{n-1}}^{(m)}}(e) \leq 2 + 1 = 2$ .

On the other hand, to show that  $b(P_n \times W_m) \geq 4$  for any  $n \geq 2$  and  $m \geq 12$ , we have to exclude any possibility for the cycle space  $\mathcal{C}(P_n \times W_m)$  to have a 3-fold basis for any  $n \geq 2$  and  $m \geq 12$ . To this end, suppose that  $\mathcal{B}$  is a 3-fold basis of the cycle space  $\mathcal{C}(P_n \times W_m)$  for any  $n \geq 2$  and  $m \geq 12$ . First, suppose that  $\mathcal{B}$  consists only of 3-cycles. Then  $|\mathcal{B}| \leq 3(m-1)n$  because any 3-cycle must contain an edge of  $E(a_i \square (v_2 v_3 \dots v_m v_2))$ , for  $i = 1, 2, \dots, n$  and each edge is of fold at most 3. This is equivalent to the inequality  $m^2(n-1) - mn + 2m - 1 \leq 3(m-1)n$  which implies that  $m^2(n-1) - 4m(n-1) - 2m + 3n - 1 \leq 0$  and so by Lemma 3.15,  $m < 6$ . This is a contradiction. Now, suppose that  $\mathcal{B}$  consists only of cycles of length greater than or equal to 4. Then  $4|\mathcal{B}| \leq 3|E(P_n \times W_m)|$  because the length of each cycle of  $\mathcal{B}$  greater than or equal to 4 and each edge is of fold at most 3. Thus,  $4(m^2(n-1) - mn + 2m - 1) \leq 3(m^2(n-1) + 2m - 2)$  which is equivalent to  $m^2(n-1) - 4m(n-1) - 2m + 2 \leq 0$  and so by Lemma 3.16,  $m < 6$ . This is a contradiction. Finally, Suppose that  $\mathcal{B}$  consists of  $r$  3-cycles and  $f$  cycles of length greater than or equal to 4. Then  $f \leq \lfloor (3(m^2(n-1) + 2m - 2) - 3r)/4 \rfloor$  because the length of each cycle of  $r$  is 3 and each cycle of  $f$  is at least 4 and the fold of each edge is at most 3. Hence,  $|\mathcal{B}| = r + f \leq r + \lfloor (3(m^2(n-1) + 2m - 2) - 3r)/4 \rfloor$  which implies that  $4(m^2(n-1) - mn + 2m - 1) \leq r + 3(m^2(n-1) + 2m - 2)$ . Thus,  $4(m^2(n-1) - mn + 2m - 1) \leq 3(m-1)n + 3(m^2(n-1) + 2m - 2)$ . By simplifying the inequality we have that  $m^2(n-1) - 7m(n-1) - 5m + 3n + 2 \leq 0$ . Thus, by Lemma 3.17  $m < 12$ . This is a contradiction. The proof is complete. ■

Now,  $C_n \times W_m$  is decomposable into  $P_n \times W_m \cup a_1 a_m [N_{m-1}] \cup \{(a_1, v_1)(a_m, v_1)\}$  where  $N_{m-1}$  is the null graph with the vertex set  $\{v_2, v_3, \dots, v_m\}$ . Thus,  $|E(C_n \times W_m)| = |E(P_n \times W_m)| + (m-1)^2 + 1$ . Hence,  $\dim \mathcal{C}(C_n \times W_m) = \dim \mathcal{C}(P_n \times W_m) + (m-1)^2 + 1 = nm^2 - mn + 1$ . By employing the same ideas as in Lemma 3.6, we have the following result.

**Lemma 3.19.**  $C_n \square v_i$  is relevant in  $C_n \times W_m$ .

**Theorem 3.20.** For each cycle  $C_n$  of order  $n$  and wheel  $W_m$  of order  $m \geq 5$ , we have that  $b(C_n \times W_m) \leq 4$ . Moreover, the equality holds if and only if  $n \geq 3$  and  $m \geq 7$ .

**Proof.** Define  $\mathcal{B}(C_n \times W_m) = \mathcal{B}(P_n \times W_m) \cup \mathcal{L}_{a_n a_1} \cup \{C_n \square v_1\}$ . By noting that  $E(\mathcal{L}_{a_n a_1}) \cap E(\mathcal{B}(P_n \times W_m)) = (a_1 \square S_m) \cup (a_n \square S_m)$  which is an edge set of a forest, we have that  $\mathcal{B}(P_n \times W_m) - \{C_n \square v_1\}$  is linearly independent. By Lemma 3.18,  $\mathcal{B}(P_n \times W_m)$  is linearly independent. Since

$$\begin{aligned} |\mathcal{B}(C_n \times W_m)| &= |\mathcal{B}(P_n \times W_m)| + |\mathcal{L}_{a_n a_1}| + 1 \\ &= nm^2 - mn + 1 \\ &= \dim \mathcal{C}(C_m \times W_n), \end{aligned}$$

$\mathcal{B}(C_m \times W_n)$  is a basis for  $\mathcal{C}(C_m \times W_n)$ . Now we can easily show that  $\mathcal{B}(C_m \times W_n)$  is a 4-fold basis. To show that  $\mathcal{C}(C_m \times W_n)$  has no 3-fold basis we argue more or less as in the last paragraph of Theorem 3.18. The proof is complete. ■

#### 4. The Minimum Cycle Bases of the Wreath Product of Graphs

In this section, we present minimum cycle bases (MCBs) for the wreath product of two paths, a cycle with a path, a path with a star, a cycle with a star, a path with a wheel and a cycle with a wheel. Moreover, we give the length of their maximum cycle.

**Theorem 4.1.**  $\mathcal{B}(P_n \times P_m)$  is a minimum cycle basis of  $P_n \times P_m$ .

**Proof.** Recall that a MCB is obtained by a greedy algorithm, that is, an algorithm that selects independent cycles starting with the shortest ones from the set of all cycles. We consider two cases:

*Case 1.*  $m$  is odd. Then the girth of  $P_n \times P_m$  is 4. Since each cycle of  $\mathcal{B}(P_n \times P_m)$  is of length 4, as a result  $\mathcal{B}(P_n \times P_m)$  is a MCB.

*Case 2.*  $m$  is even. Note that the only 3-cycles of  $P_n \times P_m$  are  $\cup_{i=1}^{n-1} \mathcal{Z}_{a_i a_{i+1}}$  and only three cycles of the four cycles of  $\mathcal{Z}_{a_i a_{i+1}}$  are linearly independent for each  $i = 1, 2, \dots, n-1$ . Thus,  $\{\mathcal{Z}_{a_i a_{i+1}}^{(1)}, \mathcal{Z}_{a_i a_{i+1}}^{(2)}, \mathcal{Z}_{a_i a_{i+1}}^{(3)} | i = 1, 2, \dots, n-1\}$  is a set consisting of the largest number of 3-cycles linearly independent of  $\mathcal{C}(P_n \times P_m)$ . Since  $\{\mathcal{Z}_{a_i a_{i+1}}^{(1)}, \mathcal{Z}_{a_i a_{i+1}}^{(2)}, \mathcal{Z}_{a_i a_{i+1}}^{(3)} | i = 1, 2, \dots, n-1\} \subseteq \mathcal{B}(P_n \times P_m)$  and  $\mathcal{B}(P_n \times P_m) - \{\mathcal{Z}_{a_i a_{i+1}}^{(1)}, \mathcal{Z}_{a_i a_{i+1}}^{(2)}, \mathcal{Z}_{a_i a_{i+1}}^{(3)} | i = 1, 2, \dots, n-1\}$  are 4-cycles,  $\mathcal{B}(P_n \times P_m)$  is MCB. The proof is complete. ■

**Corollary 4.2.**

$$l(P_n \times P_m) = \begin{cases} 4(mn - m - n + 2(n-1) \lfloor m/2 \rfloor + 1), & \text{if } n \text{ is odd,} \\ 4mn - 4m - 7n + 8(n-1) \lfloor m/2 \rfloor + 7, & \text{if } n \text{ is even.} \end{cases}$$

$$\lambda(P_n \times P_m) = 4.$$

**Theorem 4.3.** For each  $n \geq 4$ ,  $\mathcal{B}(C_n \times P_m)$  is a minimum cycle basis of  $C_n \times P_m$ .

**Proof.** By Lemma 3.6 and following, word by word, the same arguments as in the proof of Theorem 4.1 by taking into account that in Case 2 the set  $\{\mathcal{Z}_{a_i a_{i+1}}^{(1)}, \mathcal{Z}_{a_i a_{i+1}}^{(2)}, \mathcal{Z}_{a_i a_{i+1}}^{(3)} | i = 1, 2, \dots, n-1\} \cup \{\mathcal{Z}_{a_n a_1}^{(1)}, \mathcal{Z}_{a_n a_1}^{(2)}, \mathcal{Z}_{a_n a_1}^{(3)}\}$  is consisting of the largest number of 3-cycles linearly independent of  $\mathcal{C}(C_n \times P_m)$ , we have the result. The proof is complete. ■

**Corollary 4.4.**

$$\text{For } n \geq 4, l(C_n \times P_m) = \begin{cases} 4mn - 3n + 8n \lfloor m/2 \rfloor, & \text{if } n \text{ is odd,} \\ 4mn - 6n + 8n \lfloor m/2 \rfloor, & \text{if } n \text{ is even.} \end{cases}$$

and  $\lambda(C_n \times P_m) = n$ .

By noting that each of  $P_n \times S_m$  and  $C_r \times S_m$  has no 3-cycle for each  $r \geq 4$  and by Theorems 3.11 and 3.13 and Lemma 3.12, we have the following result.

**Theorem 4.5.** For each  $r \geq 4$ ,  $\mathcal{B}(P_n \times S_m)$  and  $\mathcal{B}(C_r \times S_m)$  are minimum cycle bases.

**Corollary 4.6.** For each  $r \geq 4$ ,  $l(P_n \times S_m) = 4(m^2 n - 2mn - m^2 + 2m + n - 1)$ ,  $l(C_r \times S_m) = 4(m^2 r - 2mr + r + 1)$ ,  $\lambda(P_n \times P_m) = 4$  and  $\lambda(C_r \times S_m) = r$ .

The proof of the following result is a straightforward.

**Lemma 4.7.** Let  $H$  be a subgraph of the graph  $G$ . Let  $A$  and  $B$  be a cycle basis and a minimum cycle basis of  $H$  and  $G$ , respectively. If  $A \subseteq B$ , then  $A$  is a minimum cycle basis of  $H$ .

In the following result  $\mathcal{B}_{a_i \square W_m}$  denotes to the cycle basis of the wheel  $a_i \square W_m$  consisting of 3-cycles.

**Theorem 4.8.**  $\mathcal{B}^*(P_n \times W_m) = (\cup_{i=1}^{n-1} \cup_{j=2}^m \mathcal{P}_{a_i a_{i+1}}^{(j)}) \cup (\cup_{i=1}^{n-1} \mathcal{P}_{a_{i+1} a_i}^{(m)}) \cup (\cup_{i=1}^n \mathcal{B}_{a_i \square W_m}) \cup (\cup_{i=1}^{n-1} \mathcal{S}_{a_i a_{i+1}})$  and  $\mathcal{B}^*(C_n \times W_m) = \mathcal{B}^*(P_n \times W_m) \cup$

$(\cup_{j=2}^m \mathcal{P}_{a_n a_1}^{(j)}) \cup \mathcal{P}_{a_1 a_n}^{(m)} \cup \mathcal{S}_{a_n a_1} \cup \{C_n \times v_1\}$  are minimum bases of  $P_n \times W_m$  and  $C_n \times W_m$ , respectively.

**Proof.** By Lemma 4.7, it is enough to show that  $\mathcal{B}^*(C_n \times W_m)$  is a minimum cycle basis of  $C_n \times W_m$  and  $\mathcal{B}^*(P_n \times W_m)$  is a cycle basis of  $P_n \times W_m$ . By Lemma 3.14, each one of the two sets  $(\cup_{j=2}^m \mathcal{P}_{a_i a_{i+1}}^{(j)}) \cup \mathcal{P}_{a_{i+1} a_i}^{(m)}$  and  $(\cup_{j=2}^m \mathcal{P}_{a_n a_1}^{(j)}) \cup \mathcal{P}_{a_1 a_n}^{(m)}$  is linearly independent. Note that

$$\begin{aligned} E \left( \left( \cup_{j=2}^m \mathcal{P}_{a_k a_{k+1}}^{(j)} \right) \cup \mathcal{P}_{a_{k+1} a_k}^{(m)} \right) \cap E \left( \cup_{i=1}^{k-1} \left( \left( \cup_{j=2}^m \mathcal{P}_{a_i a_{i+1}}^{(j)} \right) \cup \mathcal{P}_{a_{i+1} a_i}^{(m)} \right) \right) \\ = E(a_k \square v_2 v_3 \dots v_m) \end{aligned}$$

which is an edge set of path for each  $k = 1, 2, \dots, n-1$  and

$$\begin{aligned} E \left( \left( \cup_{j=2}^m \mathcal{P}_{a_n a_1}^{(j)} \right) \cup \mathcal{P}_{a_1 a_n}^{(m)} \right) \cap E \left( \cup_{i=1}^{n-1} \left( \left( \cup_{j=2}^m \mathcal{P}_{a_i a_{i+1}}^{(j)} \right) \cup \mathcal{P}_{a_{i+1} a_i}^{(m)} \right) \right) \\ = E(a_1 \square v_2 v_3 \dots v_m) \cup E(a_n \square v_2 v_3 \dots v_m) \end{aligned}$$

which is an edge set of a forest. Thus,  $(\cup_{i=1}^{n-1} \cup_{j=2}^m \mathcal{P}_{a_i a_{i+1}}^{(j)}) \cup (\cup_{i=1}^{n-1} \mathcal{P}_{a_{i+1} a_i}^{(m)}) \cup (\cup_{j=2}^m \mathcal{P}_{a_n a_1}^{(j)}) \cup \mathcal{P}_{a_1 a_n}^{(m)}$  is linearly independent set. Now, for each  $i = 1, 2, \dots, n$ ,  $\mathcal{B}_{a_i \times W_m}$  is a cycle basis of  $a_i \square W_m$ . Since  $E(\mathcal{B}_{a_i \square W_m}) \cap E(\mathcal{B}_{a_j \square W_m}) = \phi$  whenever  $i \neq j$ ,  $\cup_{i=1}^n \mathcal{B}_{a_i \square W_m}$  is linearly independent. Now any linear combination of  $\cup_{i=1}^n \mathcal{B}_{a_i \square W_m}$  contains an edge of  $\cup_{i=1}^n E(a_i \square (W-S))$  which is not in any cycle of  $(\cup_{i=1}^{n-1} \cup_{j=2}^m \mathcal{P}_{a_i a_{i+1}}^{(j)}) \cup (\cup_{i=1}^{n-1} \mathcal{P}_{a_{i+1} a_i}^{(m)}) \cup (\cup_{j=2}^m \mathcal{P}_{a_n a_1}^{(j)}) \cup \mathcal{P}_{a_1 a_n}^{(m)}$  where  $S$  is the star graph which is obtained from  $W_m$  by deleting the edges of the cycle  $v_2 v_3 \dots v_m v_2$ , as a result  $(\cup_{j=1}^n \cup_{i=1}^{m-1} \mathcal{P}_{a_i a_{i+1}}^{(j)}) \cup (\cup_{i=1}^{m-1} \mathcal{P}_{a_n a_1}^{(j)}) \cup (\cup_{j=2}^m \mathcal{P}_{a_n a_1}^{(j)}) \cup \mathcal{P}_{a_1 a_n}^{(m)} \cup (\cup_{i=1}^n \mathcal{B}_{a_i \square W_m})$  is linearly independent. Now,  $(\cup_{i=1}^{n-1} \mathcal{S}_{a_i a_{i+1}}) \cup \mathcal{S}_{a_n a_1}$  is a cycle basis of the planar graph  $P_n \square v_1 v_2$  which obtained by pasting all the cycle of  $(\cup_{i=1}^{n-1} \mathcal{S}_{a_i a_{i+1}}) \cup \mathcal{S}_{a_n a_1}$ , which are 4-cycles, at the common edges of the successive cycles. Note that any linear combinations of cycles of  $(\cup_{i=1}^{n-1} \mathcal{S}_{a_i a_{i+1}}) \cup \mathcal{S}_{a_n a_1}$  contains an edge of  $E(P_n \square v_1)$  which is not in any cycle of  $(\cup_{j=1}^n \cup_{i=1}^{m-1} \mathcal{P}_{a_i a_{i+1}}^{(j)}) \cup (\cup_{i=1}^{m-1} \mathcal{P}_{a_n a_1}^{(j)}) \cup (\cup_{j=2}^m \mathcal{P}_{a_n a_1}^{(j)}) \cup \mathcal{P}_{a_1 a_n}^{(m)} \cup (\cup_{i=1}^n \mathcal{B}_{a_i \square W})$ , thus  $(\cup_{j=1}^n \cup_{i=1}^{m-1} \mathcal{P}_{a_i a_{i+1}}^{(j)}) \cup (\cup_{i=1}^{m-1} \mathcal{P}_{a_n a_1}^{(j)}) \cup (\cup_{j=2}^m \mathcal{P}_{a_n a_1}^{(j)}) \cup \mathcal{P}_{a_1 a_n}^{(m)} \cup (\cup_{i=1}^n \mathcal{B}_{a_i \square W}) \cup (\cup_{i=1}^{n-1} \mathcal{S}_{a_i a_{i+1}}) \cup \mathcal{S}_{a_n a_1}$  is linearly independent. Now By Lemma 3.19,  $C_n \square v_1$  is relevant. Thus,  $\mathcal{B}^*(C_m \times W_n)$  is a linearly independent. Since

$$\begin{aligned}
|\mathcal{B}^*(C_n \times W_m)| &= (m-1)(m-2)(n-1) + (n-1)(m-2) + (m-1)n \\
&\quad + (n-1) + (m-2)(m-1) + (m-2) + 1 + 1 \\
&= m^2n - mn + 1 = \dim \mathcal{C}(C_m \times W_n)
\end{aligned}$$

$\mathcal{B}^*(C_n \times W_m)$  is a cycle basis of  $C_n \times W_m$ . Since each cycle of  $\mathcal{B}^*(C_m \times W_n) - \{(\cup_{i=1}^{n-1} \mathcal{S}_{a_i a_{i+1}}) \cup \mathcal{S}_{a_n a_1} \cup (C_n \square v_1)\}$  is of length three and since the smallest cycle contains any edge of  $(a_i, v_1)(a_{i+1}, v_1)$ ,  $(a_1, v_1)(a_n, v_1)$  is of length 4 and by Lemma 3.19, we have that each cycle of  $\mathcal{B}^*(C_n \times W_m)$  is relevant in  $C_n \times W_m$ . Therefore,  $\mathcal{B}^*(C_m \times W_n)$  is a minimum cycle basis  $C_m \times W_n$ . Since  $\mathcal{B}^*(P_m \times W_n) \subset \mathcal{B}^*(C_m \times W_n)$  and  $|\mathcal{B}^*(P_m \times W_n)| = m^2n - mn - m^2 + 2m - 1 = \dim \mathcal{C}(P_m \times W_n)$ , we have that  $\mathcal{B}^*(P_m \times W_n)$  is a cycle basis of  $P_m \times W_n$ . The proof is complete. ■

**Corollary 4.9.**  $l(P_n \times W_m) = 3m^2n - 3mn - 3m^2 + 6m - 3$ ,  $l(C_n \times W_m) = 3m^2n - 3mn + n$ ,  $\lambda(P_n \times W_m) = 4$  and  $\lambda(C_n \times W_m) = n$ .

## References

- [1] M. Anderson and M. Lipman, *The wreath product of graphs*, in: Graphs and Applications (Boulder, Colo., 1982), (Wiley-Intersci. Publ., Wiley, New York, 1985) 23–39.
- [2] A.A. Ali, *The basis number of complete multipartite graphs*, Ars Combin. **28** (1989) 41–49.
- [3] A.A. Ali, *The basis number of the direct product of paths and cycles*, Ars Combin. **27** (1989) 155–163.
- [4] A.A. Ali and G.T. Marougi, *The basis number of cartesian product of some graphs*, J. Indian Math. Soc. **58** (1992) 123–134.
- [5] A.S. Alsardary and J. Wojciechowski, *The basis number of the powers of the complete graph*, Discrete Math. **188** (1998) 13–25.
- [6] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications* (America Elsevier Publishing Co. Inc., New York, 1976).
- [7] W.-K. Chen, *On vector spaces associated with a graph*, SIAM J. Appl. Math. **20** (1971) 525–529.
- [8] D.M. Chickering, D. Geiger and D. Heckerman, *On finding a cycle basis with a shortest maximal cycle*, Inform. Process. Lett. **54** (1994) 55–58.

- [9] L.O. Chua and L. Chen, *On optimally sparse cycles and coboundary basis for a linear graph*, IEEE Trans. Circuit Theory **20** (1973) 54–76.
- [10] G.M. Downs, V.J. Gillet, J.D. Holliday and M.F. Lynch, *Review of ring perception algorithms for chemical graphs*, J. Chem. Inf. Comput. Sci. **29** (1989) 172–187.
- [11] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition* (Wiley, New York, 2000).
- [12] W. Imrich and P. Stadler, *Minimum cycle bases of product graphs*, Australas. J. Combin. **26** (2002) 233–244.
- [13] M.M.M. Jaradat, *On the basis number of the direct product of graphs*, Australas. J. Combin. **27** (2003) 293–306.
- [14] M.M.M. Jaradat, *The basis number of the direct product of a theta graph and a path*, Ars Combin. **75** (2005) 105–111.
- [15] M.M.M. Jaradat, *An upper bound of the basis number of the strong product of graphs*, Discuss. Math. Graph Theory **25** (2005) 391–406.
- [16] M.M.M. Jaradat, M.Y. Alzoubi and E.A. Rawashdeh, *The basis number of the Lexicographic product of different ladders*, SUT J. Math. **40** (2004) 91–101.
- [17] A. Kaveh, *Structural Mechanics, Graph and Matrix Methods*. Research Studies Press (Exeter, UK, 1992).
- [18] G. Liu, *On connectivities of tree graphs*, J. Graph Theory **12** (1988) 435–459.
- [19] S. MacLane, *A combinatorial condition for planar graphs*, Fundamenta Math. **28** (1937) 22–32.
- [20] M. Plotkin, *Mathematical basis of ring-finding algorithms in CIDS*, J. Chem. Doc. **11** (1971) 60–63.
- [21] E.F. Schmeichel, *The basis number of a graph*, J. Combin. Theory (B) **30** (1981) 123–129.
- [22] P. Vismara, *Union of all the minimum cycle bases of a graph*, Electr. J. Combin. **4** (1997) 73–87.
- [23] D.J.A. Welsh, *Kruskal's theorem for matroids*, Proc. Cambridge Phil. Soc. **64** (1968) 3–4.

Received 3 March 2005

Revised 9 September 2005