

DEFINING SETS IN (PROPER) VERTEX COLORINGS  
OF THE CARTESIAN PRODUCT OF A CYCLE WITH  
A COMPLETE GRAPH

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**Abstract**

In a given graph  $G = (V, E)$ , a set of vertices  $S$  with an assignment of colors to them is said to be a defining set of the vertex coloring of  $G$ , if there exists a unique extension of the colors of  $S$  to a  $c \geq \chi(G)$  coloring of the vertices of  $G$ . A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by  $d(G, c)$ .

The  $d(G = C_m \times K_n, \chi(G))$  has been studied. In this note we show that the exact value of defining number  $d(G = C_m \times K_n, c)$  with  $c > \chi(G)$ , where  $n \geq 2$  and  $m \geq 3$ , unless the defining number  $d(K_3 \times C_{2r}, 4)$ , which is given an upper and lower bounds for this defining number. Also some bounds of defining number are introduced.

**Keywords:** graph coloring, defining set, cartesian product.

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1. INTRODUCTION

A  $c$ -coloring (*proper  $c$ -coloring*) of a graph  $G$  is an assignment of  $c$  different colors to the vertices of  $G$ , such that no two adjacent vertices receive the same color. The *vertex chromatic number* of a graph  $G$ , denoted by  $\chi(G)$ , is the minimum number  $c$ , for which there exists a  $c$ -coloring for  $G$ . The maximum degree of the vertices in  $G$  is  $\Delta(G)$  and the minimum degree is  $\delta(G)$  and  $G$  is *regular* if  $\Delta(G) = \delta(G)$ . It is  *$k$ -regular* graph if the common

degree is  $k$  (see [9]). In a given graph  $G = (V, E)$ , a set of vertices  $S$  with an assignment of colors to them is said to be a *defining set of the vertex coloring of  $G$* , if there exists a unique extension of the colors of  $S$  to a  $c \geq \chi(G)$  coloring of the vertices of  $G$ . A defining set with the minimum cardinality is called a *minimum defining set* and its cardinality is the *defining number*, denoted by  $d(G, c)$ . We will use standard notations such as  $K_n$  for the complete graph on  $n$  vertices,  $C_m$  for the cycle of size  $m$  and  $G \times H$  for cartesian product of  $G$  and  $H$ . There are some papers on defining set of graphs, especially  $d(K_n \times K_n, \chi)$  (the critical set of Latin squares of order  $n$ ),  $d(C_m \times K_n, \chi)$ ,  $d(G, \chi = k)$  where  $G$  is a  $k$ -regular graph and defining set on block designs. The interested reader may see [1, 4, 5, 7, 8] and their references.

The following results can be found in [3]:

- (1)  $d(C_m \times K_3, \chi) = \lfloor \frac{m}{2} \rfloor + 1$ ,
- (2)  $m \leq d(C_m \times K_4, \chi) \leq m + 1$ ,
- (3)  $d(C_m \times K_5, \chi) = 2m$  for even  $m$  and  $2m \leq d(C_m \times K_5, \chi) \leq 2m + 1$  for odd  $m$ .

The following results can be found in [7]:

- (4)  $d(C_m \times K_5, \chi) = 2m$ , for odd  $m(\geq 5)$ ,
- (5)  $d(C_m \times K_4, \chi) = m + 1$ .

The following results can be found in [6]:

- (6)  $d(C_m \times K_n, \chi) = m(n - 3)$  for  $n \geq 6$ ,
- (7)  $d(C_{2n+1} \times K_2, \chi) = n + 1$ .

The followings are useful.

**Definition A** [2]. A graph  $G$  with  $n$  vertices, is called a uniquely 2-list colorable graph, if there exists  $S_1, S_2, \dots, S_n$ , a list of colors on its vertices, each of size 2, such that there is a unique coloring for  $G$  from this list of colors.

**Theorem B** [2]. A connected graph is uniquely 2-list colorable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph.

Let  $G$  be a  $k$ -regular graph and vertex colored with  $k$  colors. Let  $C$  be a cycle in  $G$ , then each vertex of  $C$  has at least two choice for coloring, in other

words  $C$  is at least 2-list vertex colorable, if all vertices of  $V(G) \setminus V(C)$  have been already colored. So by Theorem B the cycle  $C$  is not uniquely 2-list colorable. Now we have

**Lemma C** [6]. *If  $G$  is  $k$ -regular graph and which is colored with  $k$  colors, then every cycle in  $G$  has a vertex in defining set of  $G$ .*

If  $G = C_m \times K_n$  then, each subgraph  $K_n$  of  $G$  is said to be a *row* and each subgraph  $C_m$  of  $G$  is said to be a *column*. If  $G = K_n \times C_m$  then, each subgraph  $K_n$  of  $G$  is said to be a column and each subgraph  $C_m$  of  $G$  is said to be a row.

It is well known that  $\chi(C_m \times K_n) = \chi(K_n \times C_m) = n$  for  $n > 2$  or for  $n = 2$  and even  $m$ . Also  $\chi(C_{2r+1} \times K_2) = \chi(K_2 \times C_{2r+1}) = 3$ .

## 2. $d(C_m \times K_n, n + i)$

In this section we derive  $d(C_m \times K_n, n + i)$  for  $n, m \geq 4$  and  $i \geq 0$ . We start with the following lemma.

**Lemma 2.1.** *If  $G = C_m \times K_n$  is colored with  $n + i$  colors for  $0 \leq i \leq 3$ , then for each row, there exist at least,  $n + i - 3$  vertices in defining set.*

**Proof.** Assume that, there exists a row for which the defining set contains  $k < n + i - 3$  vertices and all other rows are completely colored. The induced subgraph of the non colored vertices of this row is a complete graph and cannot be uniquely colored by Theorem B. ■

In the following arrays the non indexed labels denote the colors of the vertices in the defining set of the graph  $C_m \times K_n$ , the indexed labels denote the colors of the vertices out of defining set and the indices denote the ordering of the coloring of these vertices.

**Theorem 2.1.** *For  $n, m \geq 4$ ,  $d(C_m \times K_n, n + 1) = m(n - 2)$ .*

**Proof.** Let  $G = C_m \times K_n$ . From Lemma 2.1 we obtain  $d(G, n + 1) \geq m(n - 2)$ . To show equality we give a defining set  $S$  of size  $m(n - 2)$  as in following arrays.

(1) For  $m \geq 4$  and  $n = 4$ , consider the arrays

$$\begin{bmatrix} 1 & 2 & 4_1 & 5_2 \\ 2_4 & 4_3 & 5 & 3 \\ 4 & 1 & 2_5 & 5_6 \\ 2_8 & 5_7 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4_1 & 5_2 \\ 2_4 & 4_3 & 5 & 3 \\ 3_5 & 1 & 4_6 & 5 \\ 4 & 3_7 & 2 & 1_8 \\ 2_9 & 5 & 3 & 4_{10} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 4_1 & 5_2 \\ 2_4 & 4_3 & 5 & 3 \\ 4 & 1 & 3_6 & 2_5 \\ 3_7 & 2_8 & 1 & 5 \\ 2 & 4 & 5_9 & 3_{10} \\ 4_{12} & 5_{11} & 3 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 4_1 & 5_2 \\ 2_4 & 4_3 & 5 & 3 \\ 5_6 & 1 & 3_5 & 4 \\ 3 & 5_7 & 2 & 1_8 \\ 5_{10} & 4 & 1_9 & 3 \\ 2 & 3_{12} & 5 & 1_{11} \\ 5_{13} & 1_{14} & 3 & 4 \end{bmatrix}$$

for  $C_4 \times K_4$ ,  $C_5 \times K_4$ ,  $C_6 \times K_4$  and  $C_7 \times K_4$  respectively with  $5 = 4 + 1$  colors.

(2) For  $m \geq 4$  and  $n = 5$ , consider the arrays

$$\begin{bmatrix} 1 & 2 & 3 & 6_2 & 4_1 \\ 2_4 & 3_3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6_6 & 4_5 \\ 2_8 & 3_7 & 6 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4_1 & 5_2 \\ 3_3 & 1_4 & 4 & 5 & 6 \\ 2 & 4_5 & 5 & 3 & 1_6 \\ 5_8 & 6 & 1_7 & 4 & 3 \\ 3_{10} & 1 & 2 & 6 & 4_9 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 4_2 & 5_1 \\ 2_4 & 3_3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6_5 & 4_6 \\ 1_8 & 2_7 & 6 & 4 & 5 \\ 2 & 3 & 1 & 5_9 & 6_{10} \\ 3_{12} & 1_{11} & 5 & 6 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 6_2 & 4_1 \\ 3_3 & 1_4 & 4 & 5 & 6 \\ 2 & 4_5 & 5 & 3 & 1_6 \\ 5_8 & 6 & 1_7 & 4 & 3 \\ 4_9 & 1 & 2 & 6 & 5_{10} \\ 3 & 4_{11} & 5 & 1_{12} & 2 \\ 4_{13} & 6 & 2 & 3_{14} & 5 \end{bmatrix}$$

for  $C_4 \times K_5$ ,  $C_5 \times K_5$ ,  $C_6 \times K_5$  and  $C_7 \times K_5$  respectively with  $6 = 5 + 1$  colors.

(3) For  $m \geq 4$  and  $n \geq 6$ , consider the following arrays,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_2 & (n-1)_1 \\ 2_4 & 3_3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\ 4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_5 & 2 & (n-1)_6 \\ 5_7 & 4_8 & 6 & 7 & 8 & \cdots & n-1 & 2 & n+1 & 3 & n \end{bmatrix},$$

$$\begin{bmatrix} 1 & & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_2 & (n-1)_1 \\ 2_4 & & 3_3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\ 4 & & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_5 & 2 & (n-1)_6 \\ 2_8 & & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5_7 & 3 \\ (n+1)_9 & & 5 & 6 & 7 & 8 & \cdots & n-1 & 2_{10} & 3 & 4 & n \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_2 & (n-1)_1 \\ 2_4 & 3_3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\ 4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_5 & 2 & (n-1)_6 \\ 2_8 & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5_7 & 3 \\ 5 & 1 & 6 & 7 & 8 & \cdots & n-1 & 2 & n_9 & 4 & (n+1)_{10} \\ 2_{12} & 4_{11} & 5 & 6 & 7 & \cdots & n-2 & n+1 & 3 & n-1 & n \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_2 & (n-1)_1 \\ 2_4 & 3_3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\ 4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_5 & 2 & (n-1)_6 \\ 2_8 & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5_7 & 3 \\ 1_9 & 5 & 6 & 7 & 8 & \cdots & n-1 & 3 & 2_{10} & 4 & n \\ n+1 & 6_{11} & 7 & 8 & 9 & \cdots & n & 2 & 3 & 5_{12} & 1 \\ 3_{13} & 4 & 5 & 6 & 7 & \cdots & n-2 & n-1 & 2 & 1_{14} & n \end{bmatrix}$$

for  $C_4 \times K_n$ ,  $C_5 \times K_n$ ,  $C_6 \times K_n$  and  $C_7 \times K_n$  respectively with  $n+1$  colors ( $n \geq 6$ ). The above arrays show that  $d(C_m \times K_n, n+1) = m(n-2)$  for ( $4 \leq m \leq 7$ ) and  $n \geq 4$ .

To obtain a defining set for  $C_m \times K_n$ , with  $m \geq 8$ , one can write  $m = 4t + r$  where  $4 \leq r \leq 7$  and  $t \geq 1$  are integers. We successively treat the  $t$  above arrays for  $C_4 \times K_n$  and then treat to with the one for  $C_r \times K_n$ . So  $d(C_m \times K_n, n+1) = m(n-2)$  for  $n$ ,  $m \geq 4$ . ■

**Theorem 2.2.** For  $n, m \geq 4$ ,  $d(C_m \times K_n, n+2) = m(n-1)$ .

**Proof.** Let  $G = C_m \times K_n$ . From Lemma 2.1 we obtain  $d(G, n+2) \geq m(n-1)$ . To show equality we give a defining set,  $S$  of size  $m(n-1)$  as in following arrays.

(1) For  $m \geq 4$  and  $n = 4$ , consider the arrays

$$\begin{bmatrix} 1 & 2 & 3 & 6_1 \\ 4 & 3_2 & 6 & 5 \\ 6_3 & 1 & 2 & 3 \\ 5 & 6 & 1_4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 6_1 \\ 2 & 4 & 6_2 & 5 \\ 3 & 6_3 & 1 & 2 \\ 2_4 & 5 & 4 & 1 \\ 6 & 3_5 & 1 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 6_1 \\ 2 & 4 & 6_2 & 5 \\ 3 & 6_3 & 1 & 2 \\ 2_4 & 5 & 4 & 1 \\ 6 & 4_5 & 1 & 2 \\ 2_6 & 3 & 5 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 6_1 \\ 2 & 4 & 6_2 & 5 \\ 3 & 6_3 & 1 & 2 \\ 6_4 & 5 & 4 & 1 \\ 2 & 6_5 & 1 & 4 \\ 4 & 3 & 6_6 & 2 \\ 6 & 1_7 & 5 & 4 \end{bmatrix}$$

for  $C_4 \times K_4$ ,  $C_5 \times K_4$ ,  $C_6 \times K_4$  and  $C_7 \times K_4$  respectively with  $6 = 4 + 2$  colors.

(2) For  $m \geq 4$  and  $n = 5$ , consider the arrays

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 7_1 \\ 2 & 3 & 5 & 7_2 & 6 \\ 3 & 4 & 7_3 & 1 & 2 \\ 7 & 1_4 & 6 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 & 7_1 \\ 2 & 3 & 5 & 7_2 & 6 \\ 3 & 4 & 7_3 & 1 & 2 \\ 7 & 5_4 & 6 & 2 & 1 \\ 4_5 & 3 & 2 & 6 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 7_1 \\ 2 & 3 & 5 & 7_2 & 6 \\ 3 & 4 & 7_3 & 1 & 2 \\ 7 & 5_4 & 6 & 2 & 1 \\ 5_5 & 3 & 1 & 6 & 4 \\ 2 & 4 & 6_6 & 7 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 & 7_1 \\ 2 & 3 & 5 & 7_2 & 6 \\ 3 & 4 & 7_3 & 1 & 2 \\ 5 & 7_4 & 6 & 3 & 1 \\ 7_5 & 2 & 4 & 6 & 3 \\ 1 & 5_6 & 6 & 3 & 4 \\ 2 & 7 & 4_7 & 1 & 5 \end{bmatrix}$$

for  $C_4 \times K_5$ ,  $C_5 \times K_5$ ,  $C_6 \times K_5$  and  $C_7 \times K_5$  respectively with  $7 = 5 + 2$  colors.

(3) For  $m \geq 4$  and  $n \geq 6$ , consider the following arrays,

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_1 \\ 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_2 & n+1 \\ 4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_3 & 1 & 2 \\ 5_4 & 6 & 7 & \cdots & n-1 & n+1 & n+2 & 3 & 2 & n \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_1 \\ 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_2 & n+1 \\ 4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_3 & 1 & 2 \\ 2 & 6 & 7 & \cdots & n-1 & n & 4_4 & 3 & 5 & 1 \\ 4_5 & 5 & 6 & \cdots & n-2 & n-1 & n+2 & n+1 & 3 & n \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_1 \\ 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_2 & n+1 \\ 4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_3 & 1 & 2 \\ 5 & 6 & 7 & \cdots & n-1 & n & (n+2)_4 & 3 & 2 & 4 \\ 4 & 7 & 8 & \cdots & n & 5 & 1 & (n+1)_5 & 6 & 2 \\ 5_6 & 6 & 7 & \cdots & n-1 & n+1 & 2 & n+2 & 3 & n \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_1 \\ 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_2 & n+1 \\ 4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_3 & 1 & 2 \\ 5 & 6 & 7 & \cdots & n-1 & n & (n+2)_4 & 3 & 2 & 4 \\ 4 & 7 & 8 & \cdots & n & n+1 & 1 & (n+2)_5 & 6 & 2 \\ (n+2)_6 & 6 & 7 & \cdots & n-1 & n & 2 & 5 & n+1 & 1 \\ 3 & 5 & 6 & \cdots & n-2 & n+2 & 1 & 2 & 4_7 & n \end{bmatrix}$$

for  $C_4 \times K_n$ ,  $C_5 \times K_n$ ,  $C_6 \times K_n$  and  $C_7 \times K_n$  respectively with  $n+2$  colors where  $n \geq 6$ . The above arrays show that  $d(C_m \times K_n, n+2) = m(n-1)$  for  $(4 \leq m \leq 7)$  and  $n \geq 4$ .

To obtain a defining set for  $C_m \times K_n$ , with  $m \geq 8$ , one can write  $m = 4t + r$  where  $4 \leq r \leq 7$  and  $t \geq 1$  are integers. We successively treat the  $t$  above arrays for  $C_4 \times K_n$  and then treat to with the one for  $C_r \times K_n$ . So  $d(C_m \times K_n, n+2) = m(n-1)$  for  $n, m \geq 4$ . ■

**Lemma 2.2.** *Let  $G = (V, E)$  be a graph with  $c \geq \Delta(G) + 2$ . Then  $d(G, c) = |V|$ .*

**Proof.** Let  $S$  be a defining set of  $G$  and  $v$  be a vertex for which  $v \notin S$ . So if all of the neighbors of vertex  $v$  are colored then the vertex  $v$  has at least two choices for coloring. ■

**Theorem 2.3.** *For  $n, m \geq 4$ ,  $d(C_m \times K_n, n + i) = mn$  where  $i \geq 3$ .*

**Proof.** The degree of any vertex in  $C_m \times K_n$  is  $n + 1$ ,  $|V(C_m \times K_n)| = mn$  and for  $i \geq 3$ ,  $n + i \geq \Delta(C_m \times K_n) + 2$ . Now use the Lemma 2.2. ■

$$3. \quad d(K_3 \times C_m, c > \chi)$$

Note that  $\chi(K_3 \times C_m) = 3$ .

**Lemma 3.1.** *Let  $G = K_3 \times C_r$ . Then  $d(G, 4) \geq r + 1$ .*

**Proof.** On the contrary assume that  $d(G, 4) \leq r$ . If  $S$  is a defining set of  $G$  with cardinality at most  $r$  and  $V$  is the set of vertices of  $G$  then the induced subgraph  $\langle V \setminus S \rangle$  of  $G$  has  $3r - d(G, 4)$  vertices and has at least  $6r - 4d(G, 4)$  edges. Since  $r - d(G, 4) \geq 0$  we have  $6r - 4d(G, 4) \geq 3r - d(G, 4)$ . Therefore  $\langle V \setminus S \rangle$  has a cycle and we use Lemma C. ■

**Theorem 3.1.** *Let  $G = K_3 \times C_r$ . Then  $d(G, 4) = r + 1$  for even  $r$  and  $r + 1 \leq d(G, 4) \leq r + 2$  for odd  $r$ .*

**Proof.** Let  $G = K_3 \times C_r$ . From Lemma 3.1 we obtain  $d(G, 4) \geq r + 1$ . We give a defining set  $S$  of size  $r + 1$  for even  $r$  and a defining set  $S$  of size  $r + 2$  for odd  $r$ .

Let  $v_1, v_2, \dots, v_r$  are the vertices of first row,  $u_1, u_2, \dots, u_r$  the vertices of the second row and  $w_1, w_2, \dots, w_r$  the vertices of the third row.

If  $r = 2n$  then we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{6}, \\ 2 & \text{if } m \equiv 3 \pmod{6}, \\ 3 & \text{if } m \equiv 5 \pmod{6} \end{cases}$$

except for  $m = 2n - 1$  when  $2n \equiv 2 \pmod{6}$ ,



$$c(u_m) = \begin{cases} 3 & \text{if } m \equiv 2(\text{mod } 6), \\ 1 & \text{if } m \equiv 4(\text{mod } 6), \\ 2 & \text{if } m \equiv 6(\text{mod } 6) \end{cases}$$

except for  $m = 2n$  when  $2n \equiv 2(\text{mod } 6)$ . In this case we set  $c(u_{2n}) = 1$  when  $2n \equiv 2(\text{mod } 6)$ .

Finally, let  $c(w_1) = 2$  if  $2n \equiv 0$  or  $4(\text{mod } 6)$  and if  $2n \equiv 2(\text{mod } 6)$ , we set  $c(w_1) = 2$  and  $c(w_{2n-1}) = 3$ . In each case we have  $d(G, 4) = r + 1$  if  $r$  is even.

If  $r = 2n + 1$  then we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if } m \equiv 1(\text{mod } 6), \\ 2 & \text{if } m \equiv 3(\text{mod } 6), \\ 3 & \text{if } m \equiv 5(\text{mod } 6) \end{cases}$$

except for  $m = 2n + 1$  when  $2n + 1 \equiv 1(\text{mod } 6)$ . In this case we set  $c(v_{2n+1}) = 4$  when  $2n + 1 \equiv 1(\text{mod } 6)$ ,

$$c(u_m) = \begin{cases} 3 & \text{if } m \equiv 2(\text{mod } 6), \\ 1 & \text{if } m \equiv 4(\text{mod } 6), \\ 2 & \text{if } m \equiv 6(\text{mod } 6) \end{cases}$$

and if  $2n + 1 \equiv 3$  or  $5(\text{mod } 6)$  we set  $c(w_1) = 2$  and  $c(w_{2n}) = 4$ .

Finally we set  $c(w_1) = 2$  and  $c(w_{2n}) = 3$  if  $2n + 1 \equiv 1(\text{mod } 6)$ . Thus  $r + 1 \leq d(G, 4) \leq r + 2$  when  $r$  is odd. ■

We have the following

**Conjecture.**  $d(K_3 \times C_r, 4) = r + 2$  for odd  $r$ .

**Lemma 3.2.** *Let  $G = (V, E)$  be a graph. Let  $S$  be a defining set of  $G$  with  $c = \Delta(G) + 1$ . If  $v$  is a vertex and  $\deg(v) \leq \Delta(G) - 1$  then  $v \in S$  and if  $\deg(v) = \Delta(G)$  then  $v \in S$  or all neighbors of  $v$  are in  $S$ .*

**Proof.** If  $v$  is a vertex with  $\deg(v) \leq \Delta(G) - 1$  and  $v \notin S$  then there exists at least two choices of colors for  $v$  eventually all of neighbors are colored. If  $\deg(v) = \Delta(G)$ , vertex  $u$  is a neighbor of  $v$ ,  $(u, v \notin S)$  and all the other neighbors of  $v$  are in  $S$  then we have two choices of colors for  $u$  and  $v$ . ■

**Theorem 3.2.** *Let  $G = K_3 \times C_r$ . Then  $d(G, 5) = 2r$ .*

**Proof.** Let  $G = K_3 \times C_r$ . From Lemma 3.2 we obtain  $d(G, 5) \geq 2r$ . To show equality we give a defining set,  $S$  of size  $2r$ .

Let  $v_1, v_2, \dots, v_r$  are the vertices of first row,  $u_1, u_2, \dots, u_r$  the vertices of the second row and  $w_1, w_2, \dots, w_r$  the vertices of the third row.

If  $r = 2n$  then we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{10}, \\ 2 & \text{if } m \equiv 3 \pmod{10}, \\ 3 & \text{if } m \equiv 5 \pmod{10}, \\ 4 & \text{if } m \equiv 7 \pmod{10}, \\ 5 & \text{if } m \equiv 9 \pmod{10} \end{cases}$$

and  $c(v_{2n}) = 5$  when  $2n \equiv 2$  or  $8 \pmod{10}$ ,

$$c(u_m) = \begin{cases} 5 & \text{if } m \equiv 2 \pmod{10}, \\ 4 & \text{if } m \equiv 4 \pmod{10}, \\ 2 & \text{if } m \equiv 6 \pmod{10}, \\ 1 & \text{if } m \equiv 8 \pmod{10}, \\ 3 & \text{if } m \equiv 0 \pmod{10} \end{cases}$$

for  $2 \leq m \leq 2n$ , except  $m \neq 2n$  when  $2n \equiv 2$  or  $8 \pmod{10}$ . In this case we set  $c(u_{2n}) = 2$  when  $2n \equiv 2 \pmod{10}$  and  $c(u_{2n}) = 3$  when  $2n \equiv 8 \pmod{10}$ ,

$$c(w_m) = \begin{cases} 1 & \text{if } m \equiv 3 \text{ or } 5 \pmod{10}, \\ 5 & \text{if } m \equiv 4 \text{ or } 6 \pmod{10}, \\ 3 & \text{if } m \equiv 2 \text{ or } 7 \pmod{10}, \\ 2 & \text{if } m \equiv 0 \text{ or } 8 \pmod{10}, \\ 4 & \text{if } m \equiv 1 \text{ or } 9 \pmod{10} \end{cases}$$

for  $m \neq 1, 2, 2n - 1$  and  $2n$ . Finally, the following cases conclude the even case.

If  $2n \equiv 4$  or  $6 \pmod{10}$  we set  $c(w_1) = 3, c(w_2) = 4, c(w_{2n-1}) = 1$  and  $c(w_{2n}) = 5$ .

If  $2n \equiv 2 \pmod{10}$  we set  $c(w_1) = 3, c(w_2) = 4$  and  $c(w_{2n-1}) = 4$ .

If  $2n \equiv 8 \pmod{10}$  we set  $c(w_1) = 4, c(w_2) = 3$  and  $c(w_{2n-1}) = 1$ .

If  $2n \equiv 0 \pmod{10}$  we set  $c(w_1) = 4, c(w_2) = 3, c(w_{2n}) = 2$  and  $c(w_{2n-1}) = 4$ .

For  $r = 2n + 1$  we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if } m \equiv 1(\text{mod } 10), \\ 2 & \text{if } m \equiv 3(\text{mod } 10), \\ 3 & \text{if } m \equiv 5(\text{mod } 10), \\ 4 & \text{if } m \equiv 7(\text{mod } 10), \\ 5 & \text{if } m \equiv 9(\text{mod } 10) \end{cases}$$

for  $1 \leq m \leq 2n + 1$  and  $m \neq 2n + 1$  when  $2n + 1 \equiv 1(\text{mod } 10)$ . And we set  $c(v_{2n+1}) = 2$  when  $2n + 1 \equiv 1(\text{mod } 10)$ ,

$$c(u_m) = \begin{cases} 5 & \text{if } m \equiv 2(\text{mod } 10), \\ 4 & \text{if } m \equiv 4(\text{mod } 10), \\ 2 & \text{if } m \equiv 6(\text{mod } 10), \\ 1 & \text{if } m \equiv 8(\text{mod } 10), \\ 3 & \text{if } m \equiv 0(\text{mod } 10) \end{cases}$$

for  $1 \leq m \leq 2n$ .

Furthermore let  $c(u_{2n+1}) = 4$  when  $2n + 1 \equiv 1, 3$  or  $9(\text{mod } 10)$ , let  $c(u_{2n+1}) = 2$  when  $2n + 1 \equiv 5(\text{mod } 10)$  and let  $c(u_{2n+1}) = 3$  when  $2n + 1 \equiv 7(\text{mod } 10)$

$$c(w_m) = \begin{cases} 1 & \text{if } m \equiv 3 \text{ or } 5(\text{mod } 10), \\ 5 & \text{if } m \equiv 4 \text{ or } 6(\text{mod } 10), \\ 3 & \text{if } m \equiv 2 \text{ or } 7(\text{mod } 10), \\ 2 & \text{if } m \equiv 0 \text{ or } 8(\text{mod } 10), \\ 4 & \text{if } m \equiv 1 \text{ or } 9(\text{mod } 10) \end{cases}$$

for  $m \neq 1, 2, 2n$  and  $2n + 1$ . Again some special cases completes the proof.

If  $2n + 1 \equiv 1$  or  $3(\text{mod } 10)$  we set  $c(w_1) = 3, c(w_2) = 4, c(w_{2n}) = 1$ .

If  $2n + 1 \equiv 9(\text{mod } 10)$  we set  $c(w_1) = 3, c(w_2) = 4$  and  $c(w_{2n}) = 2$ .

If  $2n + 1 \equiv 5(\text{mod } 10)$  we set  $c(w_1) = 4, c(w_2) = 3$  and  $c(w_{2n}) = 5$ .

If  $2n + 1 \equiv 7(\text{mod } 10)$  we set  $c(w_1) = 2, c(w_2)$  and  $c(w_{2n}) = 5$ . ■

#### 4. $d(K_2 \times C_m, c > \chi)$

Note that  $\chi(K_2 \times C_m) = 3$  if  $m$  is odd and  $\chi(K_2 \times C_m) = 2$  if  $m$  is even.

**Lemma 4.1.** *Let  $G = K_2 \times C_r$ . Then  $d(G, 3) \geq \lfloor \frac{r}{2} \rfloor + 1$ .*

**Proof.** On the contrary, assume that  $d(G, 3) \leq \lfloor \frac{r}{2} \rfloor$ . If  $S$  is a defining set of  $G$  with cardinality at most  $\lfloor \frac{r}{2} \rfloor$  and  $V$  is the set of vertices of  $G$  then the induced subgraph  $\langle V \setminus S \rangle$  of  $G$  has  $2r - d(G, 3)$  vertices and has at least  $3r - 3d(G, 3)$  edges. Since  $\lfloor \frac{r}{2} \rfloor - d(G, 3) \geq 0$  we have  $r \geq 2\lfloor \frac{r}{2} \rfloor \geq 2d(G, 3)$  and  $3r - 3d(G, 3) \geq 2r - d(G, 3)$ . Therefore  $\langle V \setminus S \rangle$  has a cycle and we use Lemma C. ■

**Theorem 4.1.** *Let  $G = K_2 \times C_{2n}$ . Then  $d(G, 3) = n + 1$ .*

**Proof.** Let  $G = K_2 \times C_{2n}$ . From Lemma 4.1 we obtain  $d(G, 3) \geq n + 1$ . To show equality we give a defining set,  $S$  of size  $n + 1$ .

If  $v_1, v_2, \dots, v_{2n}$  are the vertices of first row and  $u_1, u_2, \dots, u_{2n}$  the vertices of the second row. We determine the defining set with their colors as in following tables:

$$c(v_m) = \begin{cases} 1 & \text{if } m = 1 \text{ and } m = 2n - 2, \\ 2 & \text{if } m \equiv 0 \pmod{4} \text{ and } 1 \leq m \leq 2n - 3 \end{cases}$$

also

$$c(u_m) = 2 \text{ if } m \equiv 2 \pmod{4}, \quad (m \leq 2n - 3) \text{ and } m = 2n - 1.$$

For  $2n = 4$  we say  $c(v_1) = c(u_3) = 1$  and  $c(v_4) = 2$ . ■

**Theorem 4.2.** *If  $G = K_2 \times C_r$  then  $d(G, 4) = 2\lceil \frac{r}{2} \rceil$ .*

**Proof.** Let  $G = K_2 \times C_r$ . From Lemma 3.2 we obtain  $d(G, 4) \geq 2\lceil \frac{r}{2} \rceil$ . To show equality we give a defining set,  $S$  of size  $2\lceil \frac{r}{2} \rceil$ .

Let  $v_1, v_2, \dots, v_r$  are the vertices of first row,  $u_1, u_2, \dots, u_r$  the vertices of the second row.

If  $r = 2n$  we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{8}, \\ 2 & \text{if } m \equiv 3 \pmod{8}, \\ 3 & \text{if } m \equiv 5 \pmod{8}, \\ 4 & \text{if } m \equiv 7 \pmod{8} \end{cases}$$

except  $m = 2n - 1$  when  $n \equiv 1 \pmod{4}$ . In this case we set  $c(v_{2n-1}) = 2$ ,

$$c(u_m) = \begin{cases} 3 & \text{if } m \equiv 2(\text{mod } 8), \\ 4 & \text{if } m \equiv 4(\text{mod } 8), \\ 1 & \text{if } m \equiv 6(\text{mod } 8), \\ 2 & \text{if } m \equiv 0(\text{mod } 8) \end{cases}$$

except  $m = 2n - 2$ ,  $m = 2n$  when  $n \equiv 1(\text{mod } 4)$  and  $m = 2n$  when  $n \equiv 3(\text{mod } 4)$ , in this case we say  $c(u_{2n}) = 4$ ,  $c(u_{2n-2}) = 3$  when  $n \equiv 1(\text{mod } 4)$  and we say  $c(u_{2n}) = 2$  when  $n \equiv 3(\text{mod } 4)$ .

If  $r = 2n + 1$  we determine the defining set with their colors as in following tables:

$$c(v_m) = \begin{cases} 1 & \text{if } m \equiv 1(\text{mod } 8), \\ 2 & \text{if } m \equiv 3(\text{mod } 8), \\ 3 & \text{if } m \equiv 5(\text{mod } 8), \\ 4 & \text{if } m \equiv 7(\text{mod } 8) \end{cases}$$

except  $m = 2n + 1$  when  $n \equiv 0(\text{mod } 4)$ . In this case we set  $c(v_{2n+1}) = 2$ ,

$$c(u_m) = \begin{cases} 3 & \text{if } m \equiv 2(\text{mod } 8), \\ 4 & \text{if } m \equiv 4(\text{mod } 8), \\ 1 & \text{if } m \equiv 6(\text{mod } 8), \\ 2 & \text{if } m \equiv 0(\text{mod } 8) \end{cases}$$

except  $m = 2n$  when  $n \equiv 0, (\text{mod } 4)$  in this case  $c(u_{2n}) = 3$  and  $c(u_{2n+1}) = 4$  when  $n \equiv 0$  or  $1(\text{mod } 4)$ ,  $c(u_{2n+1}) = 2$ , when  $n \equiv 2$  or  $3(\text{mod } 4)$ . ■

**Corollary 4.3.**  $d(K_2 \times C_r, 5) = 2r$ .

**Proof.** By Lemma 2.2, each of column has at least 2 vertices in defining set. Therefore all vertices are in defining set. ■

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