SPECTRAL INTEGRAL VARIATION OF TREES*

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Abstract

In this paper, we determine all trees with the property that adding a particular edge will result in exactly two Laplacian eigenvalues increasing respectively by 1 and the other Laplacian eigenvalues remaining fixed. We also investigate a situation in which the algebraic connectivity is one of the changed eigenvalues.

Keywords: tree, Laplacian eigenvalues, spectral integral variation, algebraic connectivity.

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1. INTRODUCTION

Let \( G = (V, E) \) be a simple graph with vertex set \( V = V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = E(G) = \{e_1, \ldots, e_m\} \). Denote by \( d(v) \) the degree of \( v \in V \) in the graph \( G \). Then the Laplacian matrix of \( G \) is \( L(G) = D(G) - A(G) \), where \( D(G) \) is the diagonal matrix \( \text{diag}(d(v_1), d(v_2), \ldots, d(v_n)) \), and \( A(G) \) is the \((0,1)\) adjacency matrix of \( G \). There is a wealth of literature on Laplacian matrices for graphs (see [10] for a comprehensive overview). It is known that \( L(G) \) is singular and positive semidefinite; and its eigenvalues can be arranged as follows: \( \lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G) = 0 \). The spectrum of \( G \) is defined by the multi-set \( S(G) = \{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\} \).

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Harary and Schwenk [8] initiated the study of those graphs $G$ such that $A(G)$ has integral spectrum. The analogous problem for $L(G)$ is also interesting [6]. A graph $G$ is said to be Laplacian integral if $S(G)$ consists entirely of integers. Merris [11] has shown that the degree maximal graphs are Laplacian integral. For some related results, one can refer to [6, 7]. It seems to be difficult to characterize Laplacian integral graphs or Laplacian integral eigenvalues. Assume $G$ is Laplacian integral. In order to preserve Laplacian integrality of $G$ by adding an edge, observe first that by Lemma 3.1 in following Section 3 the eigenvalues do not decrease, and therefore the changed eigenvalues of $G$ must move up respectively by an integer as one of the following two cases (see [13, 2]):

(A) one eigenvalue of $G$ increasing by 2 (and other $n - 1$ eigenvalues remain unchanged);

(B) two eigenvalue of $G$ increasing by 1 (and other $n - 2$ eigenvalues remain unchanged).

Now dropping the assumption of $G$ be Laplacian integral, and adopting the terminology of [2], we say that the spectral integral variation occurs to $G$ in one or two places by adding an edge if case (A) or case (B) occurs to $G$. The problem of characterizing spectral integral variation occurring in one place was solved by So [13]. Subsequently, for certain subclasses of graphs, Fan [2, 3] has characterized spectral integral variation occurring in two places. Recently, Kirkland [9] characterizes all graphs with spectral integral variation occurring in two places. The characterization is written in the form of matrix equations and can be rephrased in graph theoretic language; see Theorem 2.5 in Section 2.

In this paper, we focus on the problem of determining all trees with spectral integral variation occurring in two places by adding a particular edge. By Fan’s result [2] and Kirkland’s result [9], we solve the problem and find all these trees. In addition, we also investigate a situation in which the algebraic connectivity is one of the changed eigenvalues.

2. Spectral Integral Variation of Trees

Lemma 2.1 [13]. Let $G = (V, E)$ be a simple graph with $V = \{v_1, v_2, \ldots, v_n\}$. Then spectral integral variation occurs to $G$ in one place by adding an edge $e = \{v_i, v_j\} \notin E$ if and only if $N(v_i) = N(v_j)$, where $N(v) = \{u \in V : \{u, v\} \in E\}$. 
Lemma 2.2 [2]. Let $G = (V, E)$ be a simple graph with $V = \{v_1, v_2, \ldots, v_n\}$. If spectral integral variation occurs to $G$ in two places by adding an edge $e = \{v_i, v_j\} \notin E$ and the changed eigenvalues of $G$ are $\lambda_k, \lambda_l$, then

$$\lambda_k + \lambda_l = d(v_i) + d(v_j) + 1, \lambda_k \lambda_l = d(v_i) d(v_j) + d_{ij},$$

where $d_{ij}$ is the cardinality of the set $N(v_i) \cap N(v_j)$.

Theorem 2.3 (Matrix-Tree Theorem, see [1, p. 39]). Let $G$ be a simple graph on $n$ vertices, and $t(G)$ the number of the spanning trees of $G$. Then

$$t(G) = (1/n) \prod_{i=1}^{n-1} \lambda_i(G).$$

Lemma 2.4. Let $T = (V, E)$ be a tree with $V = \{v_1, v_2, \ldots, v_n\}$ and $e = \{v_i, v_j\} \notin E$. Let $\delta$ be the distance from $v_i$ to $v_j$. If spectral integral variation occurs to $T$ in two places by adding $e$, and the changed eigenvalues of $T$ are $\lambda_k, \lambda_l$ ($\lambda_k \geq \lambda_l$), then

$$d(v_i) = d(v_j) = 1; \quad \delta = 4; \quad \lambda_k = 1/\lambda_l = (3 + \sqrt{5})/2.$$

**Proof.** If $\delta = 2$, then by Lemma 2.2, we have

$$\lambda_k + \lambda_l = d(v_i) + d(v_j) + 1, \lambda_k \lambda_l = d(v_i) d(v_j) + d_{ij}. \tag{2.1}$$

Note that the number of spanning trees of $T + e$ is $\delta + 1$ as $T + e$ has a unique cycle with length $\delta + 1$. By Theorem 2.3, we have

$$\frac{t(G + e)}{t(G)} = \frac{(\lambda_k + 1)(\lambda_l + 1)}{\lambda_k \lambda_l} = \delta + 1 = 3.$$

Then by (2.1) we have $d(v_i) + d(v_j) = 2d(v_i) d(v_j)$, and hence $d(v_i) = d(v_j) = 1$. Therefore $N(v_i) = N(v_j)$, which is a contradiction by Lemma 2.1.

Otherwise, $\delta \geq 3$. Then $d_{ij} = 0$ in Lemma 2.2. By a similar discussion to former case, we have

$$4 \geq \frac{1}{d(v_i)} + \frac{1}{d(v_j)} + \frac{2}{d(v_i) d(v_j)} = \delta \geq 3.$$
Then $\delta = 4$ if and only if $d(v_i) = d(v_j)=1$, and hence

$$\lambda_k = 1/\lambda_l = (3 + \sqrt{5})/2.$$  

It is obvious that the case of $\delta = 3$ cannot happen. \hfill \blacksquare

Next we introduce Kirkland’s result [9], which gives a characterization of the spectral integral variation occurring to a graph in two places.

**Theorem 2.5** [9]. Let $G$ be a graph on $n$ vertices $v_1, v_2, \ldots, v_n$, with Laplacian matrix $L$ given by

$$L = \begin{bmatrix} d_1 & 0 & -1^T & 0^T & -1^T & 0^T \\ 0 & d_2 & 0^T & -1^T & -1^T & 0^T \\ -1 & 0 & L_{11} & L_{12} & L_{13} & L_{14} \\ 0 & -1 & L_{21} & L_{22} & L_{23} & L_{24} \\ -1 & -1 & L_{31} & L_{32} & L_{33} & L_{34} \\ 0 & 0 & L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix},$$

where $d_1 = d(v_1), d_2 = d(v_2)$, the blocks $L_{11}, \ldots, L_{44}$ are respectively of sizes $d_1 - d_{12}, d_2 - d_{12}, d_{12}, n - 2 - d_1 - d_2 - d_{12}$, and $1, 0$ are respectively column vectors of all 1's and all 0's of suitable size. Suppose that $d_1 \geq d_2$. From $G + e$ from $G$ by adding the edge between the vertices $v_1$ and $v_2$. Then spectral integral variation occurs in two places under the addition of that edge if and only if the follow conditions hold:

$$L_{11}1 - L_{12}1 = (d_2 + 1)1,$$

$$L_{21}1 - L_{22}1 = -(d_1 + 1)1,$$

$$L_{31}1 - L_{32}1 = -(d_1 - d_2)1,$$

$$L_{41}1 - L_{42}1 = 0.$$  

Denote by $P_n = P_{v_1v_2\cdots v_n}$ a path on vertices $v_1, v_2, \ldots, v_n$ with edges $\{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, n - 1$.

**Theorem 2.6.** Let $T = (V, E)$ be a tree with $V = \{v_1, v_2, \ldots, v_n\}$ and $e = \{v_1, v_2\} \notin E$. Then spectral integral variation occurs to $T$ in two places by adding the edge $e$ if and only if $T$ has following properties:

1. $d(v_1) = d(v_2) = 1$;
2. the path from $v_1$ to $v_2$ has length 4 (say it to be $P_{v_1v_3v_5v_4v_2}$);
(3) $T$ is obtained from the path $P_{v_1v_3v_5v_4v_2}$ by identifying $v_5$ with some vertex of a tree on $n - 4$ vertices; or equivalently $T$ has the structure of the tree of Figure 2.1 where the additional edge is $\{v_1, v_2\}$.

Figure 2.1. $T_1$ is a tree on $n - 4$ vertices with some vertex identified with the vertex $v_5$.

**Proof.** Assume that spectral integral variation occurs to $T$ in two places by adding the edge $e = \{v_1, v_2\}$. By Lemma 2.4, $d(v_1) = d(v_2) = 1$; and $T$ contains a path of length 4 which joins $v_1$ and $v_2$, say it to be $P_{v_1v_3v_5v_4v_2}$. By Theorem 2.5, in the matrix (2.2), we find that $L_{11} = d(v_3), L_{22} = d(v_4)$, both of size 1; and $L_{33}$, together with the row and column that it lies, are vanished; and $L_{44}$ is of size $n - 4$. Then

$$L(T) = \begin{bmatrix} 1 & 0 & -1 & 0 & o^T \\ 0 & 1 & 0 & -1 & o^T \\ -1 & 0 & d(v_3) & 0 & L_{14} \\ 0 & -1 & 0 & d(v_4) & L_{24} \\ 0 & 0 & L_{41} & L_{42} & L_{44} \end{bmatrix}. $$

By (2.3),

$$d(v_3) = d(v_2) + 1 = 2, d(v_4) = d(v_1) + 1 = 2, N(v_3) \cap N(v_4) = \{v_5\};$$

and the necessity holds. The sufficiency is easily verified by (2.3) of Theorem 2.5.

3. **Changed Algebraic Connectivity**

Let $G = (V, E)$ be a graph on $n$ vertices $v_1, v_2, \ldots, v_n$. For convenience, we adopt the following terminology from [5]: for a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we say $x$ gives a valuation of the vertices of $V$, that is, for each vertex
vi, we associate the value $x_i$, i.e., $x(v_i) = x_i$. Then $\lambda$ is an eigenvalue of $G$ corresponding to the eigenvector $x$ if and only if $x \neq 0$ and for each $i = 1, 2, \ldots, n$,

\begin{equation}
[d(v_i) - \lambda]x(v_i) = \sum_{\{v_i, v_j\} \in E} x(v_j).
\end{equation}

Recall that the algebraic connectivity of $G$ is $\alpha(G) = \lambda_{n-1}(G)$ [4]. In particular the algebraic connectivity $\alpha(G) > 0$ if and only if $G$ is connected.

Suppose that spectral integral variation occurs to a tree $T$ in two places with $\lambda_k$ and $\lambda_l$ ($\lambda_k \geq \lambda_l$) both increasing 1 by adding a particular edge. This section gives an equivalent condition that algebraic connectivity of $T$ is a changed eigenvalue (that is, $\lambda_l = \alpha(T) = (3 - \sqrt{5})/2$ by Lemma 2.4).

**Lemma 3.1** [12]. Let $G$ be a simple graph on $n$ vertices, and let $G + e$ be the graph obtained from $G$ by adding an edge $e$. Then

$$
\lambda_1(G + e) \geq \lambda_1(G) \geq \lambda_2(G + e) \geq \lambda_2(G) \geq \lambda_3(G + e) \geq \ldots \geq \lambda_n(G + e) = \lambda_n(G) = 0.
$$

**Lemma 3.2.** Let $T$ be a tree and $v$ be a pendant vertex of $T$. Then $\alpha(T - v) \geq \alpha(T)$.

**Proof.** Let $e$ be the pendant edge incident to $v$. Then $T - e$ contains exactly two components: $v$, and $T - v$ on $n - 1$ vertices; and

$$
0 = \lambda_n(T - e) = \lambda_{n-1}(T - e) = \lambda_{n-1}(T - v),
$$

$$
\lambda_{n-2}(T - e) = \lambda_{n-2}(T - v) = \alpha(T - v).
$$

Then by Lemma 3.1, $\lambda_{n-2}(T - e) \geq \lambda_{n-1}(T)$ and the result follows. \hfill \blacksquare

Consider the graph $H_1$ of Figure 3.1. Let $\lambda$ be an eigenvalue of $H_1$ corresponding to the eigenvector $x$. Observing the symmetric property of $H_1$ and by (3.1), we may assume that $x$ satisfies one of the following conditions (3.2) and (3.3):

\begin{align}
&x(v_1) = x(v_2) =: y_1, \quad x(v_3) = x(v_4) =: y_2, \\
&x(v_5) =: y_3, \quad x(v_6) =: y_4, \quad x(v_7) = x(v_8) =: y_5;
\end{align}
Now assume $\lambda \neq 1$. If $x$ satisfies (3.3), by (3.1),

$$(1 - \lambda)x(v_1) = x(v_2), (2 - \lambda)x(v_2) = x(v_1).$$

We get $\lambda = (3 \pm \sqrt{5})/2$ as $x(v_1) \neq 0, x(v_2) \neq 0$. If $x$ satisfies (3.2), by (3.1) we have

$$
\begin{cases}
(1 - \lambda) y_1 = y_2, \\
(2 - \lambda) y_2 = y_1 + y_3, \\
(3 - \lambda) y_3 = 2y_2 + y_4, \\
(3 - \lambda) y_4 = 2y_5 + y_3, \\
(1 - \lambda) y_5 = y_4.
\end{cases}
$$

Finding the solutions of $\lambda$ of (3.4) is equivalent to find the roots of the polynomial $f(\lambda)$ as follows:

$$f(\lambda) = \det \begin{bmatrix}
1 - \lambda & -1 & 0 & 0 & 0 \\
-1 & 2 - \lambda & -1 & 0 & 0 \\
0 & -2 & 3 - \lambda & -1 & 0 \\
0 & 0 & -1 & 3 - \lambda & -2 \\
0 & 0 & 0 & -1 & 1 - \lambda
\end{bmatrix}.
$$

We get that

$$f(\lambda) = \lambda(-8 + 35\lambda - 32\lambda^2 + 10\lambda^3 - \lambda^4) =: \lambda g(\lambda),$$

where $g(\lambda)$ is the polynomial obtained by expanding $f(\lambda)$. The roots of $\lambda g(\lambda)$ are the eigenvalues of the graph.
and \( g(0) = -8, g((3 - \sqrt{5})/2) = \sqrt{5} - 1 > 0 \). Therefore \( g(\lambda) \), hence \( f(\lambda) \), has a root less than \( (3 - \sqrt{5})/2 \). So \( \alpha(H_1) < (3 - \sqrt{5})/2 \).

Suppose that spectral integral variation occurs to a tree \( T \) in two places and one changed eigenvalue is \( \alpha(T) \). Then by Lemma 2.4, \( \alpha(T) = (3 - \sqrt{5})/2 \). This implies that tree \( T \) cannot contain \( H_1 \) as a subgraph; otherwise by Lemma 3.2, under a sequential deletion of the pendent vertices, we get \( \alpha(T) \leq \alpha(H_1) < (3 - \sqrt{5})/2 \). We call \( H_1 \) a forbidden subgraph of \( T \).

**Lemma 3.3** ([1, p. 187], or [10]). Let \( T \) be a tree with diameter \( d \). Then

\[
\alpha(T) \leq 2\{1 - \cos[\pi/(d + 1)]\}.
\]

**Theorem 3.4.** Let \( T = (V, E) \) be a tree with \( V = \{v_1, v_2, \ldots, v_n\} \) and \( e = \{v_1, v_2\} \notin E \). Suppose that spectral integral variation occurs to \( T \) in two places with changed eigenvalues \( \lambda_k \) and \( \lambda_l \) (\( \lambda_k \geq \lambda_l \)) by adding the edge \( e \). Then \( \lambda_l = \alpha(T) \) if and only if \( T \) is obtained from a vertex, \( k \) (\( \geq 2 \)) paths of length 2 and \( p \) (\( \geq 0 \)) paths of length 1 by identifying that vertex with one pendent vertex of each path; or equivalently, \( T \) has the structure of \( H_2 \) of Figure 3.1, where that vertex is \( w \), \( k \) paths of length 2 are \( P_{u_{11}w}, P_{u_{12}w}, \ldots, P_{u_{1k}w} \), and \( p \) paths of length 1 are \( P_{v_{31}w}, \ldots, P_{v_{3p}w} \), and the additional edge is \( \{v_1, v_2\} \).

**Proof.** By Theorem 2.6, \( T \) has the structure of the graph in Figure 2.1; and by Lemma 2.4, \( \lambda_l = (3 - \sqrt{5})/2 \). Assume that \( \lambda_l = \alpha(T) \). Then \( \alpha(T) = (3 - \sqrt{5})/2 \). By Lemma 3.3, the diameter of \( T \) is at most 4. Since the graph \( H_1 \) of Figure 3.1 is forbidden in \( T \) by the prior discussion, \( T \) has the structure of \( H_2 \) of Figure 3.1 and the necessity follows.

Next assume that \( T = H_2 \) of Figure 3.1. We shall prove that \( \lambda_l = \alpha(T) = \alpha(H_2) \). This is equivalent to show \( \alpha(H_2) = (3 - \sqrt{5})/2 \). Suppose that \( \lambda \) is an eigenvalue of \( T \) corresponding to the eigenvector \( x \). For convenience, we relabel the vertices of \( H_2 \) as in Figure 3.1. Then we may assume that \( x \) has one of the following properties:

(A) \( x(v_{11}) = \cdots = x(v_{1k}) =: y_1, x(v_{21}) = \cdots = x(v_{2k}) =: y_2, x(v_{31}) = \cdots = x(v_{3p}) =: y_3; \)

(B) \( x(v_{11}) + \cdots + x(v_{1k}) = 0, x(v_{21}) + \cdots + x(v_{2k}) = 0, x(v_{31}) + \cdots + x(v_{3p}) = 0, x(w) = 0. \)
Now assume that $\lambda \neq 1$ and $p \geq 1$. If $x$ satisfies (B), then by (3.1), for each $i = 1, 2, \ldots, k$,
\[
(1 - \lambda)x(v_{1i}) = x(v_{2i}), \quad (2 - \lambda)x(v_{2i}) = x(v_{1i});
\]
and hence $\lambda = (3 \pm \sqrt{5})/2$. If $x$ satisfies (A), let $x(w) = y_4$, and by (3.1) we get
\[
\begin{align*}
(1 - \lambda)y_1 &= y_2, \\
(2 - \lambda)y_2 &= y_1 + y_4, \\
(1 - \lambda)y_3 &= y_4, \\
(k + p - \lambda)y_4 &= ky_2 + py_3.
\end{align*}
\]
Let
\[
f(\lambda) = \det \begin{bmatrix}
1 - \lambda & -1 & 0 & 0 \\
-1 & 2 - \lambda & 0 & -1 \\
0 & 0 & 1 - \lambda & -1 \\
0 & -k & -p & k + p - \lambda
\end{bmatrix}.
\]
Then
\[
f(\lambda) = \lambda[-(1 + 2k + p) + (4 + 3k + 3p)\lambda - (4 + k + p)\lambda^2 + \lambda^3] =: \lambda g(\lambda).
\]
$g((3 - \sqrt{5})/2) = -k < 0$, $g(1) = p > 0$, $g(3) = 2 - 2k - p < 0$ and $g(k + p + 2) = (k + p)^2 + p - 1 > 0$. So $g(\lambda)$, and hence $f(\lambda)$ has no eigenvalues less than $(3 - \sqrt{5})/2$. By above discussion, $\alpha(H_2) = (3 - \sqrt{5})/2$, and the sufficiency holds.

If $\lambda \neq 1$ and $p = 0$, then by (B) we also get $\lambda = (3 \pm \sqrt{5})/2$. From (A) we obtain 3 equations from (3.5) by dropping the 3rd equation and replacing $p$ by 0. By a similar discussion, we also get $\alpha(H_2) = (3 - \sqrt{5})/2$. The result follows.

References


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