Discussiones Mathematicae
Graph Theory 26 (2006) 49–58

SPECTRAL INTEGRAL VARIATION OF TREES*

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Abstract

In this paper, we determine all trees with the property that adding a particular edge will result in exactly two Laplacian eigenvalues increasing respectively by 1 and the other Laplacian eigenvalues remaining fixed. We also investigate a situation in which the algebraic connectivity is one of the changed eigenvalues.

Keywords: tree, Laplacian eigenvalues, spectral integral variation, algebraic connectivity.

2000 Mathematics Subject Classification: 05C50, 15A18.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, \ldots, e_m\}$. Denote by $d(v)$ the degree of $v \in V$ in the graph G. Then the *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix $\text{diag}\{d(v_1), d(v_2), \ldots, d(v_n)\}$, and $A(G)$ is the $(0, 1)$ adjacency matrix of G. There is a wealth of literature on Laplacian matrices for graphs (see [10] for a comprehensive overview). It is known that $L(G)$ is singular and positive semidefinite; and its eigenvalues can be arranged as follows: $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G) = 0$. The *spectrum* of G is defined by the multi-set $S(G) = \{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$.

*Supported by Anhui Provincial Natural Science Foundation (050460102), NSF of Department of Education of Anhui province (2004kj027, 2005kj005zd), Fund of Innovation for graduates of Anhui University, and Project of Innovation Team on Basic Mathematics of Anhui University.

Harary and Schwenk [8] initiated the study of those graphs G such that $A(G)$ has integral spectrum. The analogous problem for $L(G)$ is also interesting [6]. A graph G is said to be *Laplacian integral* if $S(G)$ consists entirely of integers. Merris [11] has shown that the degree maximal graphs are Laplacian integral. For some related results, one can refer to [6, 7]. It seems to be difficult to characterize Laplacian integral graphs or Laplacian integral eigenvalues. Assume G is Laplacian integral. In order to preserve Laplacian integrality of G by adding an edge, observe first that by Lemma 3.1 in following Section 3 the eigenvalues do not decrease, and therefore the changed eigenvalues of G must move up respectively by an integer as one of the following two cases (see [13, 2]):

(A) one eigenvalue of G increasing by 2 (and other $n-1$ eigenvalues remain unchanged);

(B) two eigenvalue of G increasing by 1 (and other $n-2$ eigenvalues remain unchanged).

Now dropping the assumption of G be Laplacian integral, and adopting the terminology of [2], we say that the *spectral integral variation* occurs to G in one or two places by adding an edge if case (A) or case (B) occurs to G. The problem of characterizing spectral integral variation occurring in one place was solved by So [13]. Subsequently, for certain subclasses of graphs, Fan [2, 3] has characterized spectral integral variation occurring in two places. Recently, Kirkland [9] characterizes all graphs with spectral integral variation occurring in two places. The characterization is written in the form of matrix equations and can be rephrased in graph theoretic language; see Theorem 2.5 in Section 2.

In this paper, we focus on the problem of determining all trees with spectral integral variation occurring in two places by adding a particular edge. By Fan's result [2] and Kirkland's result [9], we solve the problem and find all these trees. In addition, we also investigate a situation in which the algebraic connectivity is one of the changed eigenvalues.

2. Spectral Integral Variation of Trees

Lemma 2.1 [13]. *Let $G=(V,E)$ be a simple graph with $V=\{v_1,v_2,\dots,v_n\}$. Then spectral integral variation occurs to G in one place by adding an edge $e=\{v_i,v_j\}\notin E$ if and only if $N(v_i)=N(v_j)$, where $N(v)=\{u\in V:\{u,v\}\in E\}$.*

Lemma 2.2 [2]**.** *Let $G = (V, E)$ be a simple graph with $V = \{v_1, v_2, \ldots, v_n\}$. If spectral integral variation occurs to G in two places by adding an edge $e = \{v_i, v_j\} \notin E$ and the changed eigenvalues of G are λ_k, λ_l, then*

$$\lambda_k + \lambda_l = d(v_i) + d(v_j) + 1, \lambda_k\lambda_l = d(v_i)d(v_j) + d_{ij},$$

where d_{ij} is the cardinality of the set $N(v_i) \cap N(v_j)$.

Theorem 2.3 (Matrix-Tree Theorem, see [1, p. 39])**.** *Let G be a simple graph on n vertices, and $t(G)$ the number of the spanning trees of G. Then $t(G) = (1/n)\prod_{i=1}^{n-1} \lambda_i(G)$.*

Lemma 2.4. *Let $T = (V, E)$ be a tree with with $V = \{v_1, v_2, \ldots, v_n\}$ and $e = \{v_i, v_j\} \notin E$. Let δ be the distance from v_i to v_j. If spectral integral variation occurs to T in two places by adding e, and the changed eigenvalues of T are λ_k, λ_l $(\lambda_k \geq \lambda_l)$, then*

$$d(v_i) = d(v_j) = 1;\ \ \delta = 4;\ \ \lambda_k = 1/\lambda_l = (3 + \sqrt{5})/2.$$

Proof. If δ=2, then by Lemma 2.2, we have

$$\lambda_k + \lambda_l = d(v_i) + d(v_j) + 1, \lambda_k\lambda_l = d(v_i)d(v_j) + d_{ij}. \tag{2.1}$$

Note that the number of spanning trees of $T+e$ is δ+1 as $T+e$ has a unique cycle with length δ+1. By Theorem 2.3, we have

$$\frac{t(G+e)}{t(G)} = \frac{(\lambda_k + 1)(\lambda_l + 1)}{\lambda_k\lambda_l} = \delta + 1 = 3.$$

Then by (2.1) we have $d(v_i)+d(v_j) = 2d(v_i)d(v_j)$, and hence $d(v_i) = d(v_j) = 1$. Therefore $N(v_i) = N(v_j)$, which is a contradiction by Lemma 2.1.

Otherwise, $\delta \geq 3$. Then d_{ij}=0 in Lemma 2.2. By a similar discussion to former case, we have

$$4 \geq \frac{1}{d(v_i)} + \frac{1}{d(v_j)} + \frac{2}{d(v_i)d(v_j)} = \delta \geq 3.$$

Then $\delta = 4$ if and only if $d(v_i) = d(v_j)$=1, and hence

$$\lambda_k = 1/\lambda_l = (3 + \sqrt{5})/2.$$

It is obvious that the case of $\delta = 3$ cannot happen. ∎

Next we introduce Kirkland's result [9], which gives a characterization of the spectral integral variation occurring to a graph in two places.

Theorem 2.5 [9]**.** *Let G be a graph on n vertices $v_1, v_2, \ldots, v_n$, with Laplacian matrix L given by*

$$L = \begin{bmatrix} d_1 & 0 & -\mathbf{1}^T & \mathbf{0}^T & -\mathbf{1}^T & \mathbf{0}^T \\ 0 & d_2 & \mathbf{0}^T & -\mathbf{1}^T & -\mathbf{1}^T & \mathbf{0}^T \\ -\mathbf{1} & \mathbf{0} & L_{11} & L_{12} & L_{13} & L_{14} \\ \mathbf{0} & -\mathbf{1} & L_{21} & L_{22} & L_{23} & L_{24} \\ -\mathbf{1} & -\mathbf{1} & L_{31} & L_{32} & L_{33} & L_{34} \\ \mathbf{0} & \mathbf{0} & L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix}, \tag{2.2}$$

where $d_1 = d(v_1), d_2 = d(v_2)$, the blocks $L_{11}, \ldots, L_{44}$ are respectively of sizes $d_1 - d_{12}, d_2 - d_{12}, d_{12}, n - 2 - d_1 - d_2 - d_{12}$, and $\mathbf{1}, \mathbf{0}$ are respectively column vectors of all 1's and all 0's of suitable size. Suppose that $d_1 \geq d_2$. From $G+e$ from G by adding the edge between the vertices v_1 and v_2. Then spectral integral variation occurs in two places under the addition of that edge if and only if the follow conditions hold:

$$\begin{aligned} L_{11}\mathbf{1} - L_{12}\mathbf{1} &= (d_2 + 1)\mathbf{1}, \\ L_{21}\mathbf{1} - L_{22}\mathbf{1} &= -(d_1 + 1)\mathbf{1}, \\ L_{31}\mathbf{1} - L_{32}\mathbf{1} &= -(d_1 - d_2)\mathbf{1}, \\ L_{41}\mathbf{1} - L_{42}\mathbf{1} &= \mathbf{0}. \end{aligned} \tag{2.3}$$

Denote by $P_n = \mathcal{P}v_1v_2\cdots v_n$ a path on vertices $v_1, v_2, \cdots, v_n$ with edges $\{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, n-1$.

Theorem 2.6. *Let $T = (V, E)$ be a tree with $V = \{v_1, v_2, \ldots, v_n\}$ and $e = \{v_1, v_2\} \notin E$. Then spectral integral variation occurs to T in two places by adding the edge e if and only if T has following properties:*

(1) $d(v_1) = d(v_2) = 1$;

(2) *the path from v_1 to v_2 has length 4 (say it to be $\mathcal{P}v_1v_3v_5v_4v_2$);*

(3) *T is obtained from the path $\mathcal{P}v_1v_3v_5v_4v_2$ by identifying v_5 with some vertex of a tree on $n-4$ vertices; or equivalently T has the structure of the tree of Figure* 2.1 *where the additional edge is $\{v_1, v_2\}$.*

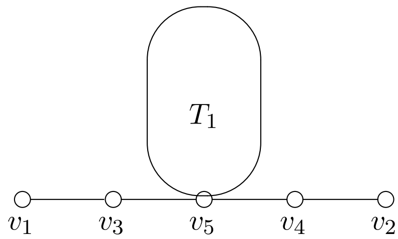


Figure 2.1. T_1 is a tree on $n-4$ vertices with some vertex identified with the vertex v_5.

Proof. Assume that spectral integral variation occurs to T in two places by adding the edge $e = \{v_1, v_2\}$. By Lemma 2.4, $d(v_1) = d(v_2) = 1$; and T contains a path of length 4 which joins v_1 and v_2, say it to be $\mathcal{P}v_1v_3v_5v_4v_2$. By Theorem 2.5, in the matrix (2.2), we find that $L_{11} = d(v_3)$, $L_{22} = d(v_4)$, both of size 1; and L_{33}, together with the row and column that it lies, are vanished; and L_{44} is of size $n-4$. Then

$$L(T) = \begin{bmatrix} 1 & 0 & -1 & 0 & \mathbf{0}^T \\ 0 & 1 & 0 & -1 & \mathbf{0}^T \\ -1 & 0 & d(v_3) & 0 & L_{14} \\ 0 & -1 & 0 & d(v_4) & L_{24} \\ \mathbf{0} & \mathbf{0} & L_{41} & L_{42} & L_{44} \end{bmatrix}.$$

By (2.3),

$$d(v_3) = d(v_2) + 1 = 2, d(v_4) = d(v_1) + 1 = 2, N(v_3) \cap N(v_4) = \{v_5\};$$

and the necessity holds. The sufficiency is easily verified by (2.3) of Theorem 2.5. ∎

3. CHANGED ALGEBRAIC CONNECTIVITY

Let $G = (V, E)$ be a graph on n vertices $v_1, v_2, \ldots, v_n$. For convenience, we adopt the following terminology from [5]: for a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we say x gives a valuation of the vertices of V, that is, for each vertex

v_i, we associate the value x_i, i.e., $x(v_i) = x_i$. Then λ is an eigenvalue of G corresponding to the eigenvector x if and only if $x \neq 0$ and for each $i = 1, 2, \dots, n$,

$$[d(v_i) - \lambda]x(v_i) = \sum_{\{v_i, v_j\} \in E} x(v_j). \tag{3.1}$$

Recall that the algebraic connectivity of G is $\alpha(G) = \lambda_{n-1}(G)$ [4]. In particular the algebraic connectivity $\alpha(G) > 0$ if and only if G is connected. Suppose that spectral integral variation occurs to a tree T in two places with λ_k and λ_l ($\lambda_k \geq \lambda_l$) both increasing 1 by adding a particular edge. This section gives an equivalent condition that algebraic connectivity of T is a changed eigenvalue (that is, $\lambda_l = \alpha(T) = (3 - \sqrt{5})/2$ by Lemma 2.4).

Lemma 3.1 [12]**.** *Let G be a simple graph on n vertices, and let $G + e$ be the graph obtained from G by adding an edge e. Then*

$$\lambda_1(G+e) \geq \lambda_1(G) \geq \lambda_2(G+e) \geq \lambda_2(G) \geq \lambda_3(G+e)$$
$$\geq \dots \geq \lambda_n(G+e) = \lambda_n(G) = 0.$$

Lemma 3.2. *Let T be a tree and v be a pendant vertex of T. Then $\alpha(T - v) \geq \alpha(T)$.*

Proof. Let e be the pendant edge incident to v. Then $T - e$ contains exactly two components: v, and $T - v$ on $n - 1$ vertices; and

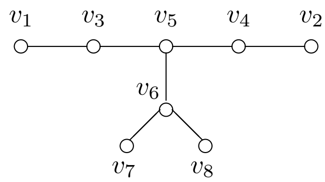
$$0 = \lambda_n(T-e) = \lambda_{n-1}(T-e) = \lambda_{n-1}(T-v),$$
$$\lambda_{n-2}(T-e) = \lambda_{n-2}(T-v) = \alpha(T-v).$$

Then by Lemma 3.1, $\lambda_{n-2}(T-e) \geq \lambda_{n-1}(T)$ and the result follows. ■

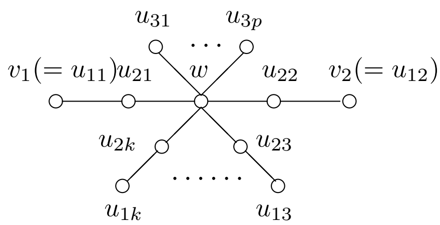
Consider the graph H_1 of Figure 3.1. Let λ be an eigenvalue of H_1 corresponding to the eigenvector x. Observing the symmetric property of H_1 and by (3.1), we may assume that x satisfies one of the following conditions (3.2) and (3.3):

$$\begin{aligned} &x(v_1) = x(v_2) =: y_1, x(v_3) = x(v_4) =: y_2,\\ &x(v_5) =: y_3, x(v_6) =: y_4, x(v_7) = x(v_8) =: y_5; \end{aligned} \tag{3.2}$$

(3.3) $x(v_1) = -x(v_2), x(v_3) = -x(v_4), x(v_7) = -x(v_8), x(v_5) = x(v_6) = 0.$



H_1



H_2 $(k \geq 2, p \geq 0)$

Figure 3.1

Now assume $\lambda \neq 1$. If x satisfies (3.3), by (3.1),

$$(1-\lambda)x(v_1) = x(v_2), (2-\lambda)x(v_2) = x(v_1).$$

We get $\lambda = (3 \pm \sqrt{5})/2$ as $x(v_1) \neq 0, x(v_2) \neq 0$. If x satisfies (3.2), by (3.1) we have

$$\left\{\begin{array}{rcl}(1-\lambda)y_1 &=& y_2,\\ (2-\lambda)y_2 &=& y_1+y_3,\\ (3-\lambda)y_3 &=& 2y_2+y_4,\\ (3-\lambda)y_4 &=& 2y_5+y_3,\\ (1-\lambda)y_5 &=& y_4.\end{array}\right. \tag{3.4}$$

Finding the solutions of λ of (3.4) is equivalent to find the roots of the polynomial $f(\lambda)$ as follows:

$$f(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 & 0 & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 & 0 \\ 0 & -2 & 3-\lambda & -1 & 0 \\ 0 & 0 & -1 & 3-\lambda & -2 \\ 0 & 0 & 0 & -1 & 1-\lambda \end{bmatrix}.$$

We get that

$$f(\lambda) = \lambda(-8 + 35\lambda - 32\lambda^2 + 10\lambda^3 - \lambda^4) =: \lambda g(\lambda),$$

and $g(0) = -8$, $g((3-\sqrt{5})/2) = \sqrt{5}-1 > 0$. Therefore $g(\lambda)$, hence $f(\lambda)$, has a root less than $(3-\sqrt{5})/2$. So $\alpha(H_1) < (3-\sqrt{5})/2$.

Suppose that spectral integral variation occurs to a tree T in two places and one changed eigenvalue is $\alpha(T)$. Then by Lemma 2.4, $\alpha(T) = (3-\sqrt{5})/2$. This implies that tree T cannot contain H_1 as a subgraph; otherwise by Lemma 3.2, under a sequential deletion of the pendent vertices, we get $\alpha(T) \le \alpha(H_1) < (3-\sqrt{5})/2$. We call H_1 a *forbidden subgraph* of T.

Lemma 3.3 ([1, p. 187], or [10]). *Let T be a tree with diameter d. Then*

$$\alpha(T) \le 2\{1-\cos[\pi/(d+1)]\}.$$

Theorem 3.4. *Let $T = (V, E)$ be a tree with $V = \{v_1, v_2, \ldots, v_n\}$ and $e = \{v_1, v_2\} \notin E$. Suppose that spectral integral variation occurs to T in two places with changed eigenvalues λ_k and λ_l ($\lambda_k \ge \lambda_l$) by adding the edge e. Then $\lambda_l = \alpha(T)$ if and only if T is obtained from a vertex, k (≥ 2) paths of length 2 and p (≥ 0) paths of length 1 by identifying that vertex with one pendent vertex of each path; or equivalently, T has the structure of H_2 of Figure 3.1, where that vertex is w, k paths of length 2 are $\mathcal{P}u_{11}u_{21}w$ ($u_{11} = v_1$), $\mathcal{P}u_{12}u_{22}w$ ($u_{12} = v_2$), $\cdots$, $\mathcal{P}u_{1k}u_{2k}w$, and p paths of length 1 are $\mathcal{P}u_{31}w$, $\cdots$, $\mathcal{P}u_{3p}w$, and the additional edge is $\{v_1, v_2\}$.*

Proof. By Theorem 2.6, T has the structure of the graph in Figure 2.1; and by Lemma 2.4, $\lambda_l = (3-\sqrt{5})/2$. Assume that $\lambda_l = \alpha(T)$. Then $\alpha(T) = (3-\sqrt{5})/2$. By Lemma 3.3, the diameter of T is at most 4. Since the graph H_1 of Figure 3.1 is forbidden in T by the prior discussion, T has the structure of H_2 of Figure 3.1 and the necessity follows.

Next assume that $T = H_2$ of Figure 3.1. We shall prove that $\lambda_l = \alpha(T) = \alpha(H_2)$. This is equivalent to show $\alpha(H_2) = (3-\sqrt{5})/2$. Suppose that λ is an eigenvalue of T corresponding to the eigenvector x. For convenience, we relabel the vertices of H_2 as in Figure 3.1. Then we may assume that x has one of the following properties:

(A) $x(v_{11}) = \cdots = x(v_{1k}) =: y_1$, $x(v_{21}) = \cdots = x(v_{2k}) =: y_2$, $x(v_{31}) = \cdots = x(v_{3p}) =: y_3$;

(B) $x(v_{11}) + \cdots + x(v_{1k}) = 0$, $x(v_{21}) + \cdots + x(v_{2k}) = 0$, $x(v_{31}) + \cdots + x(v_{3p}) = 0$, $x(w) = 0$.

Now assume that $\lambda \neq 1$ and $p \geq 1$. If x satisfies (B), then by (3.1), for each $i = 1, 2, \ldots, k$,

$$(1-\lambda)x(v_{1i}) = x(v_{2i}),\ (2-\lambda)x(v_{2i}) = x(v_{1i});$$

and hence $\lambda = (3 \pm \sqrt{5})/2$. If x satisfies (A), let $x(w) = y_4$, and by (3.1) we get

$$\begin{cases} (1-\lambda)y_1 &= y_2, \\ (2-\lambda)y_2 &= y_1 + y_4, \\ (1-\lambda)y_3 &= y_4, \\ (k+p-\lambda)y_4 &= ky_2 + py_3. \end{cases} \tag{3.5}$$

Let

$$f(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 & 0 & 0 \\ -1 & 2-\lambda & 0 & -1 \\ 0 & 0 & 1-\lambda & -1 \\ 0 & -k & -p & k+p-\lambda \end{bmatrix}.$$

Then

$$f(\lambda) = \lambda[-(1+2k+p) + (4+3k+3p)\lambda - (4+k+p)\lambda^2 + \lambda^3] =: \lambda g(\lambda).$$

$g((3-\sqrt{5})/2) = -k < 0$, $g(1) = p > 0$, $g(3) = 2 - 2k - p < 0$ and $g(k+p+2) = (k+p)^2 + p - 1 > 0$. So $g(\lambda)$, and hence $f(\lambda)$ has no eigenvalues less than $(3-\sqrt{5})/2$. By above discussion, $\alpha(H_2) = (3-\sqrt{5})/2$, and the sufficiency holds.

If $\lambda \neq 1$ and $p = 0$, then by (B) we also get $\lambda = (3 \pm \sqrt{5})/2$. From (A) we obtain 3 equations from (3.5) by dropping the 3rd equation and replacing p by 0. By a similar discussion, we also get $\alpha(H_2) = (3-\sqrt{5})/2$. The result follows. ■

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Received 11 October 2004
Revised 8 January 2005