AN ANTI-RAMSEY THEOREM ON EDGE-CUTS

JUAN JOSÉ MONTELLANO-BALLESTEROS

Instituto de Matemáticas, U.N.A.M.
Ciudad Universitaria, Coyoacán 04510
México, D.F. México

e-mail: juancho@math.unam.mx

Abstract

Let $G = (V(G), E(G))$ be a connected multigraph and let $h(G)$ be the minimum integer $k$ such that for every edge-colouring of $G$, using exactly $k$ colours, there is at least one edge-cut of $G$ all of whose edges receive different colours. In this note it is proved that if $G$ has at least 2 vertices and has no bridges, then $h(G) = |E(G)| - |V(G)| + 2$.

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In this note we consider finite undirected graphs with multiple edges allowed. Let $G = (V(G), E(G))$ be a connected multigraph. Given $Z \subseteq E(G)$, $G - Z$ denotes the graph obtained from $G$ by deleting the edges in $Z$. A set $Z \subseteq E(G)$ will be called an edge-cut if $G - Z$ is a disconnected or a trivial graph, and an edge $e \in E(G)$ will be called a bridge if $\{e\}$ is an edge-cut. A subgraph $H$ of $G$ is said to be a cut-subgraph if $E(H)$ is an edge-cut of $G$.

By an edge-colouring of $G$ we will understand a function $c : E(G) \to \mathcal{C}$ where $\mathcal{C}$ is a set of "colours". If $|c[E(G)|] = k$, then $c$ will be called a $k$-edge-colouring of $G$. Given an edge-colouring of $G$, a subgraph $H$ of $G$ is said to be Totally Multicoloured (TMC) if no pair of edges of $H$ have the same colour. Problems concerning TMC subgraphs in edge-colourings of a host graph are called anti-Ramsey problems (see [1, 2, 3, 4, 5, 6, 7]). Typically, the host graph is a complete graph or some graph with a nice structure, and the property which defines the set of TMC subgraphs in consideration is that they are isomorphic to some graph $H$. When the host graph is a graph with no specific structure, the problem becomes rather intractable unless the
graph \( H \) is very special (see [5]) or, as it happens in this note, the property which defines the set of TMC subgraphs involves strongly the structure of the host graph. Given a graph \( G \), the problem of determining the minimum integer \( h(G) \) such that every \( h(G) \)-edge-colouring of \( G \) produces at least one TMC cut-subgraph of \( G \), is presented in this note. Observe that if \( G \) has only one vertex, there is no edge-cut in \( G \), and in the case that \( G \) has a bridge, \( h(G) = 1 \). The remaining cases are considered in the following theorem.

**Theorem 1.** Let \( G = (V(G), E(G)) \) be a connected graph of order at least 2 which has no bridges. Then \( h(G) = |E(G)| - |V(G)| + 2 \).

Before presenting the proof, let us introduce some definitions. A \( k \)-edge-colouring of \( G \) which produces no TMC cut-subgraph will be called a good \( k \)-colouring of \( G \). A vertex \( x \in V(G) \) will be called a cut-vertex if the graph obtained from \( G \) by deleting \( x \) and all its incident edges is a disconnected graph. \( G \) will be called a block if it is connected and has no cut-vertices. A set \( P_1, \ldots, P_r \) of subgraphs of \( G \) will be called a decomposition of \( G \) if \( E(P_1), \ldots, E(P_r) \) is a partition of \( E(G) \), and will be called an ear-decomposition of \( G \) if it is a decomposition of \( G \) such that: \( P_1 \) is a cycle; for \( 2 \leq j \leq r \), \( P_j \) is a non-trivial path; and for every \( 2 \leq j \leq r \), \( V(P_j) \) intersects \( \bigcup_{i=1}^{j-1} V(P_i) \) in exactly the endpoints of \( P_j \). It is known (see [8]) that \( G \) is a block different from \( K_2 \) if and only if \( G \) has an ear-decomposition.

**Proof of Theorem 1.** Let \( G \) be a connected graph of order at least 2 which has no bridges and let \( k(G) = |E(G)| - |V(G)| + 1 \).

Given a \( (k(G) + 1) \)-edge-colouring of \( G \), let \( H \) be a TMC subgraph of \( G \) of size \( k(G) + 1 \). Since the graph \( G' = G - E(H) \) has \( |V(G')| - 2 \) edges, it must be disconnected and thus \( H \) is a TMC cut-subgraph of \( G \). Therefore \( h(G) \leq k(G) + 1 \).

To finish the proof we only need to show a good \( k(G) \)-colouring of \( G \). First suppose that \( G \) is a block (which is different from \( K_2 \) since \( G \) has no bridges) and let \( P_1, \ldots, P_r \) be an ear-decomposition of \( G \). Observe that

\[
|E(G)| = \sum_{i=1}^{r} |E(P_i)| = |V(P_1)| + \sum_{i=2}^{r} (|V(P_i)| - 1) = |V(G)| + (r - 1)
\]

which implies that \( r = k(G) \). Let \( c \) be a \( k(G) \)-edge-colouring of \( G \) defined as \( c(e) = i \) if and only if \( e \in E(P_i) \). It is not difficult to see that any edge-cut of \( G \) uses at least a pair of edges of some \( P_i \) and, therefore, \( c \) is a good \( k(G) \)-colouring of \( G \).
If $G$ has cut-vertices, then $G$ can be decomposed in $G_0, \ldots, G_t$ blocks, none of them isomorphic to $K_2$ since $G$ has no bridges. For each $j \leq t$, let $P_{j}^1, \ldots, P_{j}^{r_j}$ be an ear-decomposition of $G_j$. Let $c$ be an edge-colouring of $G$ defined as $c(e) = (j, i)$ if and only if $e \in E(P_j^i)$. As in the previous case, it can be seen that each block $G_j$ receives $k(G_j)$ colours and has no TMC cut-subgraphs. Therefore, the number of colours used by $c$ is $\sum_{j=0}^{t} k(G_j) = \sum_{j=0}^{t} (|E(G_j)| - |V(G_j)| + 1) = |E(G)| - (|V(G) + t| + t + 1) = k(G)$, and, since any edge-cut of $G$ contains an edge-cut of some $G_j$, $c$ is a good $k(G)$-colouring of $G$.

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References


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