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AN ANTI-RAMSEY THEOREM ON EDGE-CUTS

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Abstract

Let $G = (V(G), E(G))$ be a connected multigraph and let $h(G)$ be the minimum integer k such that for every edge-colouring of G, using exactly k colours, there is at least one edge-cut of G all of whose edges receive different colours. In this note it is proved that if G has at least 2 vertices and has no bridges, then $h(G) = |E(G)| - |V(G)| + 2$.

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In this note we consider finite undirected graphs with multiple edges allowed. Let $G = (V(G), E(G))$ be a connected graph. Given $Z \subseteq E(G)$, $G - Z$ denotes the graph obtained from G by deleting the edges in Z. A set $Z \subseteq E(G)$ will be called an *edge-cut* if $G - Z$ is a disconnected or a trivial graph, and an edge $e \in E(G)$ will be called a *bridge* if $\{e\}$ is an edge-cut. A subgraph H of G is said to be a *cut-subgraph* if $E(H)$ is an edge-cut of G.

By an *edge-colouring of* G we will understand a function $c : E(G) \to \mathcal{C}$ where $\mathcal{C}$ is a set of "colours". If $|c[E(G)]| = k$, then c will be called a *k-edge-colouring of* G. Given an edge-colouring of G, a subgraph H of G is said to be *Totally Multicoloured* (*TMC*) if no pair of edges of H have the same colour. Problems concerning TMC subgraphs in edge-colourings of a host graph are called *anti-Ramsey problems* (see [1, 2, 3, 4, 5, 6, 7]). Typically, the host graph is a complete graph or some graph with a nice structure, and the property which defines the set of TMC subgraphs in consideration is that they are *isomorphic to some graph H*. When the host graph is a graph with no specific structure, the problem becomes rather intractable unless the graph H is very special (see [5]) or, as it happens in this note, the property which defines the set of TMC subgraphs involves strongly the structure of the host graph. Given a graph G, the problem of determining the minimum integer $h(G)$ such that every $h(G)$-edge-colouring of G produces at least one TMC cut-subgraph of G, is presented in this note. Observe that if G has only one vertex, there is no edge-cut in G, and in the case that G has a bridge, $h(G) = 1$. The remaining cases are considered in the following theorem.

Theorem 1. *Let $G = (V(G), E(G))$ be a connected graph of order at least 2 which has no bridges. Then $h(G) = |E(G)| - |V(G)| + 2$.*

Before presenting the proof, let us introduce some definitions. A k-edge-colouring of G which produces no TMC cut-subgraph will be called a *good k-colouring* of G. A vertex $x \in V(G)$ will be called a *cut-vertex* if the graph obtained from G by deleting x and all its incident edges is a disconnected graph. G will be called a *block* if it is connected and has no cut-vertices. A set $P_1, \ldots, P_r$ of subgraphs of G will be called a *decomposition* of G if $E(P_1), \ldots, E(P_r)$ is a partition of $E(G)$, and will be called an *ear-decomposition* of G if it is a decomposition of G such that: P_1 is a cycle; for $2 \leq j \leq r$, P_j is a non-trivial path; and for every $2 \leq j \leq r$, $V(P_j)$ intersects $\bigcup_{i=1}^{j-1} V(P_i)$ in exactly the endpoints of P_j. It is known (see [8]) that G is a block different from K_2 if and only if G has an ear-decomposition.

***Proof of Theorem* 1.** Let G be a connected graph of order at least 2 which has no bridges and let $k(G) = |E(G)| - |V(G)| + 1$.

Given a $(k(G)+1)$-edge-colouring of G, let H be a TMC subgraph of G of size $k(G) + 1$. Since the graph $G' = G - E(H)$ has $|V(G')| - 2$ edges, it must be disconnected and thus H is a TMC cut-subgraph of G. Therefore $h(G) \leq k(G) + 1$.

To finish the proof we only need to show a good $k(G)$-colouring of G. First suppose that G is a block (which is different from K_2 since G has no bridges) and let $P_1, \ldots, P_r$ be an ear-decomposition of G. Observe that $|E(G)| = \sum_{i=1}^{r} |E(P_i)| = |V(P_1)| + \sum_{i=2}^{r} (|V(P_i)| - 1) = |V(G)| + (r - 1)$ which implies that $r = k(G)$. Let c be a $k(G)$-edge-colouring of G defined as $c(e) = i$ if and only if $e \in E(P_i)$. It is not difficult to see that any edge-cut of G uses at least a pair of edges of some P_i and, therefore, c is a good $k(G)$-colouring of G.

If G has cut-vertices, then G can be decomposed in $G_0, \ldots, G_t$ blocks, none of them isomorphic to K_2 since G has no bridges. For each $j \leq t$, let $P_1^j, \ldots, P_{r_j}^j$ be an ear-decomposition of G_j. Let c be an edge-colouring of G defined as $c(e) = (j, i)$ if and only if $e \in E(P_i^j)$. As in the previous case, it can be seen that each block G_j receives $k(G_j)$ colours and has no TMC cut-subgraphs. Therefore, the number of colours used by c is $\sum_{j=0}^{t} k(G_j) = \sum_{j=0}^{t}(|E(G_j)| - |V(G_j)| + 1) = |E(G)| - (|V(G) + t) + (t + 1) = k(G)$, and, since any edge-cut of G contains an edge-cut of some G_j, c is a good $k(G)$-colouring of G.

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