ON SHORT CYCLES THROUGH PRESCRIBED VERTICES OF A POLYHEDRAL GRAPH

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Abstract

Guaranteed upper bounds on the length of a shortest cycle through \( k \leq 5 \) prescribed vertices of a polyhedral graph or plane triangulation are proved.

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1. Introduction and Results

G.A. Dirac [2] proved that for a given integer \( c \geq 2 \) any \( k \) \((1 \leq k \leq c)\) prescribed vertices of a \( c \)-connected graph belong to a common cycle. However, the complete bipartite graph \( K_{c,c+1} \) shows that this is not true for \( c+1 \) prescribed vertices. In [3] we investigated the length of short cycles through \( k \) prescribed vertices with \( 1 \leq k \leq \min\{c,3\} \) in a \( c \)-connected graph \( G \). From A.K. Kelmans and M.V. Lomonosov [6] we know that any five vertices of a polyhedral graph (that is a planar and 3-connected graph) belong to a common cycle which is best possible.

For given integers \( k, l \) with \( 1 \leq k \leq 5, \ 3 \leq l \) and \( k \leq l \) let \( n_k(l) \) denote the minimum number \( n \) such that there exists a polyhedral graph \( G \) of order \( n \) having a subset of \( k \) vertices with the property that the length of every cycle containing those \( k \) vertices is at least \( l \). In [3] we proved

(i) \( n_1(l) = 3l - 5 \) for \( l \geq 3 \),
(ii) \( n_2(l) = \left\lfloor \frac{3l-1}{2} \right\rfloor \) for \( l \geq 3 \),

(iii) \( n_3(l) = \left\lfloor \frac{3l-1}{2} \right\rfloor \) for \( l \geq 5 \),

and the following results which will be proven here is a continuation of the investigation [3] of short cycles through prescribed vertices for a polyhedral graph.

**Theorem 1.**

\[
n_4(l) = \begin{cases} 
  l & \text{if } l \in \{4, 8\}, \\
  l + 1 & \text{if } l \in \{5, 6, 7, 9, 10\}, \\
  l + 2 & \text{if } l \in \{11, 12\}, \\
  \left\lceil \frac{4l-5}{3} \right\rceil & \text{if } l \geq 13.
\end{cases}
\]

**Theorem 2.**

\[
n_5(l) = \begin{cases} 
  l & \text{if } l = 5 \text{ or } l \geq 8, \\
  l + 1 & \text{if } l = 6 \text{ or } 7.
\end{cases}
\]

For integers \( k, l \) with \( 2 \leq k \leq 5 \), \( 3 \leq l \) and \( k \leq l \) denote by \( t_k(l) \) the minimum number \( n \) such that there exists a plane triangulation \( T \) of order \( n \) with certain \( k \) vertices such that the length of every cycle containing them is at least \( l \). Then we have \( n_k(l) \leq t_k(l) \) since every plane triangulation is 3-connected and thus a polyhedral graph. Notice that even \( n_k(l) = t_k(l) \) holds in every considered case. If, namely, \( G \) is any one of the here or in [3], respectively, constructed graphs to prove an upper bound for \( n_k(l) \) with certain \( k \) and \( l \), then we were able to construct a plane triangulation \( T \) from \( G \) by adding edges only such that the length of a shortest cycle containing the prescribed \( k \) vertices is at least \( l \).

### 2. Proofs

For terminology and notation not defined here we refer to [5]. Let \( G \) be a graph and \( A, B \subseteq V(G) \). A path \( P \) of \( G \) with one end-vertex in \( A \) and \( B \), respectively, and with \( |V(P) \cap A| = |V(P) \cap B| = 1 \) is called an \( A-B \)-path. If \( A \) or \( B \) consists of a single vertex \( x \) we write \( x \) instead of \( \{x\} \). We use \( [x, y] \) to denote an \( x-y \)-path and, moreover, \( [x, y) \) or \( (x, y) \) to denote the segments obtained from \( [x, y] \) by removing \( y \) or both end-vertices, respectively. A path
system is a set of internally disjoint paths. For a path system \( P \) let \( [P] \) and \( EV(P) \) denote the union of all paths and the set of all end-vertices of paths of \( P \), respectively. For some \( a \in V(G) \) and \( B \subseteq V(G) \setminus \{a\} \) a path system \( P \) of \( a\)-\( B \)-paths is called an \( a\)-\( B \)-fan if \( P \cap Q = \{a\} \) for different \( P, Q \in P \).

We need the following lemma which is proved in [3] in more general form.

**Lemma 1.** Let \( G \) be a \( c \)-connected graph with \( a \in V(G) \), \( B \subseteq V(G) \setminus \{a\} \) and a path system \( P \) of \( c - 1 \) \( a\)-\( B \)-paths. Let \( B' = B \setminus EV(P) \) if this is not empty, and \( B' \) be an arbitrary nonempty subset of \( B \) otherwise. Then there is a vertex \( b \in B' \) and a path system \( Q \) of \( c \) \( a\)-\( B \)-paths such that \( EV(Q) = EV(P) \cup \{b\} \), all vertices of \( B \setminus \{b\} \) are end-vertices of as many paths of \( P \) as of \( Q \), and \( Q \) has one more path with end-vertex \( b \) than does \( P \).

We define five polyhedral graphs containing the vertices of a prescribed 4-element set \( X \) as follows. Let \( F_1 \) be the complete graph \( K_4 \) on \( X \). Let \( F_2 \) denote the graph which is obtained from a 4-cycle \( C \) on \( X \) by connecting an additional vertex \( a \notin X \) with all vertices of \( C \). Let \( F_3 \) denote the graph which results from \( C \) and two adjacent vertices \( a, b \notin X \) by connecting two adjacent vertices of \( C \) with \( a \) and the remaining two vertices of \( C \) with \( b \). The graph \( F_3 \) is obtained if two non-adjacent vertices \( a, b \notin X \) are connected with three vertices of a 4-path \( P \) on \( X \), respectively, such that every vertex of \( X \) becomes degree 3. Eventually, let \( F_5 \) denote the cube graph containing the vertices of \( X \) such that no two vertices of \( X \) are adjacent.

**Lemma 2.** Every polyhedral graph \( G \) with \( X = \{x_1, x_2, x_3, x_4\} \subseteq V(G) \) has a subgraph \( H \) which is a subdivision of some \( F_i \) with \( 1 \leq i \leq 5 \).

**Proof of Lemma 2.** Lemma 1 implies that \( G \) has an \( x_1\)-\( x_2 \)-path system \( \{P_1, P_2, P_3\} \) which contains \( x_3 \) by planarity of \( G \), i.e., we may assume that \( x_3 \in V(P_1) \). Moreover, Lemma 1 yields an \( x_3\)-\( V(P_2 \cup P_3) \)-fan \( Q = \{[x_1, x_3], [x_2, x_3], [a, x_3]\} \), where we may assume that \( a \in V(P_2) \). Thus, \( G \) has a path system \( P = \{[x_1, x_2], [x_1, x_3], [x_2, x_3], [a, x_1], [a, x_2], [a, x_3]\} \).

Suppose first, that \( x_4 \) is contained in \( [P] \). Considering symmetries we have to examine three different cases.

**Case 1.** \( x_4 = a \).

Then \( [P] \) is a subdivision of \( F_1 \).
Case 2. $x_4 \in (x_1, x_2)$.
By Lemma 1 there is an $x_4$-$V([P] \setminus (x_1, x_2))$-fan $Q = \{[x_1, x_4], [x_2, x_4], [b, x_4]\}$ where $b \in V([P] \setminus (x_1, x_2))$. Let $H$ denote the subgraph $[P \cup Q] \setminus (x_1, x_2)$ of $G$, then by symmetries there are following subcases. If $b = x_3$ or $b = a$ then $H$ is a subdivision of $F_1$ or $F_2$, respectively. If $b \in (x_1, x_3)$ or $b \in (a, x_1)$ then $H$ is a subdivision of $F_4$ or $F_3$, respectively.

Case 3. $x_4 \in (a, x_1)$.
Applying Lemma 1 again there is an $x_4$-$V([P] \setminus (a, x_1))$-fan $Q = \{[x_1, x_4], [a, x_4], [b, x_4]\}$ where $b \in V([P] \setminus (a, x_1))$. Let $H$ denote the subgraph $[P \cup Q] \setminus (a, x_1)$ of $G$. Considering symmetries we have: If $b \in (x_1, x_2)$ or $b \in [x_2, a)$ then $H$ is a subdivision of $F_4$ or $F_1$, respectively.

Suppose now, that $x_4$ is not contained in $[P]$ and in any other such path system of $G$. Applying Lemma 1 we obtain an $x_4$-$V([P])$-fan $Q = \{[b, x_4], [c, x_4], [d, x_4]\}$ such that each path of $P$ contains at most one vertex of $EV(Q)$ and that at most one path of $P$ with end vertex $a$ contains a vertex of $EV(Q)$. Thereby and since $G$ is planar we may assume that $b \in (x_1, x_2)$, $c \in (x_2, x_3)$ and $d \in (x_1, x_3)$ which implies that $[P \cup Q]$ is a subdivision of $F_5$.

Figure 1 contains further three polyhedral graphs which contain the vertices of $X = \{x_1, x_2, x_3, x_4\}$ and which are needed to prove Theorem 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

**Proof of Theorem 1.** For $l = 6, 7, 11$ and $l \geq 13$ connect a vertex $a$ with each vertex of a 4-cycle $C = x_1x_2x_3x_4x_1$. Put $\alpha = \frac{l-5}{r}$ and suppose $l \equiv r (\text{mod } 3)$ where $r \in \{0, 1, 2\}$. Subdivide every edge $e$ of $C$ with respect to $r$ by the number of new vertices given in Table 1. Connect every new vertex with $a$ and denote the so constructed polyhedral graph by $G$. 
A simple calculation shows that the length of a shortest cycle in $G$ containing $X = \{x_1, x_2, x_3, x_4\}$ is $l$ and that the order of $G$ is $\lceil \frac{4l-5}{3} \rceil$, in every case.

For $l = 4, 5, 8, 9, 10, 12$, let $G$ be $F_1, F_4, F_5, F_6, F_7, F_8$, respectively, with $X \subseteq V(G)$. In these special cases it is not hard to see that the length of a shortest cycle of $G$ containing $X$ is $l$. That together with $n_4(l) \leq |G|$ completes the proof of the upper bound.

Suppose, now, that $G$ is a polyhedral graph of order $n$ with a 4-element subset $X = \{x_1, x_2, x_3, x_4\}$ of $V(G)$ such that the length of a shortest cycle containing $X$ is at least $l$. Because of Lemma 2 it is sufficient to estimate for $i = 1, \ldots, 5$ the order of a subgraph $H$ of $G$ which is a subdivision of $F_i$ with $X \subseteq V(F_i)$ and to deduce a lower bound for $n_4(l)$.

$i = 1$: $H$ has three different cycles $C_1, C_2, C_3$ passing each vertex of $F_1$. Every vertex of $V(H) \setminus V(F_1)$ occurs in precisely two of these three cycles. Thus, $2|H| + 4 \geq |C_1| + |C_2| + |C_3| \geq 3l$ and, consequently, $|H| \geq \lceil \frac{3l-4}{2} \rceil$.

$i = 2$: $H$ has four cycles $C_1, \ldots, C_4$ containing all vertices of $F_2$ and one cycle $C_5$ containing $X$ but no other vertex of $F_2$. Every vertex of $V(H) \setminus V(F_2) \setminus V(C_5)$ occurs in precisely two and every vertex of $V(C_5) \setminus V(F_2)$ in precisely three of the cycles $C_1, \ldots, C_4$. Thus, $2|H| + |C_5| + 4 \cdot 1 + 2 \geq |C_1| + \ldots + |C_4| \geq 4l$ and, thereby, $2|H| + |C_5| + 6 \geq 4l$. From $|C_5| \leq |H| - 1$ we further obtain $|H| \geq \lceil \frac{4l-5}{3} \rceil$.

$i = 3, 4$: $H$ has three different cycles $C_1, C_2, C_3$ passing each vertex of $F_3$. Every vertex of $V(H) \setminus V(F_1)$ occurs in precisely two of these three cycles. Thus, $2|H| + 6 \geq |C_1| + |C_2| + |C_3| \geq 3l$ and, consequently, $|H| \geq \lceil \frac{3l-6}{2} \rceil$.

$i = 5$: $H$ has six different cycles $C_1, \ldots, C_6$ passing each vertex of $F_5$. Every vertex of $V(H) \setminus V(F_5)$ occurs in precisely four of these six cycles. Thus, $4|H| + 2 \cdot 8 \geq |C_1| + \ldots + |C_6| \geq 6l$ and, consequently, $|H| \geq \lceil \frac{3l-8}{2} \rceil$.

Because of $|G| \geq \min\{|H_i| : 1 \leq i \leq 5\}$ and $|G| \geq l$ we obtain
In the special cases \( l = 5, 6 \) one can observe that since \( G \) has a subgraph \( H \) which is a subdivision of \( F_i \) for some \( i \in \{1, \ldots, 5\} \) the order of \( G \) can not be smaller than 6 or 7, respectively. That proves the lower bound.

**Proof of Theorem 2.** For \( l = 5, 6, 7, 8, 9 \) let \( G_l \) be the polyhedral graphs with \( X = \{x_1, \ldots, x_5\} \subseteq V(G_l) \) given in Figure 2.

![Figure 2](image-url)

For \( l > 9 \) let \( G_l \) be the polyhedral graph which results from \( G_9 \) by subdividing \( x_1x_2 \) by \( l - 9 \) new vertices and connecting each of them with \( a \notin X \).

Notice that \( |G_l| = l \) if \( l = 5 \) or \( l \geq 8 \) and \( |G_l| = l + 1 \) if \( l = 6 \) or 7. It is not hard to see that for every \( l \geq 5 \) the length of any cycle of \( G_l \) passing all the vertices of \( X \) is at least \( l \).

So, it remains to prove \( n_5(l) > l \) for \( l = 6, 7 \). Let \( l = 6 \) and suppose that there exists a polyhedral graph \( G \) of order 6 with \( V(G) = X \cup \{a\} \) such that every cycle which contains the vertices of \( X \) is a hamiltonian one. Let \( \mathcal{C}(G) \) denote the set of all cycles of \( G \). Then we may suppose that \( x_1x_2x_3x_4x_5ax_1 \in \mathcal{C}(G) \). Clearly, \( x_1x_5 \notin E(G) \) which implies that \( x_1x_3 \).
or $x_1x_4 \in E(G)$. If $x_1x_3 \in E(G)$ then $x_2x_5 \notin E(G)$ because otherwise $x_1x_2x_3x_4x_3x_1 \in \mathcal{C}(G)$. Thus, $x_3x_5 \in E(G)$ and also $x_1x_4,x_2x_4 \notin E(G)$ because otherwise $x_1x_2x_3x_4x_1 \in \mathcal{C}(G)$, respectively. Thereby, $x_2$ and $x_4$ are connected with a which yields that $\{x_3,a\}$ is a cutset, a contradiction. So, we have that $x_1x_3 \notin E(G)$ and $x_1x_4 \in E(G)$ which implies that $x_3x_5 \notin E(G)$ because otherwise $x_1x_2x_3x_4x_1 \in \mathcal{C}(G)$. That implies $x_2x_5 \in E(G)$ and thereby $d_G(x_3) = 2$, a contradiction.

Now, let $l = 7$ and suppose that there exists a polyhedral graph $G$ of order 7 with $V(G) = X \cup \{a,b\}$ such that every cycle which contains the vertices of $X$ is a hamiltonian one. We may assume that $\mathcal{C}(G)$ contains one of the cycles $C_1 = x_1x_2x_3x_4x_5bx_1$, $C_2 = x_1x_2x_3x_4ax_5bx_1$, $C_3 = x_1x_2x_3ax_4x_5bx_1$.

**Case 1.** $C_1 \in \mathcal{C}(G)$.

Clearly, $x_1x_5,x_1a,x_5b \notin E(G)$. If $x_1x_3 \in E(G)$ then $x_2x_5$, $x_2a \notin E(G)$ because otherwise $x_1x_2x_3x_4x_3x_1 \in \mathcal{C}(G)$, respectively. Thus, $x_3x_5 \in E(G)$ which yields $x_1x_4$, $x_2x_4$, $x_1b \notin E(G)$ because otherwise $x_1x_2x_3x_4x_3x_1$ or $x_1x_2x_4x_3x_1$ or $x_1x_2x_3x_5x_4bx_1 \in \mathcal{C}(G)$, respectively. That implies $x_2b,x_4a \in E(G)$ which means that $\{x_3,a\}$ or $\{x_3,b\}$ would be a cutset of $G$, a contradiction. If $x_1x_3 \notin E(G)$ we have $x_1x_4 \in E(G)$ and $x_3x_5$, $x_3a \notin E(G)$ because otherwise $x_1x_2x_3x_4x_1$ or $x_1x_2x_3x_5x_4bx_1 \in \mathcal{C}(G)$, respectively. That implies $x_2x_5 \in E(G)$ which means by planarity that $x_3b \notin E(G)$. Thus, $d_G(x_3) = 2$, a contradiction.

**Case 2.** $C_2 \in \mathcal{C}(G)$.

Clearly, $x_1x_5,x_4x_5 \notin E(G)$. Suppose, first, $x_1x_3 \in E(G)$ then $x_2x_5 \notin E(G)$ because otherwise $x_1x_2x_5ax_4x_3x_1 \in \mathcal{C}(G)$. Thereby, $x_3x_5 \in E(G)$ which implies that $x_1x_4,x_2x_4 \notin E(G)$ because otherwise $x_1x_2x_3x_5ax_4x_1$ or $x_1x_2x_4x_3x_1 \in \mathcal{C}(G)$, respectively. Thus, $x_1b \in E(G)$ which yields by planarity $x_1a,x_2a \notin E(G)$, i.e., $\{x_3,b\}$ would be a cutset of $G$, a contradiction. Suppose, now, $x_1x_3 \notin E(G)$ and $x_1x_4 \in E(G)$. Then $x_2x_5$, $x_3x_5 \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4bx_1$ or $x_1x_2x_3x_5ax_4x_1 \in \mathcal{C}(G)$, respectively. That yields $d_G(x_5) = 2$, a contradiction. Suppose $x_1x_3,x_1x_4 \notin E(G)$ then $x_1a \in E(G)$. If, here, $x_2x_5 \in E(G)$ then $x_3x_5 \notin E(G)$ because otherwise $x_1x_2x_5x_3x_1ax_1 \in \mathcal{C}(G)$. By planarity, $x_3b,x_4b \notin E(G)$ which means that $\{x_2,a\}$ would be a cutset of $G$, a contradiction. If $x_2x_5 \notin E(G)$ then $x_3x_5 \in E(G)$ and, consequently, $x_2x_4 \notin E(G)$ because otherwise $x_1x_2x_3x_5ax_1 \in \mathcal{C}(G)$. Planarity implies $x_4b \notin E(G)$ and, hence, $d_G(x_4) = 2$, a contradiction.
Case 3. $C_3 \in C(G)$.

Clearly, $x_1x_5, x_3x_4 \not\in E(G)$. Suppose, first, $x_1x_3 \in E(G)$ then $x_2x_4, x_2x_5 \not\in E(G)$ because otherwise $x_1x_3x_2x_4x_5bx_1$ or $x_1x_3x_4x_5x_2x_1 \in C(G)$, respectively. That implies $x_1x_4$ or $x_4b \in E(G)$. If $x_1x_4 \in E(G)$ then $x_2b \not\in E(G)$ because otherwise $x_1x_3x_2bx_5x_4x_1 \in C(G)$. Thereby, $x_2a \in E(G)$ which implies $x_3x_5, x_3b \not\in E(G)$ because otherwise $x_1x_2x_3x_5x_4x_1$ or $x_1x_2x_3bx_5x_4x_1 \in C(G)$, respectively. That gives $d_G(x_3) = 2$, a contradiction. If $x_1x_4 \not\in E(G)$ then $x_4b \in E(G)$ which implies $x_3x_5 \not\in E(G)$ because otherwise $x_1x_2x_3x_5x_4bx_1 \in C(G)$. Thus, $x_5a \in E(G)$ and $\{a, b\}$ would be a cutset of $G$, a contradiction.

Suppose, now, $x_1x_3 \not\in E(G)$ and $x_1x_4 \in E(G)$. Then $x_3x_5, x_3b \not\in E(G)$ because otherwise $x_1x_2x_3x_5x_4x_1$ or $x_1x_2x_3bx_5x_4x_1 \in C(G)$, respectively. That implies $d_G(x_3) = 2$, a contradiction.

Suppose, eventually, $x_1x_3, x_1x_4 \not\in E(G)$ then $x_1a \in E(G)$. That implies $x_3x_5 \not\in E(G)$ because otherwise $x_1x_2x_3x_5ax_1 \in C(G)$). Thereby, $x_3b \in E(G)$ and by planarity $x_2x_4, x_2x_5 \not\in E(G)$ which means that $\{a, b\}$ would be a cutset of $G$, a contradiction, and the proof is complete.\vspace{1em}

References


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