A NOTE ON MAXIMAL COMMON SUBGRAPHS OF THE DIRAC’S FAMILY OF GRAPHS

JOZEF BUCKO∗

PETER MIHÓK∗

Technical University of Košice
Faculty of Economics
Němcovej 32, 040 01 Košice, Slovakia

e-mail: peter.mihok@tuke.sk
e-mail: jozef.bucko@tuke.sk

JEAN-FRANÇOIS SACLÉ

LRI, Bât. 490, Université de Paris-Sud
91405 Orsay, France

e-mail: sacle@lri.fr

AND

MARIUSZ WOŹNIAK

AGH University of Science and Technology
Department of Applied Mathematics
Al. Mickiewicza 30, 30–059 Kraków, Poland

e-mail: mwozniak@agh.edu.pl

Abstract

Let \( F^n \) be a given set of unlabeled simple graphs of order \( n \). A maximal common subgraph of the graphs of the set \( F^n \) is a common subgraph \( F \) of order \( n \) of each member of \( F^n \), that is not properly contained in any larger common subgraph of each member of \( F^n \). By well-known Dirac’s Theorem, the Dirac’s family \( D F^n \) of the graphs of order \( n \) and minimum degree \( \delta \geq \frac{n}{2} \) has a maximal common subgraph

∗Research supported by Slovak VEGA Grant 2/4134/24.
containing $C_n$. In this note we study the problem of determining all maximal common subgraphs of the Dirac’s family $\mathcal{DF}^{2n}$ for $n \geq 2$.

**Keywords:** maximal common subgraph, Dirac’s family, Hamiltonian cycle.

**2000 Mathematics Subject Classification:** 05C75, 05C45.

We follow the definitions and terminology of [1]. Let $\mathcal{F}^n$ be a given set of unlabeled simple graphs of order $n$. A maximal common subgraph of the graphs of the set $\mathcal{F}^n$ is a common subgraph $F$ of order $n$ of each member of $\mathcal{F}^n$, that is not properly contained in any larger common subgraph of each member of $\mathcal{F}^n$. By well-known Dirac’s Theorem, the Dirac’s family $\mathcal{DF}^n$ of the graphs of order $n$ and minimum degree $\delta \geq \frac{n}{2}$ has a maximal common subgraph containing $C_n$ (see [2, 3, 4]). The cycles $C_4$ and $C_6$ are maximal common subgraphs of $\mathcal{DF}^4$ and $\mathcal{DF}^6$, respectively. While $C_4$ is the unique maximal common subgraph of $\mathcal{DF}^4$, for $\mathcal{DF}^6$ it is easy to check that there are exactly two maximal common subgraphs: $C_6$ and the graph $F_6$ (see Figure 1).

Figure 1. Maximal common subgraphs of $\mathcal{DF}^6$.

In this note we study the problem of determining maximal common subgraphs of the Dirac’s family $\mathcal{DF}^{2n}$ for $n \geq 2$. It is easy to see that to determine all maximal common subgraphs of the Dirac’s family $\mathcal{DF}^n$, it is enough to consider the maximal common subgraphs of the family of the minimal elements of the set $\mathcal{DF}^n$ partially ordered by the relation $\subseteq$ - to be a subgraph. The minimal Dirac’s graphs of order 8 are presented in Figure 2.

Because the complete bipartite graph $K_{4,4}$ is a member of the set of minimal elements of $\mathcal{DF}^8$, each maximal common subgraph of the set $\mathcal{DF}^8$ must be a bipartite graph with a balanced regular two-colouring (i.e., four vertices in each colour class). Using this fact we determined all maximal
common subgraphs of the set $\mathcal{DF}^8$. They are presented in the Figure 3. Since they could also be found by a computer search, we omit a detailed proof here.

\begin{figure}[h]
\begin{center}
\begin{tabular}{ccc}
$G_1$ & $G_2$ & $G_3$
\end{tabular}
\end{center}
\begin{center}
\begin{tabular}{ccc}
$G_4$ & $G_5$ & $G_6$
\end{tabular}
\end{center}
\begin{center}
\begin{tabular}{ccc}
$G_7$ & $G_8$ & $G_9$
\end{tabular}
\end{center}
\begin{center}
\begin{tabular}{ccc}
$G_{10}$ & $G_{11}$ & $G_{12}$
\end{tabular}
\end{center}
\caption{Minimal Dirac’s graphs of order 8.}
\end{figure}
The problem of determining the maximal common subgraphs for the Dirac’s family $\mathcal{DF}^n$ is much more complicated for odd $n$ and we can mention only that the wheel $W_5 = K_1 + C_4$ is the unique maximal common subgraph of $\mathcal{DF}^5$, however for $\mathcal{DF}^7$ there are at least 5 different maximal common subgraphs.

As the main result of this note we will show that the Hamiltonian cycle $C_n$ is not a maximal common subgraph of the Dirac’s family $\mathcal{DF}^n$ for $n \geq 7$.

The proof is based on the following lemma.

**Lemma 1.** Let $G$ be a graph of order $n \geq 7$ satisfying Dirac’s condition $\delta(G) \geq n/2$. Let $H = abcd$ be a 4-cycle in $G$ having a tail $T = [x_0 \cdots x_k]$ of maximum length $k$. Then $k = n - 4$.

**Proof.** Without loss of generality, we may assume that $x_k = a$. Assume, to the contrary, that $V \setminus (H \cup T)$ is nonempty and let $y$ be a vertex in this set. We will produce a contradiction by finding in $G$ a 4-cycle with a longer tail.

Denote by $x_{i_1} = x_1, \ldots, x_{i_p}$ the neighbours of $x_0$ belonging to $T$.

**Case 1.** If there is an $i \in \{1, \ldots, p\}$ such that $yx_{i-1} \in E$, then $[yx_{i-1} \cdots x_0 x_i \cdots x_k]$ is a tail of length $k + 1$ for $H$. Assume henceforth the contrary. Let $q$ be the number of neighbours of $x_0$ in the set $\{b, c, d\}$. We have by hypothesis $p + q \geq n/2$. Let $q_1$ be the number of neighbours of $y$ in the set $\{a, b, c, d\}$. Note that $y$ has at most $k - p$ neighbours in the set $T \setminus \{a\}$ and at most $n - k - 5$ neighbours outside the set $H \cup T$.

**Case 2.** If $b$ and $d$ are both neighbours of $y$, then $[x_0 \cdots x_kb]$ is a tail of length $k + 1$ for the $C_4$ $ybcd$ of $G$. So we may assume that we have
q_1 \leq 3.\ Now\ we\ obtain\ by\ hypothesis\ for\ the\ number\ of\ neighbours\ of\ y:\ \n/2 \leq \deg(y) \leq (k-p) + q_1 + n - k - 5 = n + q_1 - 5 - p \leq n/2 + q + q_1 - 5,\ so\ q + q_1 \geq 5\ implying\ q \geq 2.\ So\ x_0\ must\ have\ a\ neighbour\ in\ the\ set\ \{b,\ d\}.\ By\ symmetry,\ we\ may\ suppose\ bx_0 \in E.

Case 3. If a and c are both neighbours of y, then \([x_{k-1} \cdots x_0 ba]\) is a tail of length \(k+1\) for the \(C_4\) gadc of \(G\). This being not the case, we have \(q_1 \leq 2\) therefore \(q = 3\) and \(q_1 = 2\). Now the three vertices \(b, c, d\) are neighbours of \(x_0\), and by symmetry we may suppose \(yb \in E\). There remains only two cases, according to whether \(a\) or \(c\) is the other neighbour of \(y\) in \(H\).

Case 4. If \(a\) is neighbour of \(y\), then \([x_1 \cdots x_k yb]\) is a tail of length \(k + 1\) for the cycle \(x_0bcd\).

Case 5. If \(c\) is neighbour of \(y\), then \([da \cdots x_0]\) is a tail of length \(k + 1\) for the cycle \(ybx_0c\).

\[\begin{array}{c}
\text{b} \\
y \\
\text{c} \\
\text{a} = x_k \\
\text{x}_1 \\
\text{x}_{i-1} \\
\text{x}_1 \\
\text{d}
\end{array}\]

Figure 4. Case 1 in the proof of Lemma 1.

Theorem 2. Let \(G = (V, E)\) be a graph of order \(|V| = n\), with \(n \geq 7\). If \(G\) satisfies the Dirac’s condition \(\delta(G) \geq n/2\), then \(G\) contains as a subgraph, a Hamiltonian cycle with a chord that skips two vertices on this cycle.

Proof. It is straightforward that \(G\) contains a cycle \(C_4\) as a subgraph. For a subgraph \(H\) of \(G\), a tail of \(H\) is any path \([x_0 \cdots x_k]\) in \(G\) sharing with \(H\) only the vertex \(x_k\). We now complete the proof of the theorem, by examining a 4-cycle \(H = abcd\) with a tail \(T\) of length \(k = n - 4 \geq 3\). Such a cycle exists by the previous lemma. We assume, as before, that \(x_k = a\) and keep the same notations as in the proof of the lemma. In the same way, we study and eliminate all possible cases.
Case 1. If $b$ or $d$, say $b$ by symmetry, is neighbour of $x_{i-1}$, with $i \in \{i_1, \ldots, i_p\}$ then the Hamiltonian cycle $adcbx_{i-1} \cdots x_0x_{i} \cdots x_k$ has the chord $ab$.

If this is not the case, then $q \leq 1$ and we must have for the neighbours of $b$ (or $d$) : $n/2 \leq \deg(b) \leq n - 1 - p \leq n/2 + q - 1$, so $q = 1$ and we have $cx_0 \in E$, $p = n/2 - 1$ (hence $n$ is even and $n \geq 8$ in this case). Moreover, $x_i, 1 \leq i \leq k - 1$ is neighbour of $b$ if and only if it is also neighbour of $d$ and $i + 1$ is not in the set $\{i_2, \ldots, i_p\}$. Finally, we must have, for the above inequalities being equalities, $bd \in E$.

Case 2. If $bx_{k-1} \in E$, then the Hamiltonian cycle $cdabx_{k-1} \cdots x_0c$ has the chord $cb$. Assuming the contrary we must have $ax_0 \in E$, otherwise $bx_{k-1} \notin E$ and we must have $ax_0 \in E$.

Case 3. If $bx_{k-2} \in E$, it is a chord of the Hamiltonian cycle $cdab \cdots x_0c$. At last, we may assume $x_0x_{k-1} \in E$, otherwise $bx_{k-2} \notin E$. Since $n \geq 8$, $b$ (as well as $d$) must have a neighbour $x_i$ with $1 \leq i \leq k - 3$, forming a 4-cycle $bx_idc$ with the tail $[x_{i+1} \cdots x_0 \cdots x_i]$ or the tail $[x_{i-1} \cdots x_0a \cdots x_i]$. In these configurations, $x_{i+1}$ or $x_{i-1}$ play the role of $x_0$, and $c$ keeps its own one. Therefore, after eliminating the first case, we obtain that both $x_{i-1}c$ and $x_{i+1}c$ are in $E$. Now the cycle $beda$ has the tail $x_{k-1} \cdots x_0c$, in which vertices $a$ and $c$ exchange their places. Therefore, it remains to consider only the case when both $ax_{i+1}$ and $ax_{i-1}$ are in $E$. In this case the Hamiltonian cycle $x_{i+1} \cdots x_{k-1}x_0 \cdots x_i badcx_{i+1}$ has the chord $ax_{i+1}$.

References


Received 22 June 2004
Revised 13 June 2005