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A NOTE ON MAXIMAL COMMON SUBGRAPHS OF THE DIRAC'S FAMILY OF GRAPHS

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Abstract

Let $\mathcal{F}^n$ be a given set of unlabeled simple graphs of order n. A *maximal common subgraph* of the graphs of the set $\mathcal{F}^n$ is a common subgraph F of order n of each member of $\mathcal{F}^n$, that is not properly contained in any larger common subgraph of each member of $\mathcal{F}^n$. By well-known Dirac's Theorem, the Dirac's family $\mathcal{DF}^n$ of the graphs of order n and minimum degree $\delta \geq \frac{n}{2}$ has a maximal common subgraph

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containing C_n. In this note we study the problem of determining all maximal common subgraphs of the Dirac's family $\mathcal{DF}^{2n}$ for $n \geq 2$.

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We follow the definitions and terminology of [1]. Let $\mathcal{F}^n$ be a given set of unlabeled simple graphs of order n. A *maximal common subgraph* of the graphs of the set $\mathcal{F}^n$ is a common subgraph F of order n of each member of $\mathcal{F}^n$, that is not properly contained in any larger common subgraph of each member of $\mathcal{F}^n$. By well-known Dirac's Theorem, the Dirac's family $\mathcal{DF}^n$ of the graphs of order n and minimum degree $\delta \geq \frac{n}{2}$ has a maximal common subgraph containing C_n (see [2, 3, 4]). The cycles C_4 and C_6 are maximal common subgraphs of $\mathcal{DF}^4$ and $\mathcal{DF}^6$, respectively. While C_4 is the unique maximal common subgraph of $\mathcal{DF}^4$, for $\mathcal{DF}^6$ it is easy to check that there are exactly two maximal common subgraphs: C_6 and the graph F_6 (see Figure 1).

Figure 1. Maximal common subgraphs of $\mathcal{DF}^6$.

In this note we study the problem of determining maximal common subgraphs of the Dirac's family $\mathcal{DF}^{2n}$ for $n \geq 2$. It is easy to see that to determine all maximal common subgraphs of the Dirac's family $\mathcal{DF}^n$, it is enough to consider the maximal common subgraphs of the family of the minimal elements of the set $\mathcal{DF}^n$ partially ordered by the relation $\subseteq$ - *to be a subgraph*. The minimal Dirac's graphs of order 8 are presented in Figure 2.

Because the complete bipartite graph $K_{4,4}$ is a member of the set of minimal elements of $\mathcal{DF}^8$, each maximal common subgraph of the set $\mathcal{DF}^8$ must be a bipartite graph with a balanced regular two-colouring (i.e., four vertices in each colour class). Using this fact we determined all maximal common subgraphs of the set $\mathcal{DF}^8$. They are presented in the Figure 3. Since they could also be found by a computer search, we omit a detailed proof here.

G_1 G_2 G_3

G_4 G_5 G_6

G_7 G_8 G_9

G_{10} G_{11} G_{12}

Figure 2. Minimal Dirac's graphs of order 8.

H_1 H_2 H_3

Figure 3. The maximal common subgraphs of $\mathcal{DF}^8$.

The problem of determining the maximal common subgraphs for the Dirac's family $\mathcal{DF}^n$ is much more complicated for odd n and we can mention only that the wheel $W_5 = K_1 + C_4$ is the unique maximal common subgraph of $\mathcal{DF}^5$, however for $\mathcal{DF}^7$ there are at least 5 different maximal common subgraphs.

As the main result of this note we will show that the Hamiltonian cycle C_n is not a maximal common subgraph of the Dirac's family $\mathcal{DF}^n$ for $n \geq 7$. The proof is based on the following lemma.

Lemma 1. *Let G be a graph of order $n \geq 7$ satisfying Dirac's condition $\delta(G) \geq n/2$. Let $H = abcd$ be a 4-cycle in G having a tail $T = [x_0 \cdots x_k]$ of maximum length k. Then $k = n - 4$.*

Proof. Without loss of generality, we may assume that $x_k = a$. Assume, to the contrary, that $V \setminus (H \cup T)$ is nonempty and let y be a vertex in this set. We will produce a contradiction by finding in G a 4-cycle with a longer tail.

Denote by $x_{i_1} = x_1, \ldots, x_{i_p}$ the neighbours of x_0 belonging to T.

Case 1. If there is an $i \in \{1, \cdots, p\}$ such that $yx_{i-1} \in E$, then $[yx_{i-1} \cdots x_0 x_i \cdots x_k]$ is a tail of length $k+1$ for H. Assume henceforth the contrary. Let now q be the number of neighbours of x_0 in the set $\{b, c, d\}$. We have by hypothesis $p + q \geq n/2$. Let q_1 be the number of neighbours of y in the set $\{a, b, c, d\}$. Note that y has at most $k - p$ neighbours in the set $T \setminus \{a\}$ and at most $n - k - 5$ neighbours outside the set $H \cup T$.

Case 2. If b and d are both neighbours of y, then $[x_0 \cdots x_k b]$ is a tail of length $k+1$ for the C_4 $ybcd$ of G. So we may assume that we have

$q_1 \leq 3$. Now we obtain by hypothesis for the number of neighbours of y : $n/2 \leq deg(y) \leq (k-p)+q_1+n-k-5 = n+q_1-5-p \leq n/2+q+q_1-5$, so $q+q_1 \geq 5$ implying $q \geq 2$. So x_0 must have a neighbour in the set $\{b,d\}$. By symmetry, we may suppose $bx_0 \in E$.

Case 3. If a and c are both neighbours of y, then $[x_{k-1} \cdots x_0ba]$ is a tail of length $k+1$ for the C_4 $yadc$ of G. This being not the case, we have $q_1 \leq 2$ therefore $q = 3$ and $q_1 = 2$. Now the three vertices b,c,d are neighbours of x_0, and by symmetry we may suppose $yb \in E$. There remains only two cases, according to whether a or c is the other neighbour of y in H.

Case 4. If a is neighbour of y, then $[x_1 \cdots x_kyb]$ is a tail of length $k+1$ for the cycle x_0bcd.

Case 5. If c is neighbour of y, then $[da \cdots x_0]$ is a tail of length $k+1$ for the cycle ybx_0c. ■

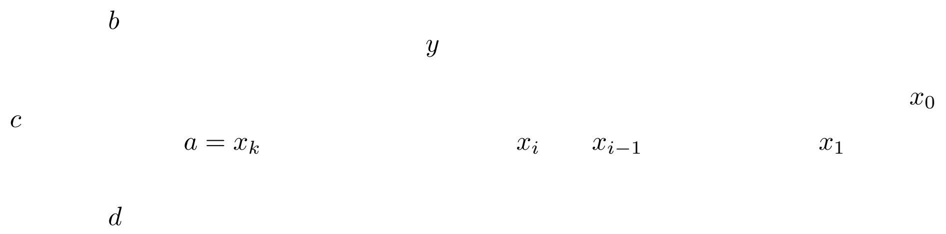

Figure 4. Case 1 in the proof of Lemma 1.

Theorem 2. *Let $G = (V,E)$ be a graph of order $|V| = n$, with $n \geq 7$. If G satisfies the Dirac's condition $\delta(G) \geq n/2$, then G contains as a subgraph, a Hamiltonian cycle with a chord that skips two vertices on this cycle.*

Proof. It is straightforward that G contains a cycle C_4 as a subgraph. For a subgraph H of G, a tail of H is any path $[x_0 \cdots x_k]$ in G sharing with H only the vertex x_k. We now complete the proof of the theorem, by examining a 4-cycle $H = abcd$ with a tail T of length $k = n-4 \geq 3$. Such a cycle exists by the previous lemma. We assume, as before, that $x_k = a$ and keep the same notations as in the proof of the lemma. In the same way, we study and eliminate all possible cases.

Case 1. If b or d, say b by symmetry, is neighbour of x_{i-1}, with $i \in \{i_1, \ldots, i_p\}$ then the Hamiltonian cycle $adcbx_{i-1} \cdots x_0 x_i \cdots x_k$ has the chord ab.

If this is not the case, then $q \leq 1$ and we must have for the neighbours of b (or d) : $n/2 \leq deg(b) \leq n - 1 - p \leq n/2 + q - 1$, so $q = 1$ and we have $cx_0 \in E$, $p = n/2 - 1$ (hence n is even and $n \geq 8$ in this case). Moreover, $x_i, 1 \leq i \leq k-1$ is neighbour of b if and only if it is also neighbour of d *and* $i+1$ is not in the set $\{i_2, \cdots, i_p\}$. Finally, we must have, for the above inequalities being equalities, $bd \in E$.

Case 2. If $bx_{k-1} \in E$, then the Hamiltonian cycle $cdabx_{k-1} \cdots x_0 c$ has the chord cb. Assuming the contrary we must have $ax_0 \in E$, otherwise $bx_{k-1} \notin E$ and we must have $ax_0 \in E$.

Case 3. If $bx_{k-2} \in E$, it is a chord of the Hamiltonian cycle $cdba \cdots x_0 c$. At last, we may assume $x_0 x_{k-1} \in E$, otherwise $bx_{k-2} \notin E$. Since $n \geq 8$, b (as well as d) must have a neighbour x_i with $1 \leq i \leq k-3$, forming a 4-cycle $bx_i dc$ with the tail $[x_{i+1} \cdots ax_0 \cdots x_i]$ or the tail $[x_{i-1} \cdots x_0 a \cdots x_i]$. In these configurations, x_{i+1} or x_{i-1} play the role of x_0, and c keeps its own one. Therefore, after eliminating the first case, we obtain that both $x_{i-1}c$ and $x_{i+1}c$ are in E. Now the cycle $bcda$ has the tail $x_{k-1} \cdots x_0 c$, in which vertices a and c exchange their places. Therefore, it remains to consider only the case when both ax_{i+1} and ax_{i-1} are in E. In this case the Hamiltonian cycle $x_{i+1} \cdots x_{k-1} x_0 \cdots x_i badcx_{i+1}$ has the chord ax_{i+1}. ■

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