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# ON THE $p$-DOMINATION NUMBER OF CACTUS GRAPHS

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**Abstract**

Let $p$ be a positive integer and $G = (V, E)$ a graph. A subset $S$ of $V$ is a $p$-dominating set if every vertex of $V - S$ is dominated at least $p$ times. The minimum cardinality of a $p$-dominating set a of $G$ is the $p$-domination number $\gamma_p(G)$. It is proved for a cactus graph $G$ that $\gamma_p(G) \leqslant (|V| + |L_p(G)| + c(G))/2$, for every positive integer $p \geqslant 2$, where $L_p(G)$ is the set of vertices of $G$ of degree at most $p - 1$ and $c(G)$ is the number of odd cycles in $G$.

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## 1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of $G$ is $n(G) = |V(G)|$ and the *degree* of a vertex $v$, denoted by $\deg_G(v)$, is the number of vertices adjacent to $v$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A vertex of $V$ is called a *cut vertex* if removing it from $G$ increases the number of components of $G$. A graph $G$ is called a *cactus graph* if each edge of $G$ is contained in at most one cycle. A *unicycle graph* is a graph with exactly one cycle. A tree $T$ is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively $p$ and $q$ leaves attached at each support vertex, is denoted by $S_{p,q}$.

For a positive integer $p$, a subset $S$ of $V(G)$ is a *$p$-dominating set* if every vertex not in $S$ is adjacent to at least $p$ vertices of $S$. The *$p$-domination number* $\gamma_p(G)$ is the minimum cardinality of a $p$-dominating set of $G$. Note that every graph $G$ has a $p$-dominating set, since $V(G)$ is such a set. Also the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. The concept of $p$-domination was introduced by Fink and Jacobson [2, 3]. For more details on domination and its variations see the books of Haynes, Hedetniemi, and Slater [4, 5].

We make a straightforward observation.

**Observation 1.** Every $p$-dominating set of a graph $G$ contains any vertex of degree at most $p - 1$.

In this paper we present an upper bound for the $p$-domination number for cactus graphs in terms of the order, the number of odd cycles and the number of vertices of degrees at most $p - 1$.

The following result due to Blidia et al. [1] will be useful for the next. Let $L_p(G)$ denote the set $\{x \in V(G) : \deg_G(x) \leqslant p - 1\}$.

**Theorem 2** (Blidia, Chellali and Volkmann [1]). *Let $p$ be a positive integer. If $G$ is a bipartite graph then*

$$\gamma_p(G) \leqslant (n + |L_p(G)|)/2 .$$

## 2. Main Results

We begin by giving an upper bound for the $p$-domination number for connected unicycle graphs.

**Theorem 3.** *Let $p \geqslant 2$ be a positive integer. If $G$ is a connected unicycle graph then*

$$\gamma_p(G) \leqslant (n + |L_p(G)| + 1)/2$$

*and this bound is sharp.*

***Proof.*** Let $G$ be a connected unicycle graph. If $G$ is bipartite then the result is valid by Theorem 2. So assume that $G$ contains an odd cycle denoted by $C$. If $G = C$, then $\gamma_p(G) = n$ if $p \geqslant 3$ and $\gamma_p(G) = (n+1)/2$ if $p = 2$, in both cases the result holds. Thus we assume that $G \neq C$, that is $G$ contains at least one leaf.

Suppose that the result does not hold and let $G$ be the smallest connected unicycle graph such that $\gamma_p(G) > (n+|L_p(G)|+1)/2$. We claim that every vertex on $C$ has degree exactly $p$. Suppose to the contrary that there is a vertex $x \in C$ such that $\deg_G(x) \neq p$ and let $y$ be one of its two neighbors on $C$. Let $G'$ be the spanning graph of $G$ obtained by removing the edge $xy$. Then $G'$ is tree and so a bipartite graph. We also have $|L_p(G')| \leqslant |L_p(G)|+1$ and $n(G) = n(G')$. According to Theorem 2, we deduce that

$$\gamma_p(G) \leqslant \gamma_p(G') \leqslant (n(G') + \left|L_p(G')\right|)/2 \leqslant (n(G) + |L_p(G)| + 1)/2,$$

a contradiction with our assumption.

Similarly, we will show that every vertex not on $C$ and different to a leaf has degree at least $p$. Assume to the contrary that there is a vertex $x \in V(G) - C$ different to a leaf with $\deg_G(x) \leqslant p - 1$ and let $z$ be its neighbor in the unique path from $x$ to $C$. Let $G_1$ be the connected unicycle subgraph of $G$ containing $x$ and obtained by removing all the edges incident to $x$ excepted the edge $xz$, and let $G_2$ be the component containing $x$ by removing the edge $xz$. Let $D_1$ and $D_2$ denote a $\gamma_p(G_1)$-set and a $\gamma_p(G_2)$-set, respectively. Clearly $G_1$ contains $C$ and $G_2$ is a tree, $x \in D_1 \cap D_2$, $x \in L_p(G_1) \cap L_p(G_2)$, $|L_p(G_1)|+|L_p(G_2)| = |L_p(G)|+1$ and $n(G_1)+n(G_2) = n(G)+1$. Furthermore, $D_1 \cup D_2$ is a $p$-dominating set of $G$. In addition, $G_1$ and $G_2$ have order less than $G$ and so satisfy the theorem, implying that

$$\begin{aligned} \gamma_p(G) &\leqslant |D_1 \cup D_2| = \gamma_p(G_1) + \gamma_p(G_2) - 1 \\ &\leqslant (n(G_1) + |L_p(G_1)| + 1)/2 + (n(G_2) + |L_p(G_2)|)/2 - 1 \\ &\leqslant (n + |L_p(G)| + 1)/2, \end{aligned}$$

contradicting the assumption.

Suppose now that $V(G) - C$ contains a support vertex. Let $a$ be a support vertex of $G$ of maximum distance from $C$. As seen above, $a$ has degree at least $p$. Let $G' = G - (L_a \cup \{a\})$. Then $\gamma_p(G') + |L_a| = \gamma_p(G)$, $n(G') = n(G) - |L_a| - 1$ and $L_p(G) \geqslant L_p(G') + |L_a| - 1$. It follows that

$$\gamma_p(G') + |L_a| = \gamma_p(G) > (n(G) + |L_p(G)| + 1)/2$$

implying that

$$\gamma_p(G') > (n(G) + |L_p(G)| + 1 - 2\,|L_a|)/2$$

and so

$$\gamma_p(G') > (n(G') + \left|L_p(G')\right| + 1)/2$$

contradicting our assumption that $G$ is the smallest graph that does not satisfy the theorem.

Consequently, every vertex of $V(G) - C$ must be a leaf and so every vertex on $C$ is adjacent to exactly $p - 2$ leaves, which implies that

$$\gamma_p(G) = n - (|V(C)| - 1)/2 = (n(G) + |L_p(G)| + 1)/2$$

a contradiction.

To see that this bound is sharp, consider the graph $G$ formed by an odd cycle $C$ where each vertex on $C$ is adjacent to exactly $p - 2$ vertices. Then $\gamma_p(G) = n - (|V(C)| - 1)/2 = (n(G) + |L_p(G)| + 1)/2$. ■

**Theorem 4.** *Let $p \geqslant 2$ be a positive integer. If $G$ is a connected cactus graph with $c(G)$ odd cycles then,*

$$\gamma_p(G) \leqslant (n + |L_p(G)| + c(G))/2,$$

*and this bound is sharp.*

***Proof.*** If $G$ is a bipartite graph, then by Theorem 2 the result holds. If $G$ is a unicycle graph then by Theorem 3 the result is also valid. So consider a cactus graph $G$ containing at least two cycles with one of odd length. Assume that the result does not hold and let $G$ be the smallest cactus graph such that $\gamma_p(G) > (n(G) + |L_p(G)| + c(G))/2$. We also assume that among all such graphs, $G$ is the one having the fewest edges.

First, let $u$ be a vertex on an odd cycle $C$ of $G$ and assume that $\deg_G(u) \neq p$. Let $G'$ be the spanning graph of $G$ obtained by removing an edge of $C$ incident with $u$. Then $|L_p(G')| \leqslant |L_p(G)|+1$ and $c(G') = c(G)-1$. Also $G'$ satisfies the result and so

$$\begin{aligned}\gamma_p(G) &\leqslant \gamma_p(G') \leqslant (n(G') + \left|L_p(G')\right| + c(G'))/2 \\ &\leqslant (n(G) + |L_p(G)| + 1 + c(G) - 1)/2 = (n + |L_p(G)| + c(G))/2\,,\end{aligned}$$

a contradiction. Thus every vertex in an odd cycle has degree exactly $p$.

Now consider a vertex $v$ different from a leaf and contained in no odd cycle. Then, either $v$ is a cut vertex or $v$ is on an even cycle and $\deg_G(v) = 2$. Suppose first that $v$ is a cut vertex with $\deg_G(v) < p$. Let $G_1$ and $G_2$ be two connected cactus subgraphs of $G$ with $V(G) = V(G_1) \cup V(G_2)$ having $v$ as a unique common vertex. Then, $c(G) = c(G_1) + c(G_2), n(G) = n(G_1) + n(G_2) - 1, |L_p(G)| = |L_p(G_1)| + |L_p(G_2)| - 1$. Now let $D_1$ and $D_2$ denote a $\gamma_p(G_1)$-set and a $\gamma_p(G_2)$-set, respectively. Then $v \in D_1 \cup D_2$ and $D_1 \cup D_2$ is a $p$-dominating set of $G$. Since $G_1$ and $G_2$ satisfy the result,

$$\begin{aligned}\gamma_p(G) &\leqslant |D_1 \cup D_2| = |D_1| + |D_2| - 1 \\ &\leqslant (n(G_1) + |L_p(G_1)| + c(G_1))/2 + (n(G_2) + |L_p(G_2)| + c(G_2))/2 - 1 \\ &\leqslant (n(G) + |L_p(G)| + c(G))/2\,,\end{aligned}$$

a contradiction. Consequently, every cut vertex contained in no odd cycle has degree at least $p$.

Now let $v$ be a vertex on an even cycle with $\deg_G(v) = 2$. Since we have assumed in the beginning of the proof that $G$ has at least two cycles, we have $p \geqslant 3$. We claim that each neighbor of $v$ has degree exactly $p$. Indeed, let $u$ be a neighbor of $v$ and assume that $\deg_G(u) \neq p$. Then every $\gamma_p(G')$-set $S$ is a $p$-dominating set of $G$ where $G'$ is obtained from $G$ by removing the edge $vu$. So

$$\gamma_p(G) \leqslant |S| \leqslant (n(G') + \left|L_p(G')\right| + c(G'))/2 = (n(G) + |L_p(G)| + c(G))/2\,,$$

a contradiction. Thus $\deg_G(u) = p$.

Now let $C$ denote an odd cycle of length at least 5 and let $w$ be a vertex on $C$, $a$ and $b$ its neighbors on $C$. Delete the edges $wa, wb$. The remaining graph has two components for otherwise $wa$ or $wb$ would be contained in two cycles. Let $G_1$ be the component containing $w$ and $G_2$ the other component where a new edge is added joining $a$ and $b$. Then both $G_1$ and $G_2$ verify the theorem. Also $\deg_{G_2}(a) = \deg_{G_2}(b) = p$, $|L_p(G_1)| + |L_p(G_2)| \leqslant |L_p(G)| + 1$

and $c(G_1)+c(G_2) = c(G)-1$. Let $D_1$ and $D_2$ be a $\gamma_p(G_1)$-set and a $\gamma_p(G_2)$-set, respectively. Then $D_1$ contains $w$ since $\deg_{G_1}(w) = p-2$. It can be checked that $D_1 \cup D_2$ is a $p$-dominating set of $G$. It follows that

$$\begin{aligned}\gamma_p(G) &\leqslant |D_1 \cup D_2| \\ &\leqslant (n(G_1)+|L_p(G_1)|+c(G_1))/2+(n(G_2)+|L_p(G_2)|+c(G_2))/2 \\ &\leqslant (n(G)+|L_p(G)|+1+c(G)-1)/2 = (n(G)+|L_p(G)|+c(G))/2\end{aligned}$$

contradicting our assumption. Thus it remains to investigate the case that each odd cycle is a triangle.

Let $C = uvw$ be a triangle of $G$. If $p = 2$ then as claimed before $G = C_3$ and the theorem is valid. So assume that $p \geqslant 3$. Let $G_u, G_v$ and $G_w$ be the three components of $G$ containing $u, v, w$, respectively, by removing the edges $uv, uw$ and $vw$. Suppose that each component contains at most one vertex of degree at least $p$ and let $j$ the number of vertices of degree at least $p$ in the three components. Then $j \leqslant 3$ and $|L_p(G)| = n-3-j$. In this case, $G_u$ is either a star of center vertex $u$ with $p-2$ leaves, or star of order at least 4 where $u$ is a leaf if $p = 3$, or a double star $S_{p-3,p-1}$ with $u$ as a support vertex if $p \geqslant 4$, or a graph formed by a cycle $C_4$ where $u \in V(C_4)$ and is adjacent to $p-4$ leaves (if $p \geqslant 4$), its neighbors on the cycle have degree 2 and the remaining vertex of the cycle is adjacent to $p-2$ leaves. Likewise $G_v$ and $G_w$. If each component is a tree then $G$ is a unicycle and the result follows by Theorem 3. So we assume that $G_u$ is a component containing the cycle $C_4$. Now it is a routine matter to check that

$$\gamma_p(G) = n-(j+1) \leqslant (n(G)+|L_p(G)|+c(G))/2 = n-1-j/2\,,$$

a contradiction.

Thus we may assume, without loss of generality, that $G_u$ contains at least two vertices of degree at least $p$. Let $G'$ be the component containing $v, w$ by removing the edges $uv, uw$. Let $G_0$ be the graph constructed from $G'$ by attaching $v$ and $w$ to the support vertices say $a, b$ of a double star $S_{p-2,p-2}$ (so $v, w, a, b$ induce a cycle $C_4$) and let $D_u$ and $D_0$ a $\gamma_p(G_u)$-set and a $\gamma_p(G_0)$-set, respectively. Then, without loss of generality, $D_0$ contains $v, w, a$ all the leaves adjacent to $a$ and $b$. Also $D_u$ contains $u$ since it has degree at most $p-2$. Obviously $D_u \cup (D_0-(\{a\} \cup L_a \cup L_b))$ is a $p$-dominating set of $G$. It is easy to check that $G_u$ contains at least $2p-1$ vertices. Thus $G_0$ has order less than $G$ since we have added $2p-2$ vertices and so both

$G_u, G_0$ verify the result. On the other hand, $n(G) = n(G_u) + n(G_0) - 2p + 2$, $L_p(G) = L_p(G_u) - 1 + L_p(G_0) - 2p + 4$, $c(G) = c(G_u) + c(G_0) + 1$. Consequently

$$\begin{aligned}\gamma_p(G) &\leqslant |D_u \cup (D_0 - (\{a\} \cup L_a \cup L_b))| = \gamma_p(G_u) + \gamma_p(G_0) - 2p + 3 \\ &\leqslant (n(G_u) + |L_p(G_u)| + c(G_u))/2 \\ &\quad + (n(G_0) + |L_p(G_0)| + c(G_0))/2 - 2p + 3 \\ &\leqslant (n(G) + |L_p(G)| + c(G))/2\,,\end{aligned}$$

a contradiction with our assumption.

That this bound is sharp may be seen by considering the graph $G_k$ formed by $k \geqslant 1$ triangles where each vertex of the triangle is attached to $p-2$ leaves, and identifying a vertex of every triangle with a vertex of a path $P_k$. Then $n(G_k) = (3p-3)k$, $|L_p(G_k)| = 3(p-2)k$, $c(G_k) = k$ and $\gamma_p(G) = (n(G_k) + |L_p(G_k)| + c(G_k))/2 = (3p-4)k$. ■

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