ON THE \( p \)-DOMINATION NUMBER OF
CACTUS GRAPHS

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Abstract

Let \( p \) be a positive integer and \( G = (V, E) \) a graph. A subset \( S \)
of \( V \) is a \( p \)-dominating set if every vertex of \( V - S \) is dominated at
least \( p \) times. The minimum cardinality of a \( p \)-dominating set \( a \) of \( G \)
is the \( p \)-domination number \( \gamma_p(G) \). It is proved for a cactus graph \( G \)
that \( \gamma_p(G) \leq (|V| + |L_p(G)| + c(G))/2 \), for every positive integer \( p \geq 2 \),
where \( L_p(G) \) is the set of vertices of \( G \) of degree at most \( p - 1 \) and
\( c(G) \) is the number of odd cycles in \( G \).

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1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of $G$ is $n(G) = |V(G)|$ and the degree of a vertex $v$, denoted by $\deg_G(v)$, is the number of vertices adjacent to $v$. A vertex of degree one is called a leaf and its neighbor is called a support vertex. A vertex of $V$ is called a cut vertex if removing it from $G$ increases the number of components of $G$. A graph $G$ is called a cactus graph if each edge of $G$ is contained in at most one cycle. A unicycle graph is a graph with exactly one cycle. A tree $T$ is a double star if it contains exactly two vertices that are not leaves. A double star with, respectively $p$ and $q$ leaves attached at each support vertex, is denoted by $S_{p,q}$.

For a positive integer $p$, a subset $S$ of $V(G)$ is a $p$-dominating set if every vertex not in $S$ is adjacent to at least $p$ vertices of $S$. The $p$-domination number $\gamma_p(G)$ is the minimum cardinality of a $p$-dominating set of $G$. Note that every graph $G$ has a $p$-dominating set, since $V(G)$ is such a set. Also the 1-domination number $\gamma_1(G)$ is the usual domination number $\gamma(G)$. The concept of $p$-domination was introduced by Fink and Jacobson [2, 3]. For more details on domination and its variations see the books of Haynes, Hedetniemi, and Slater [4, 5].

We make a straightforward observation.

**Observation 1.** Every $p$-dominating set of a graph $G$ contains any vertex of degree at most $p - 1$.

In this paper we present an upper bound for the $p$-domination number for cactus graphs in terms of the order, the number of odd cycles and the number of vertices of degrees at most $p - 1$.

The following result due to Blidia et al. [1] will be useful for the next. Let $L_p(G)$ denote the set $\{x \in V(G) : \deg_G(x) \leq p - 1\}$.

**Theorem 2** (Blidia, Chellali and Volkmann [1]). Let $p$ be a positive integer. If $G$ is a bipartite graph then

$$\gamma_p(G) \leq (n + |L_p(G)|)/2.$$
2. Main Results

We begin by giving an upper bound for the $p$-domination number for connected unicycle graphs.
Theorem 3. Let $p \geq 2$ be a positive integer. If $G$ is a connected unicycle graph then
$$
\gamma_p(G) \leq \frac{n + |L_p(G)| + 1}{2}
$$
and this bound is sharp.

Proof. Let $G$ be a connected unicycle graph. If $G$ is bipartite then the result is valid by Theorem 2. So assume that $G$ contains an odd cycle denoted by $C$. If $G = C$, then $\gamma_p(G) = n$ if $p \geq 3$ and $\gamma_p(G) = (n + 1)/2$ if $p = 2$, in both cases the result holds. Thus we assume that $G \neq C$, that is $G$ contains at least one leaf.

Suppose that the result does not hold and let $G$ be the smallest connected unicycle graph such that $\gamma_p(G) > (n + |L_p(G)| + 1)/2$. We claim that every vertex on $C$ has degree exactly $p$. Suppose to the contrary that there is a vertex $x \in C$ such that $\deg_C(x) \neq p$ and let $y$ be one of its two neighbors on $C$. Let $G'$ be the spanning graph of $G$ obtained by removing the edge $xy$. Then $G'$ is a tree and so a bipartite graph. We also have $|L_p(G')| \leq |L_p(G)| + 1$ and $n(G) = n(G')$. According to Theorem 2, we deduce that
$$
\gamma_p(G) \leq \gamma_p(G') \leq \frac{(n(G') + |L_p(G')|)/2}{(n(G) + |L_p(G)| + 1)/2},
$$
a contradiction with our assumption.

Similarly, we will show that every vertex not on $C$ and different to a leaf has degree at least $p$. Assume to the contrary that there is a vertex $x \in V(G) - C$ different to a leaf with $\deg_G(x) \leq p - 1$ and let $z$ be its neighbor in the unique path from $x$ to $C$. Let $G_1$ be the connected unicycle subgraph of $G$ containing $x$ and obtained by removing all the edges incident to $x$ excepted the edge $xz$, and let $G_2$ be the component containing $x$ by removing the edge $xz$. Let $D_1$ and $D_2$ denote a $\gamma_p(G_1)$-set and a $\gamma_p(G_2)$-set, respectively. Clearly $G_1$ contains $C$ and $G_2$ is a tree, $x \in D_1 \cap D_2$, $x \in L_p(G_1) \cap L_p(G_2)$, $|L_p(G_1)| + |L_p(G_2)| = |L_p(G)| + 1$ and $n(G_1) + n(G_2) = n(G) + 1$. Furthermore, $D_1 \cup D_2$ is a $p$-dominating set of $G$. In addition, $G_1$ and $G_2$ have order less than $G$ and so satisfy the theorem, implying that
$$
\gamma_p(G) \leq |D_1 \cup D_2| = \gamma_p(G_1) + \gamma_p(G_2) - 1
\leq (n(G_1) + |L_p(G_1)| + 1)/2 + (n(G_2) + |L_p(G_2)|)/2 - 1
\leq (n + |L_p(G)| + 1)/2,
$$
contradicting the assumption.
Suppose now that \( V(G) - C \) contains a support vertex. Let \( a \) be a support vertex of \( G \) of maximum distance from \( C \). As seen above, \( a \) has degree at least \( p \). Let \( G' = G - (L_a \cup \{a\}) \). Then \( \gamma_p(G') + |L_a| = \gamma_p(G), \ n(G') = n(G) - |L_a| - 1 \) and \( L_p(G) \geq L_p(G') + |L_a| - 1 \). It follows that
\[
\gamma_p(G') + |L_a| = \gamma_p(G) > \left( n(G) + \left| L_p(G) \right| + 1 \right)/2
\]
implying that
\[
\gamma_p(G') > (n(G) + |L_p(G)| + 1)/2
\]
and so
\[
\gamma_p(G') > \left( n(G') + \left| L_p(G') \right| + 1 \right)/2
\]
contradicting our assumption that \( G \) is the smallest graph that does not satisfy the theorem.

Consequently, every vertex of \( V(G) - C \) must be a leaf and so every vertex on \( C \) is adjacent to exactly \( p - 2 \) leaves, which implies that
\[
\gamma_p(G) = n - (|V(C)| - 1)/2 = \left( n(G) + \left| L_p(G) \right| + 1 \right)/2
\]
a contradiction.

To see that this bound is sharp, consider the graph \( G \) formed by an odd cycle \( C \) where each vertex on \( C \) is adjacent to exactly \( p - 2 \) vertices. Then
\[
\gamma_p(G) = n - (|V(C)| - 1)/2 = \left( n(G) + \left| L_p(G) \right| + 1 \right)/2.
\]

**Theorem 4.** Let \( p \geq 2 \) be a positive integer. If \( G \) is a connected cactus graph with \( c(G) \) odd cycles then,
\[
\gamma_p(G) \leq \left( n + \left| L_p(G) \right| + c(G) \right)/2,
\]
and this bound is sharp.

**Proof.** If \( G \) is a bipartite graph, then by Theorem 2 the result holds. If \( G \) is a unicycle graph then by Theorem 3 the result is also valid. So consider a cactus graph \( G \) containing at least two cycles with one of odd length. Assume that the result does not hold and let \( G \) be the smallest cactus graph such that \( \gamma_p(G) > \left( n(G) + \left| L_p(G) \right| + c(G) \right)/2 \). We also assume that among all such graphs, \( G \) is the one having the fewest edges.

First, let \( u \) be a vertex on an odd cycle \( C \) of \( G \) and assume that \( \deg_G(u) \neq p \). Let \( G' \) be the spanning graph of \( G \) obtained by removing an
edge of $C$ incident with $u$. Then $|L_p(G')| \leq |L_p(G)| + 1$ and $c(G') = c(G) - 1$.
Also $G'$ satisfies the result and so

\[
\gamma_p(G) \leq \gamma_p(G') \leq (n(G') + |L_p(G')| + c(G'))/2 \\
\leq (n(G) + |L_p(G)| + 1 + c(G) - 1)/2 = (n + |L_p(G)| + c(G))/2,
\]
a contradiction. Thus every vertex in an odd cycle has degree exactly $p$.

Now consider a vertex $v$ different from a leaf and contained in no odd cycle. Then, either $v$ is a cut vertex or $v$ is on an even cycle and $\deg_{G_2}(v) = 2$. Suppose first that $v$ is a cut vertex with $\deg_{G_2}(v) < p.$ Let $G_1$ and $G_2$ be two connected cactus subgraphs of $G$ with $V(G) = V(G_1) \cup V(G_2)$ having $v$ as a unique common vertex. Then, $c(G) = c(G_1) + c(G_2), n(G) = n(G_1) + n(G_2) - 1, |L_p(G)| = |L_p(G_1)| + |L_p(G_2)| - 1.$ Now let $D_1$ and $D_2$ denote a $\gamma_p(G_1)$-set and a $\gamma_p(G_2)$-set, respectively. Then $v \in D_1 \cup D_2$ and $D_1 \cup D_2$ is a $p$-dominating set of $G.$ Since $G_1$ and $G_2$ satisfy the result,

\[
\gamma_p(G) \leq |D_1 \cup D_2| = |D_1| + |D_2| - 1 \\
\leq (n(G_1) + |L_p(G_1)| + c(G_1))/2 + (n(G_2) + |L_p(G_2)| + c(G_2))/2 - 1 \\
\leq (n(G) + |L_p(G)| + c(G))/2,
\]
a contradiction. Consequently, every cut vertex contained in no odd cycle has degree at least $p$.

Now let $v$ be a vertex on an even cycle with $\deg_{G_2}(v) = 2.$ Since we have assumed in the beginning of the proof that $G$ has at least two cycles, we have $p \geq 3.$ We claim that each neighbor of $v$ has degree exactly $p.$ Indeed, let $u$ be a neighbor of $v$ and assume that $\deg_{G_2}(u) \neq p.$ Then every $\gamma_p(G')$-set $S$ is a $p$-dominating set of $G$ where $G'$ is obtained from $G$ by removing the edge $vu.$ So

\[
\gamma_p(G) \leq |S| \leq (n(G') + |L_p(G')| + c(G'))/2 = (n(G) + |L_p(G)| + c(G))/2,
\]
a contradiction. Thus $\deg_{G_2}(u) = p.$

Now let $C$ denote an odd cycle of length at least 5 and let $w$ be a vertex on $C,$ $a$ and $b$ its neighbors on $C.$ Delete the edges $wa, wb.$ The remaining graph has two components for otherwise $wa$ or $wb$ would be contained in two cycles. Let $G_1$ be the component containing $w$ and $G_2$ the other component where a new edge is added joining $a$ and $b.$ Then both $G_1$ and $G_2$ verify the theorem. Also $\deg_{G_2}(a) = \deg_{G_2}(b) = p, |L_p(G_1)| + |L_p(G_2)| \leq |L_p(G)| + 1$
and \(c(G_1) + c(G_2) = c(G) - 1\). Let \(D_1\) and \(D_2\) be a \(\gamma_p(G_1)\)-set and a \(\gamma_p(G_2)\)-set, respectively. Then \(D_1\) contains \(w\) since \(\deg_{G_1}(w) = p - 2\). It can be checked that \(D_1 \cup D_2\) is a \(p\)-dominating set of \(G\). It follows that

\[
\gamma_p(G) \leq |D_1 \cup D_2| \\
\leq (n(G_1) + |L_p(G_1)| + c(G_1))/2 + (n(G_2) + |L_p(G_2)| + c(G_2))/2 \\
\leq (n(G) + |L_p(G)| + 1 + c(G) - 1)/2 = (n(G) + |L_p(G)| + c(G))/2
\]

contradicting our assumption. Thus it remains to investigate the case that each odd cycle is a triangle.

Let \(C = uvw\) be a triangle of \(G\). If \(p = 2\) then as claimed before \(G = C_3\) and the theorem is valid. So assume that \(p \geq 3\). Let \(G_u, G_v\) and \(G_w\) be the three components of \(G\) containing \(u, v, w\), respectively, by removing the edges \(uw, uw\) and \(vw\). Suppose that each component contains at most one vertex of degree at least \(p\) and let \(j\) the number of vertices of degree at least \(p\) in the three components. Then \(j \leq 3\) and \(|L_p(G)| = n - 3 - j\). In this case, \(G_u\) is either a star of center vertex \(u\) with \(p - 2\) leaves, or star of order at least \(4\) where \(u\) is a leaf if \(p = 3\), or a double star \(S_{p-3,p-1}\) with \(u\) as a support vertex if \(p \geq 4\), or a graph formed by a cycle \(C_4\) where \(u \in V(C_4)\) and is adjacent to \(p - 4\) leaves (if \(p \geq 4\), its neighbors on the cycle have degree \(2\) and the remaining vertex of the cycle is adjacent to \(p - 2\) leaves. Likewise \(G_u\) and \(G_w\). If each component is a tree then \(G\) is a unicycle and the result follows by Theorem 3. So we assume that \(G_u\) is a component containing the cycle \(C_4\). Now it is a routine matter to check that

\[
\gamma_p(G) = n - (j + 1) \leq (n(G) + |L_p(G)| + c(G))/2 = n - 1 - j/2,
\]

a contradiction.

Thus we may assume, without loss of generality, that \(G_u\) contains at least two vertices of degree at least \(p\). Let \(G'\) be the component containing \(v, w\) by removing the edges \(uv, uw\). Let \(G_0\) be the graph constructed from \(G'\) by attaching \(v\) and \(w\) to the support vertices say \(a, b\) of a double star \(S_{p-2,p-2}\) (so \(v, w, a, b\) induce a cycle \(C_4\)) and let \(D_u\) and \(D_0\) a \(\gamma_p(G_u)\)-set and a \(\gamma_p(G_0)\)-set, respectively. Then, without loss of generality, \(D_0\) contains \(v, w, a\) all the leaves adjacent to \(a\) and \(b\). Also \(D_u\) contains \(u\) since it has degree at most \(p - 2\). Obviously \(D_u \cup (D_0 - (\{a\} \cup L_a \cup L_b))\) is a \(p\)-dominating set of \(G\). It is easy to check that \(G_u\) contains at least \(2p - 1\) vertices. Thus \(G_0\) has order less than \(G\) since we have added \(2p - 2\) vertices and so both
\( G_u, G_0 \) verify the result. On the other hand, 
\[ n(G) = n(G_u) + n(G_0) - 2p + 2, \]
\[ L_p(G) = L_p(G_u) - 1 + L_p(G_0) - 2p + 4, \]
\[ c(G) = c(G_u) + c(G_0) + 1. \]
Consequently
\[ \gamma_p(G) \leq |D_u \cup (D_0 - \{a\} \cup L_a \cup L_b)| = \gamma_p(G_u) + \gamma_p(G_0) - 2p + 3 \]
\[ \leq (n(G_u) + |L_p(G_u)| + c(G_u))/2 \]
\[ + (n(G_0) + |L_p(G_0)| + c(G_0))/2 - 2p + 3 \]
\[ \leq (n(G) + |L_p(G)| + c(G))/2, \]
a contradiction with our assumption.

That this bound is sharp may be seen by considering the graph \( G_k \) formed by \( k \geq 1 \) triangles where each vertex of the triangle is attached to \( p - 2 \) leaves, and identifying a vertex of every triangle with a vertex of a path \( P_k \). Then 
\[ n(G_k) = (3p - 3)k, \]
\[ |L_p(G_k)| = 3(p - 2)k, \]
\[ c(G_k) = k \]
and 
\[ \gamma_p(G) = (n(G_k) + |L_p(G_k)| + c(G_k))/2 = (3p - 4)k. \]

References


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