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ON γ-LABELINGS OF TREES

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Abstract

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of a γ-labeling f is $\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)$. The maximum value of a γ-labeling of G is defined as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

while the minimum value of a γ-labeling of G is

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

The values $\text{val}_{\max}(S_{p,q})$ and $\text{val}_{\min}(S_{p,q})$ are determined for double stars $S_{p,q}$. We present characterizations of connected graphs G of order n for which $\text{val}_{\min}(G) = n$ or $\text{val}_{\min}(G) = n + 1$.

Keywords: γ-labeling, value of a γ-labeling.

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1. Introduction

For a graph G of order n and size m, a *γ-labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by

$$f'(e) = |f(u) - f(v)| \text{ for each edge } e = uv \text{ of } G.$$

Therefore, a graph G of order n and size m has a γ-labeling if and only if $m \geq n - 1$. In particular, every connected graph has a γ-labeling. If the induced edge-labeling f' of a γ-labeling f is also one-to-one, then f is a *graceful labeling*, one of the most studied of graph labelings. An extensive survey of graph labelings as well as their applications has been given by Gallian [2].

Each γ-labeling f of a graph G of order n and size m is assigned a *value* denoted by $\text{val}(f)$ and defined by

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

Since f is a one-to-one function from $V(G)$ to $\{0, 1, 2, \ldots, m\}$, it follows that $f'(e) \geq 1$ for each edge e in G and so

$$\text{val}(f) \geq m. \qquad (1)$$

Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each γ-labeling is shown in Figure 1 as well.

For a graph G of order n and size m, the *maximum value* of a γ-labeling of a graph G is defined as

$$\text{val}_{\max}(G) = \max\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

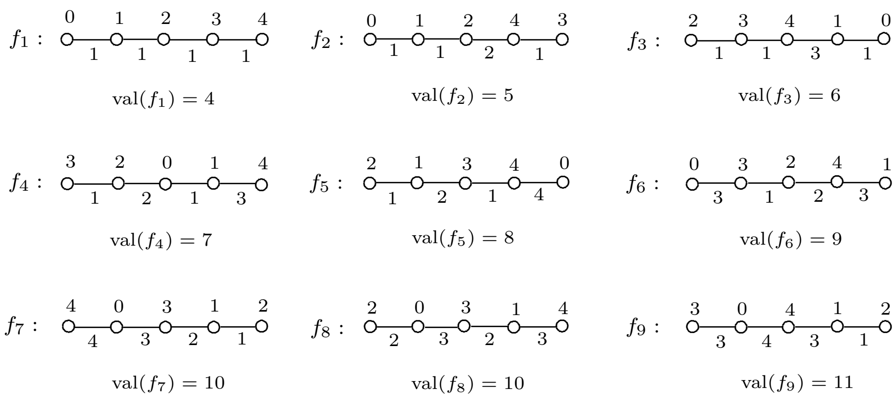

Figure 1: Some γ-labelings of P_5.

while the *minimum value* of a γ-labeling of G is

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling g of G is a γ-max *labeling* if

$$\mathrm{val}(g) = \mathrm{val}_{\max}(G)$$

and a γ-labeling h is a γ-min *labeling* if

$$\mathrm{val}(h) = \mathrm{val}_{\min}(G).$$

Since $\mathrm{val}(f_1) = 4$ for the γ-labeling f_1 of P_5 shown in Figure 1 and the size of P_5 is 4, it follows that f_1 is a γ-min labeling of P_5. Although less clear, the γ-labeling f_9 shown in Figure 1 is a γ-max labeling. The concepts of a γ-labeling of a graph and the value of a γ-labeling were introduced in [1].

For a γ-labeling f of a graph G of size m, the *complementary labeling* $\overline{f} : V(G) \to \{0, 1, 2, \ldots, m\}$ of f is defined by

$$\overline{f}(v) = m - f(v) \text{ for } v \in V(G).$$

Not only is $\overline{f}$ a γ-labeling of G as well but $\mathrm{val}(\overline{f}) = \mathrm{val}(f)$. This gives us the following observation that appeared in [1].

Observation 1.1. Let f be a γ-labeling of a graph G. Then f is a γ-max labeling (γ-min labeling) of G if and only if $\overline{f}$ is a γ-max labeling (γ-min labeling).

A more general vertex labeling of a graph was introduced by Hegde in [3]. A vertex function f of a graph G is defined from $V(G)$ to the set of nonnegative integers that induces an edge function f' defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. Such a function is called a *geodetic function* of G. A one-to-one geodetic function is a *geodetic labeling* of G if the induced edge function f' is also one-to-one. The following result was established by Hegde which provides an upper bound for $\text{val}_{\max}(G)$ (see [3]).

Theorem (Hegde)**.** *For any geodetic γ-labeling f of a graph G of order n,*

$$\sum_{e \in E(G)} f'(e) \leq \sum_{i=0}^{n-1} (2i - n + 1) f(v_i).$$

The following results were obtained in [1] for the paths P_n and stars $K_{1,n-1}$ of order n.

Theorem A. *For each integer $n \geq 2$,*

$$\text{val}_{\min}(P_n) = n - 1 \text{ and } \text{val}_{\max}(P_n) = \left\lfloor \frac{n^2 - 2}{2} \right\rfloor.$$

Theorem B. *Let G be a connected graph of order n and size m. Then*

$$\text{val}_{\min}(G) = m \textit{ if and only if } G \cong P_n.$$

Theorem C. *For each integer $n \geq 3$,*

$$\text{val}_{\min}(K_{1,n-1}) = \binom{\left\lfloor \frac{n+1}{2} \right\rfloor}{2} + \binom{\left\lceil \frac{n+1}{2} \right\rceil}{2} \text{ and } \text{val}_{\max}(K_{1,n-1}) = \binom{n}{2}.$$

Theorem D. *For each integer $n \geq 3$,*

$$\text{val}_{\min}(C_n) = 2(n - 1)$$

and

$$\text{val}_{\max}(C_n) = \begin{cases} \dfrac{n(n+2)}{2} & \text{if } n \text{ is even,} \\ \dfrac{(n-1)(n+3)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

In this paper, we investigate γ-labelings of trees, beginning with double stars.

2. γ-Labelings of Double Stars

We now turn to the double star $S_{p,q}$ containing central vertices u and v such that $\deg u = p$ and $\deg v = q$ and determine $\text{val}_{\min}(S_{p,q})$ and then $\text{val}_{\max}(S_{p,q})$.

Proposition 2.1. *For integers $p, q \geq 2$,*

$$\text{val}_{\min}(S_{p,q}) = \left(\left\lfloor \frac{p}{2} \right\rfloor + 1\right)^2 + \left(\left\lfloor \frac{q}{2} \right\rfloor + 1\right)^2 - \left(n_p \left\lfloor \frac{p+2}{2} \right\rfloor + n_q \left\lfloor \frac{q+2}{2} \right\rfloor + 1\right),$$

where

$$n_p = \begin{cases} 1 & \text{if } p \text{ is even,} \\ 0 & \text{if } p \text{ is odd} \end{cases} \quad \text{and} \quad n_q = \begin{cases} 1 & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Let $N(u) = \{v, u_1, u_2, \ldots, u_{p-1}\}$ and $N(v) = \{u, v_1, v_2, \ldots, v_{q-1}\}$. Since the proof is similar whether p and q are odd or even, we provide the proof in one of these four cases only, namely when p and q are odd. Let $p = 2s + 1$ and $q = 2t + 1$ for positive integers s and t. Define a γ-labeling f of $S_{p,q}$ by

$$f(x) = \begin{cases} s & \text{if } x = u, \\ 2s + t + 1 & \text{if } x = v, \\ i - 1 & \text{if } x = u_i,\ 1 \leq i \leq s, \\ i & \text{if } x = u_i,\ s + 1 \leq i \leq 2s, \\ 2s + i & \text{if } x = v_i,\ 1 \leq i \leq t, \\ 2s + i + 1 & \text{if } x = v_i,\ t + 1 \leq i \leq 2t. \end{cases}$$

Thus exactly two edges in $\{uu_i :\ 1 \le i \le 2s\}$ are labeled a for each integer a with $1 \le a \le s$ and exactly two edges in $\{vv_i :\ 1 \le i \le 2t\}$ are labeled b for each integer b with $1 \le b \le t$. Furthermore, the edge uv is labeled $s+t+1$. Therefore,

$$\begin{aligned}\mathrm{val}(f) &= (s+t+1) + 2(1+2+\ldots+s) + 2(1+2+\ldots+t)\\ &= (s+t+1) + 2\binom{s+1}{2} + 2\binom{t+1}{2} = (s+1)^2 + (t+1)^2 - 1.\end{aligned}$$

Therefore,

$$\mathrm{val}_{\min}(S_{p,q}) \le (s+1)^2 + (t+1)^2 - 1.$$

Next, consider an arbitrary γ-labeling g of $S_{p,q}$. We may assume that $g(u) < g(v)$; otherwise, we could consider the complementary γ-labeling $\overline{g}$ of g. We show that

$$\mathrm{val}(g) \ge (s+1)^2 + (t+1)^2 - 1.$$

First, we make the following observations:

1. At most two edges in $\{uu_i : 1 \le i \le 2s\}$ can be labeled a for each integer a with $1 \le a \le s$ and this can occur only if the labels in $\{g(u) \pm a : 1 \le i \le s\}$ are available for the vertices u_i $(1 \le a \le 2s)$.
2. At most two edges in $\{vv_i : 1 \le i \le 2t\}$ can be labeled b for each integer b with $1 \le b \le t$ and this can occur only if the labels in $\{g(v) \pm b : 1 \le b \le t\}$ are available for the vertices v_i $(1 \le i \le 2t)$.

Therefore,

$$\sum_{e \in E(G)-\{uv\}} g'(e) \ge 2\binom{s+1}{2} + 2\binom{t+1}{2}.$$

Thus if $g'(uv) = g(v) - g(u) \ge s+t+1$, then

$$\mathrm{val}(g) \ge (s+t+1) + 2\binom{s+1}{2} + 2\binom{t+1}{2} = (s+1)^2 + (t+1)^2 - 1.$$

Suppose then that $g'(uv) = s+t+1-k$ for some integer k with $1 \le k \le s+t$. Then there are $s+t-k$ vertices of $S_{p,q}$ that are labeled with integers between $g(u)$ and $g(v)$. Consequently, $s+t+k$ vertices of $S_{p,q}$ are assigned a label less than $g(u)$ or greater than $g(v)$, which implies that at least k vertices of $S_{p,q}$ are assigned a label less than $g(u)-s$ or greater than $g(v)+t$. For each vertex u_i, $1 \le i \le 2s$, assigned a label less than $g(u)-s$,

$$\sum_{i=1}^{2s} g'(uu_i) \text{ must exceed } 2\binom{s+1}{2}$$

by at least 1; while for each vertex v_i, $1 \le i \le 2s$, assigned a label greater than $g(v)+t$,

$$\sum_{i=1}^{2t} g'(vv_i) \text{ must exceed } 2\binom{t+1}{2}$$

by at least 1. Therefore,

$$\sum_{e \in E(G)-\{uv\}} g'(e) \ge 2\binom{s+1}{2} + 2\binom{t+1}{2} + k.$$

However then,

$$\begin{aligned} \text{val}(g) &= g'(uv) + \sum_{e \in E(G)-\{uv\}} g'(e) \\ &\ge (s+t+1-k) + \left[2\binom{s+1}{2} + 2\binom{t+1}{2} + k\right] \\ &= (s+1)^2 + (t+1)^2 - 1. \end{aligned}$$

In general, $\text{val}(g) \ge (s+1)^2 + (t+1)^2 - 1$. Therefore, $\text{val}_{\min}(S_{p,q}) = (s+1)^2 + (t+1)^2 - 1$. ■

Theorem 2.2 *For every pair p, q of positive integers,*

$$\text{val}_{\max}(S_{p,q}) = \frac{1}{2}\left[p^2 + q^2 + 4pq - 3p - 3q + 2\right].$$

Proof. Let u and v be the central vertices of $S_{p,q}$, where $\deg u = p$ and $\deg v = q$, and let f be the γ-labeling of $S_{p,q}$ in which we assign the label 0 to u, the label $p+q-1$ to v, the labels $1, 2, \ldots, q-1$ to the end-vertices adjacent to v, and the labels $q, q+1, \ldots, p+q-2$ to the end-vertices adjacent to u. The value of f is $(p^2+q^2+4pq-3p-3q+2)/2$, which is therefore a lower bound for $\mathrm{val}_{\max}(S_{p,q})$.

We now show that $\mathrm{val}_{\max}(S_{p,q}) \le (p^2+q^2+4pq-3p-3q+2)/2$. First, we claim that $S_{p,q}$ has a γ-max labeling for which $\{f(u), f(v)\} = \{0, p+q-1\}$. We verify this claim by induction on $p+q$. The claim is clearly true for $p+q=2$. Assume that the claim is true for $p+q=k-1$, where $k \ge 3$. Let $T = S_{p,q}$, where $p+q=k$. Let f be a γ-max labeling of T. If $\{f(u), f(v)\} = \{0, p+q-1\}$, then the claim is true. Suppose that at least one $f(u)$ and $f(v)$ is neither 0 nor $p+q-1$. By Observation1.1, we may assume that $f(w) = p+q-1$ and $w \ne u, v$. The vertex w is therefore an end-vertex of T. Let $x \in \{u, v\}$ be the vertex of T that is adjacent to w. Then $T' = T - w$ is isomorphic to $S_{p',q'}$, where $p'+q' = k-1$. By the inductive hypothesis, T' has a γ-max labeling g for which $\{g(u), g(v)\} = \{0, p+q-2\}$. By Observation1.1, we may assume that $g(x) = 0$. Now

$$\text{(2)} \quad \mathrm{val}(f) = (p+q-1-f(x)) + \sum_{e \in E(T')} f'(e) \le p+q-1+\mathrm{val}_{\max}(T').$$

We extend g to a γ-labeling h of T by defining $h(w) = p+q-1$. Then

$$\text{(3)} \qquad \mathrm{val}(h) = p+q-1+\mathrm{val}_{\max}(T').$$

By (2) and (3), $\mathrm{val}(f) \le \mathrm{val}(h)$. Since f is a γ-max labeling of T, so too is h a γ-max labeling of T. Let $y \in \{u, v\}$ for which $h(y) = p+q-2$. Thus y is not adjacent to w. Next, let ϕ be the γ-labeling of T defined by

$$\phi(z) = \begin{cases} h(z) & \text{if } z \ne w, y, \\ p+q-1 & \text{if } z = y, \\ p+q-2 & \text{if } z = w. \end{cases}$$

Then $\mathrm{val}(\phi) = \mathrm{val}(h)$ if $\deg y \le 2$; while $\mathrm{val}(\phi) > \mathrm{val}(h)$ if $\deg y \ge 3$. Since $\mathrm{val}(\phi)$ cannot exceed $\mathrm{val}(h)$, it follows that $\deg y \le 2$, and ϕ has the desired property that verifies the claim. By the claim and Observation 1.1, there is a γ-max labeling f of $S_{p,q}$ with $f(u) = 0$ and $f(v) = p+q-1$.

If there is an end-vertex t_1 of $S_{p,q}$ adjacent to v with $f(t_1) = i > q-1$, then there is an end-vertex t_2 of $S_{p,q}$ adjacent to u with $f(t_2) = j$, where $1 \le j \le q-1$. Interchanging the labels of t_1 and t_2 produces a γ-labeling f_1 with $\text{val}(f_1) > \text{val}(f)$, which is impossible. Thus f is the γ-labeling described in the first paragraph of the proof and $\text{val}(f) = (p^2+q^2+4pq-3p-3q+2)/2$. ■

3. Connected Graphs of Order n with Minimum Value n

We already mentioned (in Theorem B) that a connected graph G of order n has minimum value $n-1$ if and only if $G \cong P_n$. We now determine all those connected graphs G of order n for which $\text{val}_{\min}(G) = n$. It is useful to present several lemmas first.

Lemma 3.1. *If G is a connected graph of size m and G' is a connected subgraph of G having size m', then*

$$\text{val}_{\min}(G) \ge (m-m') + \text{val}_{\min}(G').$$

Proof. Suppose that G has order n and G' has order n'. Let f be a γ-min labeling of G. Then the restriction h of f to G' is a one-to-one function. Suppose that the vertices of G' are labeled $a_1, a_2, \cdots, a_{n'}$ by h, where $0 \le a_1 < a_2 < \cdots < a_{n'} \le m$. Thus, for $1 \le i \ne j \le n'$, $|a_i - a_j| \ge |i-j|$. Consider the one-to-one function $g : \{a_1, a_2, \cdots, a_{n'}\} \to \{0, 1, 2, \cdots, m'\}$ defined by $g(a_i) = i-1$ for $1 \le i \le n'$. Then $\phi = g \circ h : V(G') \to \{0, 1, 2, \cdots, m'\}$ is a γ-labeling of G'. Furthermore,

$$\text{val}_{\min}(G') \le \text{val}(\phi) \le \sum_{e \in E(G')} h'(e) = \sum_{e \in E(G')} f'(e).$$

Since $f'(e) \ge 1$ for every edge e in G, it follows that

$$\begin{aligned} \text{val}(f) &= \sum_{e \in E(G-G')} f'(e) + \sum_{e \in E(G')} f'(e) \\ &\ge (m-m') + \text{val}_{\min}(G'), \end{aligned}$$

as desired. ■

Lemma 3.1 can be extended to obtain the following result.

Lemma 3.2. *If G is a connected graph of size m containing pairwise edge-disjoint connected subgraphs $G_1, G_2, \cdots, G_k$, where G_i has size m_i for $1 \leq i \leq k$, then*

$$\mathrm{val}_{\min}(G) \geq \left(m - \sum_{i=1}^{k} m_i \right) + \sum_{i=1}^{k} \mathrm{val}_{\min}(G_i).$$

Lemma 3.3. *Let G be a connected graph of order n with maximum degree Δ. Then*

$$\mathrm{val}_{\min}(G) \geq \begin{cases} (n-1) + k(k-1) & \text{if } \Delta = 2k, \\ (n-1) + k^2 & \text{if } \Delta = 2k+1. \end{cases}$$

Furthermore, this bound is sharp for stars.

Proof. Let $v \in V(G)$ with $\deg v = \Delta$ and let f be a γ-min labeling of G. Note that at most two edges incident with v can be labeled i for each i with $1 \leq i \leq \lfloor \Delta/2 \rfloor$. Thus, if $\Delta = 2k$, then

$$\mathrm{val}_{\min}(G) \geq (n-1-2k) + 2(1+2+\cdots+k) = (n-1) + k(k-1);$$

while if $\Delta = 2k+1$, then

$$\mathrm{val}_{\min}(G) \geq [(n-1)-(2k+1)] + 2(1+2+\cdots+k) + (k+1) = (n-1) + k^2.$$

That this bound is sharp for stars follows from Theorem C. ■

The proof of the next lemma is straightforward and is therefore omitted.

Lemma 3.4. *Let f be a γ-labeling of a connected graph G. If P is a $u-v$ path in G, then*

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)|.$$

Lemma 3.5. *For the tree F of Figure 2,* $\mathrm{val}_{\min}(F) = 8$.

Proof. The γ-labeling f of F shown in Figure 2 has value 8 and so $\mathrm{val}_{\min}(F) \leq 8$. On the other hand, let g be γ-min labeling of F and

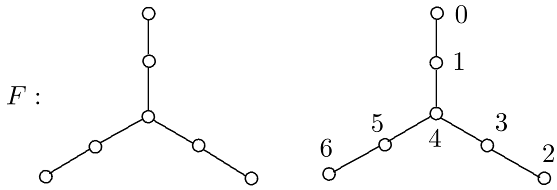

Figure 2: A tree F and a γ-labeling of F.

let $u, v \in V(F)$ such that $g(u) = 0$ and $g(v) = 6$. Suppose that P is a $u - v$ path in F. Then

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)| = 6$$

by Lemma 3.4. Since there are at least two edges of F not in P, it follows that $\text{val}_{\min}(F) = \text{val}(g) \geq 8$. ■

A *caterpillar* is a tree the removal of whose vertices results in a path. We are now able to characterize all connected graphs of order $n \geq 4$ whose minimum value is n.

Theorem 3.6. *Let G be a connected graph of order $n \geq 4$. Then $\text{val}_{\min}(G) = n$ if and only if G is a caterpillar, $\Delta(G) = 3$, and G has a unique vertex of degree 3.*

Proof. Let T be the tree obtained from the path $v_1, v_2, \cdots, v_{n-1}$ by adding the vertex v_n and joining v_n to a vertex v_k, where $2 \leq k \leq n-2$. Thus v_k is the only vertex of degree 3 in T. Define a γ-labeling f of T by

$$f(v_i) = \begin{cases} i-1 & \text{if } 1 \leq i \leq k, \\ i & \text{if } k < i \leq n-1, \\ k & \text{if } i = n. \end{cases}$$

Since $\text{val}(f) = n$, it follows that $\text{val}_{\min}(T) \leq n$ and so $\text{val}_{\min}(T) = n$ by Theorem B.

For the converse, let G be a connected graph of order $n \geq 4$ such that G is not a caterpillar with $\Delta(G) = 3$ containing a unique vertex of degree 3. We show that $\text{val}_{\min}(G) \neq n$. This is certainly true if $G \cong P_n$ or if G is not a tree by Theorem B. Hence we may assume that G is a tree T with

$\Delta(T) \geq 3$. If $\Delta(T) \geq 4$, then $\text{val}_{\min}(T) \geq (n-1)+2 = n+1$ by Lemma 3.3. Thus $\Delta(T) = 3$. We consider two cases.

Case 1. T contains two vertices u and v with degree 3.
If u and v are adjacent, then T contains the double star $S_{3,3}$ as a subgraph. By Theorem 2.2, $\text{val}_{\min}(S_{3,3}) = 7$. Since the order of $S_{3,3}$ is 6, it then follows by Lemma 3.1 that $\text{val}_{\min}(T) \geq (n-6)+7 = n+1$.

Thus we may assume that u and v are not adjacent. Let $N(u) = \{u_1, u_2, u_3\}$ and $N(v) = \{v_1, v_2, v_3\}$. Then $v \notin N(u)$ and $u \notin N(v)$. For any γ labeling g of T, $g'(e) \geq 2$ for at least one edge e in $\{uu_i : 1 \leq i \leq 3\}$ and at least one edge e in $\{vv_i : 1 \leq i \leq 3\}$. Therefore, at least two edges in T are labeled 2 or more by g and so $\text{val}_{\min}(T) \geq \text{val}(g) \geq n+1$.

Case 2. T has exactly one vertex with degree 3.
Thus T contains the graph F in Lemma 3.5 as a subgraph. Since $\text{val}_{\min}(F) = 8$ by Lemma 3.5 and the order of F is 7, it then following by Lemma 3.1 that $\text{val}_{\min}(T) \geq (n-7)+8 = n+1$. ■

4. Some Results on the Minimum Value of a Tree in Terms of Its Order and Other Parameters

In Theorem 3.6, we considered caterpillars T having maximum degree 3 and a unique vertex of degree 3. We now compute the minimum value of all such trees that are not necessarily caterpillars.

Theorem 4.1. *Let T be a tree of order $n \geq 4$ such that $\Delta(T) = 3$ and T has a unique vertex v of degree 3. If d is the distance between v and a nearest end-vertex, then*

$$\text{val}_{\min}(T) = n + d - 1.$$

Proof. Let x, y, and z be the three end-vertices of T, where $d(v,x) = d$, $d(v,y) = d'$, and $d(v,z) = d''$, where $d \leq d' \leq d''$. Let $P : v = v_0, v_1, \cdots, v_d = x$, $P' : v = u_0, u_1, \cdots, u_{d'} = y$, and $P'' : v = w_0, w_1, \cdots, w_{d''} = z$ denote the $v-x$ path, $v-y$ path, and $v-z$ path in T. Let $f : V(T) \to \{0, 1, 2, \cdots, n-1\}$ be the γ-labeling of T for which $f(w_i) = d'' - i$ for $0 \leq i \leq d''$, $f(v_i) = d'' + i$ for $1 \leq i \leq d$, and $f(u_i) = i - d' + n - 1$ for $1 \leq i \leq d'$. Since $\text{val}(f) = n + d - 1$, it follows that $\text{val}_{\min}(T) \leq n + d - 1$.

It remains therefore to show that $\text{val}_{\min}(T) \geq n + d - 1$. Let $g : V(T) \to \{0, 1, 2, \cdots, n-1\}$ be an arbitrary γ-labeling of T, and suppose that $g(v) = i$. Let

$$S = \{u \in V(T) : d(u, v) \leq d\}.$$

Thus $|S| = 3d + 1$. Let a denote the smallest label assigned by g to a vertex of S and let b denote the largest such label. We now consider two cases.

Case 1. The vertices in S labeled a and b belong to two of the three paths P, P', and P'', say P and P', respectively. Then

$$\sum_{e \in E(P)} g'(e) \geq i - a \text{ and } \sum_{e \in E(P')} g'(e) \geq b - i.$$

Thus

$$\sum_{e \in \langle S \rangle} g'(e) \geq (i - a) + (b - i) + d = b - a + d \geq 3d + d = 4d.$$

Since there are $(n - 1) - 3d$ edges of T not belonging to $\langle S \rangle$, it follows that

$$\sum_{e \in E(T)} g'(e) \geq 4d + (n - 1 - 3d) = n + d - 1.$$

Case 2. The vertices in S labeled a and b belong to one of the three paths P, P', and P'', say P. Then

$$\sum_{e \in E(P)} g'(e) \geq b - a.$$

Thus

$$\sum_{e \in \langle S \rangle} g'(e) \geq (b - a) + 2d \geq 3d + 2d = 5d.$$

Since there are $(n - 1) - 3d$ edges of T not belonging to $\langle S \rangle$, it follows that

$$\sum_{e \in E(T)} g'(e) \geq 5d + (n - 1 - 3d) = n + 2d - 1.$$

In general, $\sum_{e \in E(T)} g'(e) \geq n + d - 1$ and so $\text{val}_{\min}(T) \geq n + d - 1$. ■

Next, we generalize Theorem 3.6 to caterpillars T with $\Delta(T) = 3$ having an arbitrary number of vertices of degree 3.

Theorem 4.2. *If T is a caterpillar of order $n \geq 4$ such that $\Delta(T) = 3$ and T has exactly k vertices of degree 3, then*

$$\mathrm{val}_{\min}(T) = n + k - 1.$$

Proof. Let T be a caterpillar of order $n \geq 4$ with $\Delta(T) = 3$ such that T contains k vertices of degree 3. Then $\mathrm{diam}(T) = n - k - 1$. Let $P: v_0, v_1, v_2, \cdots, v_{n-k-1}$ be a path of length $n - k - 1$ in T. Let $i_1, i_2, \cdots, i_k$ be integers such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n - k - 2$ and $\deg v_{i_j} = 3$ for $1 \leq j \leq k$. Let u_j be the vertex not on P that is adjacent to v_{i_j}, where $1 \leq j \leq k$. Furthermore, let $f : V(T) \to \{0, 1, \cdots, n-1\}$ be the γ-labeling of T defined by

$$f(v_t) = \begin{cases} d(v_t, v_0) & \text{if } t \leq i_1, \\ d(v_t, v_0) + \max\{j : i_j < t\} & \text{otherwise} \end{cases}$$

and

$$f(u_j) = 1 + f(v_{i_j}).$$

Since $\mathrm{val}(f) = n + k - 1$, it follows that $\mathrm{val}_{\min}(T) \leq n + k - 1$.

Next, we show that $\mathrm{val}_{\min}(T) \geq n + k - 1$. Let

$$f : V(T) \to \{0, 1, 2, \cdots, n-1\}$$

be an arbitrary γ-labeling of T and let $u, v \in V(T)$ such that $f(u) = 0$ and $f(v) = n - 1$. Let P be a $u - v$ path in T. The length of P is at most $\mathrm{diam}(T) = n - k - 1$. Also, by Lemma 3.3

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)| = n - 1.$$

Since there are at least k edges of T not on P, it follows that

$$\mathrm{val}(f) = \sum_{e \in E(T)} f'(e) \geq (n-1) + k,$$

and so $\mathrm{val}_{\min}(T) \geq n + k - 1$. ∎

We now present a lower bound for the minimum value of a tree in terms of its order, maximum degree, and diameter.

Theorem 4.3. *If T is a tree of order $n \geq 4$, maximum degree Δ, and diameter d, then*

$$\mathrm{val}_{\min}(T) \geq \frac{8n + \Delta^2 - 6\Delta - 4d + \delta_\Delta}{4},$$

where

$$\delta_\Delta = \begin{cases} 0 & \text{if } \Delta \text{ is even,} \\ 1 & \text{if } \Delta \text{ is odd.} \end{cases}$$

Furthermore, this bound is sharp for paths and stars.

Proof. Let f be a γ-labeling of T and let $u, v \in V(T)$ such that $f(u) = 0$ and $f(v) = n - 1$. Let P be a $u - v$ path in T. Let x be a vertex of T with $\deg x = \Delta$. We consider two cases.

Case 1. $\Delta = 2k$ *for some integer* $k \geq 1$. Since (1) at most two edges of T incident with x can be labeled by i for each i with $1 \leq i \leq (k-1)$ and (2) the length of P is at most d, it follows that

$$\begin{aligned} \mathrm{val}(f) &\geq (n-1) + 2[1 + 2 + \cdots + (k-1)] + [(n-1-d) - (2k-2)] \\ &= 2n + k^2 - 3k - d = 2n + \frac{\Delta^2}{4} - \frac{3\Delta}{2} - d \\ &= \frac{8n + \Delta^2 - 6\Delta - 4d}{4}. \end{aligned}$$

Case 2. $\Delta = 2k + 1$ *for some integer* $k \geq 1$. By the same reasoning used in Case 1,

$$\begin{aligned} \mathrm{val}(f) &\geq (n-1) + 2[1 + 2 + \cdots + (k-1)] + k + [(n-1-d) - (2k-1)] \\ &= 2n - 1 + k^2 - 2k - d = 2n + \frac{(\Delta-1)^2}{4} - \Delta - d \\ &= \frac{8n + \Delta^2 - 6\Delta - 4d + 1}{4}. \end{aligned}$$

That this bound is sharp for paths and stars follows by Theorems B and C. ■

5. Connected Graphs of Order n with Minimum Value $n+1$

In Theorem 3.6, all connected graphs of order $n \geq 4$ having minimum value n are characterized. In particular, if T is a caterpillar of order $n \geq 4$ whose only vertex of degree exceeding 2 has degree 3, then $\text{val}_{\min}(T) = n$. In this section, we characterize those connected graphs of order $n \geq 5$ having minimum value $n+1$. First, we show that every caterpillar of order $n \geq 5$ whose unique vertex of degree exceeding 2 has degree 4 must have minimum value $n+1$.

Lemma 5.1. *Let T be a caterpillar of order $n \geq 5$. If T has a unique vertex v with degree greater than 2 and $\deg v = 4$, then*

$$\text{val}_{\min}(T) = n + 1.$$

Proof. By Lemma 3.3, $\text{val}_{\min}(T) \geq n+1$. It remains to show that $\text{val}_{\min}(T) \leq n+1$. Suppose that T is obtained from path $v_1, v_2, \cdots, v_{n-2}$ by adding the vertices v_{n-1} and v_n and joining each of v_{n-1} and v_n to a vertex v_k, where $2 \leq k \leq n-3$. Thus v_k is the only vertex of degree greater than 2 in T and $\deg v_k = 4$. Define a γ-labeling f of T by

$$f(v_i) = \begin{cases} i-1 & \text{if } 1 \leq i \leq k-1, \\ i & \text{if } i = k, \\ i+1 & \text{if } k+1 \leq i \leq n-2, \\ k-1 & \text{if } i = n-1, \\ k+1 & \text{if } i = n. \end{cases}$$

Since $\text{val}(f) = n+1$, it follows that $\text{val}_{\min}(T) \leq n+1$. ∎

For a fixed integer n, let $\mathcal{T}_1$ be the set of caterpillars T of order $n \geq 5$ such that T has a unique vertex v with degree greater than 2 and $\deg v = 4$ (as described in Lemma 5.1), let $\mathcal{T}_2$ be the set of trees T of order n such that T is a caterpillar of order $n \geq 6$ with $\Delta(T) = 3$ and T has exactly two vertices of degree 3, and let $\mathcal{T}_3$ be the set of trees T of order $n \geq 7$ such that T has a unique vertex v of degree greater than 2 and $\deg v = 3$, where the distance between v and a nearest end-vertex of T is 2. By Lemma 5.1 and Theorems 4.1 and 4.2, we have the following.

Corollary 5.2. *Let T be a tree of order n. If $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then* $\text{val}_{\min}(T) = n + 1$.

Lemma 5.3. *Each of the threes F_1, F_2, and F_3 in Figure 3 of order $n = 9, 8, 8$, respectively, has minimum value $n + 2$, that is,*

$$\text{val}_{\min}(F_1) = 11 \textit{ and } \text{val}_{\min}(F_2) = \text{val}_{\min}(F_3) = 10.$$

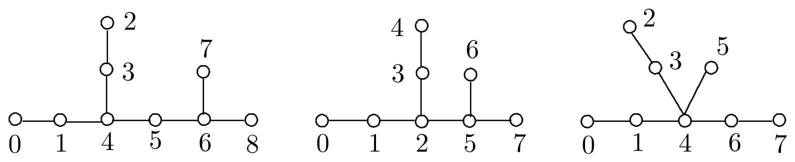

Figure 3: The graphs F_1, F_2, and F_3.

Proof. For each integer i with $1 \le i \le 3$, a γ-labeling f_i of F_i is shown in Figure 3. Since $\text{val}(f_1) = 11$ and $\text{val}(f_2) = \text{val}(f_3) = 10$, it follows that $\text{val}_{\min}(F_1) \le 11$, $\text{val}_{\min}(F_2) \le 10$, and $\text{val}_{\min}(F_3) \le 10$.

Next, we show that $\text{val}_{\min}(F_1) \ge 11$. Let g be γ-min labeling of F_1 and let $u, v \in V(F_1)$ such that $g(u) = 0$ and $g(v) = 8$. Suppose that P is a $u - v$ path in F_1. Then $\sum_{e \in E(P)} f'(e) \ge 8$ by Lemma 3.4. Since there are at least three edges of F_1 not in P, it follows that $\text{val}_{\min}(F_1) = \text{val}(g) \ge 8 + 3 = 11$. A similar argument shows that $\text{val}_{\min}(F_2) \ge 10$, and $\text{val}_{\min}(F_3) \ge 10$. ■

We now characterize all trees of order $n \ge 5$ whose minimum value is $n + 1$.

Theorem 5.4. *Let T be a tree of order $n \ge 5$. Then* $\text{val}_{\min}(T) = n + 1$ *if and only if $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.*

Proof. By Corollary 5.2, if $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then $\text{val}_{\min}(T) = n + 1$. It therefore remains to verify the converse. We begin by establishing the following three claims.

Claims. Let T be a tree of order $n \ge 7$ such that $\text{val}_{\min}(T) = n + 1$ and $T \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. Then:

(1) $3 \le \Delta(T) \le 4$.

(2) T has at most two vertices of degree greater than 2.

(3) If v is a vertex of T with $\deg v \geq 3$, then the distance between v and a nearest end-vertex in T is at most 2.

Proof of Claims. Since $\mathrm{val}_{\min}(T) = n + 1$, it follows that T is not a path by Theorem B and so $\Delta(T) \geq 3$. If $\Delta(T) \geq 5$, then $\mathrm{val}_{\min}(T) \geq (n - 1) + 2^2 = n + 3$ by Lemma 3.3, a contradiction. Thus $3 \leq \Delta(T) \leq 4$ and so Claim (1) holds.

Next we verify Claim (2). Suppose that T has $k \geq 3$ vertices of degree greater than 2. Then T contains a caterpillar T' of order n' as a subgraph with $\Delta(T') = 3$ such that T' has exactly three vertices of degree 3. By Theorem 4.2, $\mathrm{val}_{\min}(T') = n' + 2$. It then follows from Lemma 3.1 that

$$\mathrm{val}_{\min}(T) \geq [(n - 1) - (n' - 1)] + \mathrm{val}_{\min}(T') \geq (n - n') + (n' + 2) = n + 2,$$

a contradiction. Thus Claim (2) holds.

We now verify Claim (3). Let v be a vertex of T with $\deg v \geq 3$. If the distance between v and a nearest end-vertex in T is greater than 2, then T contains a subtree T'' of order n'' such that (a) $\Delta(T'') = 3$ and T'' has a unique vertex v of degree 3 and (b) the distance d between v and a nearest end-vertex in T'' is greater than 2. By Theorem 4.1,

$$\mathrm{val}_{\min}(T'') = n' + d - 1 \geq n' + 2.$$

Again, by Lemma 3.1,

$$\mathrm{val}_{\min}(T) \geq [(n - 1) - (n' - 1)] + \mathrm{val}_{\min}(T') \geq (n - n') + (n' + 2) = n + 2,$$

a contradiction. Thus Claim (3) holds. This completes the proof of the three claims.

We continue with the proof of the theorem. Assume, to the contrary, that there is a tree T of order $n \geq 7$ with $\mathrm{val}_{\min}(T) = n + 1$ such that $T \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. By Claim (1), $3 \leq \Delta(T) \leq 4$. We consider two cases, according to whether $\Delta(T) = 3$ or $\Delta(T) = 4$.

Case 1. $\Delta(T) = 3$. If T is a caterpillar, then T contains exactly two vertices of degree 3 by Theorem 4.2. However then, $T \in \mathcal{T}_2$, a contradiction. Thus T is not a caterpillar. If T has exactly one vertex x of degree 3, then the distance between x and a nearest end-vertex of T is 2 by Theorem 4.1. However then, $T \in \mathcal{T}_3$, again a contradiction. Thus T is not a caterpillar and T contains exactly two vertices u and v of degree 3 by Claim (2). Furthermore, we may assume that the distance d from u to a nearest end-vertex of T is 2 by Claim (3). We consider three subcases.

Subcase 1.1. $d(u,v) \geq 3$. Then T contains two edge-disjoint subgraphs H_1 and H_2 such that H_1 is isomorphic to the graph F in Lemma 3.5 and H_2 is isomorphic to $K_{1,3}$. Let f be a γ-min labeling of T. Since $\mathrm{val}_{\min}(H_1) = 8$ by Lemma 3.5 and $\mathrm{val}_{\min}(H_2) = 4$ by Theorem C, it follows by Lemma 3.2 that

$$\mathrm{val}_{\min}(T) \geq [(n-1) - 6 - 3] + (8 + 4) = n + 2,$$

a contradiction.

Subcase 1.2. $d(u,v) = 2$. Then T contains the graph F_1 of Lemma 5.3 as a subgraph. Since the size of F_1 is 8 and $\mathrm{val}_{\min}(F_1) = 11$ by Lemma 5.3, it follows from Lemma 3.1 that $\mathrm{val}_{\min}(T) \geq [(n-1)-8]+11 = n+2$, which produces a contradiction.

Subcase 1.3. $d(u,v) = 1$. Then T contains the graph F_2 of Lemma 5.3 as a subgraph. Since the size of F_2 is 7 and $\mathrm{val}_{\min}(F_2) = 10$ by Lemma 5.3, it follows from Lemma 3.1 that $\mathrm{val}_{\min}(T) \geq [(n-1) - 7] + 10 = n + 2$, a contradiction.

Case 2. $\Delta(T) = 4$. There are two subcases.

Subcase 2.1. *T has a unique vertex v of degree exceeding* 2. Then $\deg v = 4$. If T is a caterpillar, then $T \in \mathcal{T}_1$, a contradiction. Thus T is not a caterpillar. However then, T contains the graph F_3 of Lemma 5.3 as a subgraph. Since the size of F_3 is 7 and $\mathrm{val}_{\min}(F_3) = 10$ by Lemma 5.3, it follows from Lemma 3.1 that $\mathrm{val}_{\min}(T) \geq [(n-1) - 7] + 10 = n + 2$, a contradiction.

Subcase 2.2. *T has two vertices u and v of degree exceeding* 2. If T is not a caterpillar, then $\mathrm{val}_{\min}(T) \geq n + 2$ by the proofs of Subcases 1.1, 1.2, and 1.3 in Case 1, which is a contradiction. Thus we may assume that T is a caterpillar and $\deg u = 4$. There are two subcases.

Subcase 2.2.1. $d(u,v) \geq 2$. Then T contains two edge-disjoint subgraphs isomorphic to $K_{1,4}$ and $K_{1,3}$, respectively. Let f be a γ-min labeling of T.

Since $\text{val}_{min}(K_{1,4}) = 6$ and $\text{val}_{min}(K_{1,3}) = 4$ by Theorem C, it follows from Lemma 3.2 that $\text{val}_{\min}(T) \geq [(n-1)-4-3]+6+4 = n+2$, a contradiction.

Subcase 2.2.2. $d(u,v) = 1$. Then T contains the double star $S_{4,3}$ as a subgraph. Since the size of $S_{4,3}$ is 6 and $\text{val}_{\min}(S_{4,3}) = 9$ by Proposition 2.1, it follows by Lemma 3.1 that $\text{val}_{\min}(T) \geq [(n-1)-6]+9 = n+2$, a contradiction. ■

We next characterize all connected graphs G of order n for which $\text{val}_{\min}(G) = n+1$. First, we present two lemmas. Since the proofs are straightforward, we omit them.

Lemma 5.5. *For the graph H of Figure* 4*,* $\text{val}_{\min}(H) = 9$*.*

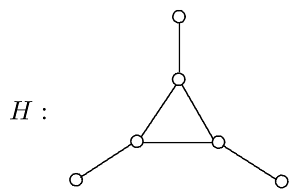

Figure 4: The graph H of Lemma 5.5.

Let $\mathcal{F}$ be the set of all graphs of order $n \geq 3$ obtained from the path $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+2} for some i with $1 \leq i \leq n-2$.

Lemma 5.6. *If $F \in \mathcal{F}$, then* $\text{val}_{\min}(F) = n+1$*.*

Theorem 5.7. *Let G be a connected graph of order n. Then* $\text{val}_{\min}(G) = n+1$ *if and only if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{F}$.*

Proof. We have seen in Theorem 5.4 and Lemma 5.6 that if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{F}$, then $\text{val}_{\min}(G) = n+1$. For the converse, let G be a connected graph for which $\text{val}_{\min}(G) = n+1$ such that $G \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. It then follows from Theorem 5.4 that G is not a tree. Hence G contains cycles. By Theorem B, G contains exactly one cycle C and so G has size n. Suppose that C is a k-cycle, where $k \geq 3$. Since $\text{val}_{\min}(G) = 2k-2$ by Theorem D, it follows by Lemma 3.1 that

$$\text{val}_{\min}(G) \geq (n-k) + (2k-2) = n+k-2.$$

Since $\text{val}_{\min}(G) = n + 1$, the cycle C is a triangle. If G contains the graph H of Figure 4 as a subgraph, then by Lemmas 5.5 and 3.1,

$$\text{val}_{\min}(G) \geq (n-6) + \text{val}_{\min}(H) = (n-6) + 9 = n+3,$$

which is impossible. Therefore, at least one vertex of C has degree 2 in G. Furthermore, G contains no vertex of degree 4 or more; for otherwise, G contains $K_{1,4}$ as a subgraph and by Lemma 3.1 and Theorem C,

$$\text{val}_{\min}(G) \geq (n-4) + \text{val}_{\min}(K_{1,4}) = (n-4) + 6 = n+2,$$

a contradiction. Also, observe that there cannot be a vertex of degree 3 that does not belong to C; for otherwise, G contains edge-disjoint subgraphs K_3 and $K_{1,3}$ and by Lemma 3.2, Theorems C and D,

$$\begin{aligned} \text{val}_{\min}(G) &\geq (n-3-3) + \text{val}_{\min}(K_3) + \text{val}_{\min}(K_{1,3}) \\ &= (n-6) + 4 + 4 = n+2, \end{aligned}$$

which is impossible. This implies that $G \in \mathcal{F}$. ■

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