DECOMPOSITIONS INTO TWO PATHS

ZDZISŁAW SKUPIEŃ

Faculty of Applied Mathematics
AGH University of Science and Technology
al. Mickiewicza 30, 30–059 Kraków, Poland
e-mail: skupien@uci.agh.edu.pl

Abstract

It is proved that a connected multigraph $G$ which is the union of two edge-disjoint paths has another decomposition into two paths with the same set, $U$, of endvertices provided that the multigraph is neither a path nor cycle. Moreover, then the number of such decompositions is proved to be even unless the number is three, which occurs exactly if $G$ is a tree homeomorphic with graph of either symbol $+$ or $\perp$. A multigraph on $n$ vertices with exactly two traceable pairs is constructed for each $n \geq 3$. The Thomason result on hamiltonian pairs is used and is proved to be sharp.

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1. Introduction

Investigations presented in what follows have been inspired by discussion within GRAPHNET [1] in February 2001 on a two-path conjecture presented then and there by Ken W. Smith of Central Michigan University. The conjecture says that a connected graph $G$ which is the edge-disjoint union of two paths of length $n$ has at least one more subgraph which is a path of length $n$. In four days the discussion concluded with a note by Doug West in which he presented a proof (based on Thomason’s paper [6]) of the
result which we state below in terms of decompositions. For notation and terminology, see [7].

The above two-path conjecture of Smith brings to mind a result due to Sloane [4]. It answers in the affirmative Shen Lin’s question of 1965 whether every 4-regular graph which is hamiltonian decomposable has another hamiltonian cycle. Lin’s paper [2] originated related investigations into hamiltonian decompositions.

Recall that a decomposition of a graph $G$ is a collection of edge-disjoint subgraphs whose union is equal to $G$. A decomposition is called hamiltonian (traceable) if all decomposition parts are hamiltonian cycles (hamiltonian paths), the decomposition is called a hamiltonian pair (traceable pair) if the number of parts in question is two. Let $h_2(G)$ and $t_2(G)$ be the numbers of respectively hamiltonian and traceable pairs of $G$, the numbers being 0 if $\Delta(G) > 4$. Similarly, let $p_2(G)$ stand for the number of decompositions of $G$ into two nontrivial paths.

In what follows by a graph, $G$, we mean a multigraph (which is loopless), the phrase simple graph is used to emphasize that multiple (or parallel) edges do not appear. The degree of a vertex is the number of incident edges.

**Theorem A** (D. West). If $G$ is connected and decomposable into two paths of length $k$ (where $k > 1$), then $G$ is decomposable into a different pair of paths. In particular, one of these two, different from the original pair, has length at least $k$.

The following theorem is a part of Thomason’s related result.

**Theorem B.** Let $G$ be a multigraph with three or more vertices that has a hamiltonian pair. Then the number $h_2(G)$ of hamiltonian decompositions of $G$ is even and at least four. Moreover, for any two edges of $G$, the number of hamiltonian pairs in which the two edges are in the same part is also even.

Note that if $G$ has a pair of parallel edges, a simple switch produces the pair of new decomposition parts. Nevertheless, there are large multigraphs with only few (i.e., two) traceable pairs. Our main result follows.

**Theorem 1.** Let $G$ be a connected multigraph that is decomposable into two nontrivial paths whose set of endvertices is denoted by $U$. If $G$ is neither a path nor a cycle, then $G$ is the union of a different pair of edge-disjoint paths with the same set $U$ of endvertices. In fact, the number $p_2(G)$ of such
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Decompositions is then even unless $G$ is homeomorphic with the graph of either symbol $+$, $-$, and then $p_2 = 3$.

**Corollary 2.** The number of traceable pairs among multigraphs of order $n \geq 3$ is even.

The following result shows that Thomason’s lower bound $h_2(G) \geq 4$ in his theorem above is sharp.

**Proposition 3.** For each $n \geq 3$ there are two $n$-vertex multigraphs $M_n$ and $M''_n$ which have exactly four hamiltonian pairs and two traceable ones, respectively.

2. Proofs and Examples

**Proof of Theorem 1.** Let $P$ and $Q$ denote the original paths whose union is $G$. Consider the only interesting case that the largest vertex valency, $\Delta(G)$, is 3 or 4 and $G$ is not homeomorphic with $+$ or $\bot$. Then both paths $P$ and $Q$ are nontrivial and $G$ has three or more edges. Moreover, $2 \leq |U| \leq 4$, the vertices in $U$ have degree 3 or less, and $U$ includes all vertices of $G$ of odd degree (1 or 3). Furthermore, at least two vertices of $G$ are of degree larger than one.

**Case 1.** $G$ has exactly one vertex of degree bigger than 2 and $|U| = 3$. Then paths $P$ and $Q$ share one endvertex and intersect at another vertex which is of degree 3 or 4 in $G$. Hence $\delta(G) = 1$. It is easily seen that $p_2(G) = 2$.

**Case 2.** $|U| = 2$. Then both vertices in $U$ are of degree two in $G$, $G$ has a vertex of degree 4 and no vertex of odd degree. Let $\hat{G}$ be obtained from $G$ by joining vertices in $U$ by two parallel edges, say $e, f$. Let $G'$ be the 4-regular homeomorph of $\hat{G}$ (obtained by contracting an edge incident to a degree-2 vertex, one after another until no such an edge remains). Then $p_2(G)$ is equal to the number of hamiltonian decompositions of $G'$ in which a fixed length-2 path $P_3$ containing the edge $e$ and its neighbor is in one decomposition part. Therefore $p_2(G)$ is even by Theorem B. Moreover, $p_2(G) = h_2(G')/2$.

**Case 3.** $|U| = 3$ and $G$ has two or more vertices of degree bigger than 2. Hence $U$ comprises a vertex, $x$, of degree 2 and two vertices, say $y, z$, of odd
degree. Let $\hat{G} = G + \{xy, xz\}$ and let $G'$ be the 4-regular homeomorph of $\hat{G}$. Then $p_2(G) = h_2(G') - h'_2$ where $h'_2$ counts the hamiltonian decompositions of $G'$ such that one part has preimage in $\hat{G}$ containing the path $yxz$. Hence, by Theorem B, $p_2(G)$ is even.

Case 4. $|U| = 4$. Then all vertices in $U$ are of odd degree (1 or 3) and $G$ has at least two vertices with degrees in the set $\{3, 4\}$. Add to $G$ a new vertex, say $w$, together with four edges joining $w$ to all vertices in $U$. Let $\hat{G}$ and $G'$ be the resulting multigraph and its 4-regular homeomorph, respectively. Then two-path decompositions of $G$ (which must keep $U$ fixed) are in one-one correspondence with hamiltonian pairs of $G'$ whence $p_2(G)$ is even.

**Proof of Proposition 3.** Given the cycle $C_n$ with $n \geq 5$ and a path $P_4 = stuv$ contained in $C_n$, let the multigraph $M_n$ be obtained from the square $C_n^2$ of $C_n$ by removing the two crossing chords $su$ and $tv$ and by doubling of edges $st$ and $uv$. Thus $M_n$ is a 4-regular multigraph with two pairs of parallel edges. Note that contracting any two parallel edges of $M_n$ with $n \geq 6$ results in $M_n - 1$. Assume that multigraphs $M_n$ and $M_3$ are obtained if this contracting is applied to $M_5$ and then to $M_4$, respectively. Hence $M_3 = 2K_3$, the doubled triangle. Let $M''_n$ be obtained from $M_n$ by removing a pair of parallel edges. Hence $M''_3 = 2P_3$ and $M''_4$ is the join of the 2-cycle $C_2$ and $2K_1$. Assume that, for each $n \geq 4$, notation in $M''_n$ is chosen so that degree-2 vertices are $u$ and $v$ (or $u'$, $v'$) and $2st$ are the two parallel edges. Note that there exists a map $M''_n \rightarrow M''_{n+1}$ for $n \geq 4$ in which the degree-2 vertex $u$ with neighbors $t$ and, say, $v_1$ ($v_1 = s$ if $n = 4$) is removed and replaced by two new vertices, say $u'$, $v'$, together with four edges $tu', u'v, v'v, v_1v_1$. It is enough to show that $t_2(M''_n) = 2$. This equality is easily seen for $n = 3, 4$. Use the map $M''_n \rightarrow M''_{n+1}$ to show by induction on $n \geq 4$ that in each traceable pair of $M''_n$ the part containing the edge $tu$ is either the $v-u$ section of the cycle $C_n$ or its switching at $2st$.  

**3. Concluding Remarks**

More examples of multigraphs $M$ on $n$ vertices (inclusive of the above examples $M''_n$) with the smallest possible nonzero number of traceable pairs $t_2(M) = 2$, and with $|U| = 2$, are given in author’s paper [3] for each $n \geq 7$. Then $p_2(M) = 2$, with vertices in $U$ being the only possible endvertices of
decomposition parts. Simple graphs $G$ of each order $n \geq 5$ with $t_2(G) = 4$ and $|U| = 3$ are given in [3], too.

At the other extreme, $t_2(2P_n) = \frac{1}{2}h_2(2C_n) = 2^{n-2}$ and this is not the largest value of $t_2$ among $n$-vertex simple graphs. It is a challenging problem to find (good estimates of) the largest value of $t_2$ (and/or $h_2$) among simple graphs (or multigraphs) on $n$ vertices.

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References


