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DECOMPOSITIONS INTO TWO PATHS

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Abstract

It is proved that a connected multigraph G which is the union of two edge-disjoint paths has another decomposition into two paths with the same set, U, of endvertices provided that the multigraph is neither a path nor cycle. Moreover, then the number of such decompositions is proved to be even unless the number is three, which occurs exactly if G is a tree homeomorphic with graph of either symbol $+$ or $\perp$. A multigraph on n vertices with exactly two traceable pairs is constructed for each $n \geq 3$. The Thomason result on hamiltonian pairs is used and is proved to be sharp.

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1. Introduction

Investigations presented in what follows have been inspired by discussion within GRAPHNET [1] in February 2001 on a *two-path conjecture* presented then and there by Ken W. Smith of Central Michigan University. The conjecture says that a connected graph G which is the edge-disjoint union of two paths of length n has at least one more subgraph which is a path of length n. In four days the discussion concluded with a note by Doug West in which he presented a proof (based on Thomason's paper [6]) of the result which we state below in terms of decompositions. For notation and terminology, see [7].

The above two-path conjecture of Smith brings to mind a result due to Sloane [4]. It answers in the affirmative Shen Lin's question of 1965 whether every 4-regular graph which is hamiltonian decomposable has another hamiltonian cycle. Lin's paper [2] originated related investigations into hamiltonian decompositions.

Recall that a decomposition of a graph G is a collection of edge-disjoint subgraphs whose union is equal to G. A decomposition is called *hamiltonian* (*traceable*) if all decomposition parts are hamiltonian cycles (hamiltonian paths), the decomposition is called a *hamiltonian pair* (*traceable pair*) if the number of parts in question is two. Let $h_2(G)$ and $t_2(G)$ be the *numbers* of respectively hamiltonian and traceable pairs of G, the numbers being 0 if $\Delta(G) > 4$. Similarly, let $p_2(G)$ stand for the *number of decompositions* of G into two nontrivial paths.

In what follows by a graph, G, we mean a multigraph (which is loopless), the phrase *simple graph* is used to emphasize that multiple (or parallel) edges do not appear. The degree of a vertex is the number of incident edges.

Theorem A (D. West). *If G is connected and decomposable into two paths of length k (where $k > 1$), then G is decomposable into a different pair of paths. In particular, one of these two, different from the original pair, has length at least k.*

The following theorem is a part of Thomason's related result.

Theorem B. *Let G be a multigraph with three or more vertices that has a hamiltonian pair. Then the number $h_2(G)$ of hamiltonian decompositions of G is even and at least four. Moreover, for any two edges of G, the number of hamiltonian pairs in which the two edges are in the same part is also even.*

Note that if G has a pair of parallel edges, a simple switch produces the pair of new decomposition parts. Nevertheless, there are large multigraphs with only few (i.e., two) traceable pairs. Our main result follows.

Theorem 1. *Let G be a connected multigraph that is decomposable into two nontrivial paths whose set of endvertices is denoted by U. If G is neither a path nor a cycle, then G is the union of a different pair of edge-disjoint paths with the same set U of endvertices. In fact, the number $p_2(G)$ of such* *decompositions is then even unless G is homeomorphic with the graph of either symbol $+$, or $\perp$, and then $p_2 = 3$.*

Corollary 2. *The number of traceable pairs among multigraphs of order $n \geq 3$ is even.* ■

The following result shows that Thomason's lower bound $h_2(G) \geq 4$ in his theorem above is sharp.

Proposition 3. *For each $n \geq 3$ there are two n-vertex multigraphs M_n and M''_n which have exactly four hamiltonian pairs and two traceable ones, respectively.*

2. Proofs and Examples

***Proof of Theorem* 1.** Let P and Q denote the original paths whose union is G. Consider the only interesting case that the largest vertex valency, $\Delta(G)$, is 3 or 4 and G is not homeomorphic with $+$ or $\perp$. Then both paths P and Q are nontrivial and G has three or more edges. Moreover, $2 \leq |U| \leq 4$, the vertices in U have degree 3 or less, and U includes all vertices of G of odd degree (1 or 3). Furthermore, at least two vertices of G are of degree larger than one.

Case 1. G has exactly one vertex of degree bigger than 2 and $|U| = 3$. Then paths P and Q share one endvertex and intersect at another vertex which is of degree 3 or 4 in G. Hence $\delta(G) = 1$. It is easily seen that $p_2(G) = 2$.

Case 2. $|U| = 2$. Then both vertices in U are of degree two in G, G has a vertex of degree 4 and no vertex of odd degree. Let $\hat{G}$ be obtained from G by joining vertices in U by two parallel edges, say e, f. Let G' be the 4-regular homeomorph of $\hat{G}$ (obtained by contracting an edge incident to a degree-2 vertex, one after another until no such an edge remains). Then $p_2(G)$ is equal to the number of hamiltonian decompositions of G' in which a fixed length-2 path P_3 containing the edge e and its neighbor is in one decomposition part. Therefore $p_2(G)$ is even by Theorem B. Moreover, $p_2(G) = h_2(G')/2$.

Case 3. $|U| = 3$ and G has two or more vertices of degree bigger than 2. Hence U comprises a vertex, x, of degree 2 and two vertices, say y, z, of odd degree. Let $\hat{G} = G + \{xy, xz\}$ and let G' be the 4-regular homeomorph of $\hat{G}$. Then $p_2(G) = h_2(G') - h_2'$ where h_2' counts the hamiltonian decompositions of G' such that one part has preimage in $\hat{G}$ containing the path yxz. Hence, by Theorem B, $p_2(G)$ is even.

Case 4. $|U| = 4$. Then all vertices in U are of odd degree (1 or 3) and G has at least two vertices with degrees in the set $\{3, 4\}$. Add to G a new vertex, say w, together with four edges joining w to all vertices in U. Let $\hat{G}$ and G' be the resulting multigraph and its 4-regular homeomorph, respectively. Then two-path decompositions of G (which must keep U fixed) are in one-one correspondence with hamiltonian pairs of G' whence $p_2(G)$ is even. ■

***Proof of Proposition* 3.** Given the cycle C_n with $n \geq 5$ and a path $P_4 = stuv$ contained in C_n, let the multigraph M_n be obtained from the square ${C_n}^2$ of C_n by removing the two crossing chords su and tv and by doubling of edges st and uv. Thus M_n is a 4-regular multigraph with two pairs of parallel edges. Note that contracting any two parallel edges of M_n with $n \geq 6$ results in M_{n-1}. Assume that multigraphs M_4 and M_3 are obtained if this contracting is applied to M_5 and then to M_4, respectively. Hence $M_3 = {}^2K_3$, the doubled triangle. Let M_n'' be obtained from M_n by removing a pair of parallel edges. Hence $M_3'' = {}^2P_3$ and M_4'' is the join of the 2-cycle C_2 and $2K_1$. Assume that, for each $n \geq 4$, notation in M_n'' is chosen so that degree-2 vertices are u and v (or u', v') and ${}^2st$ are the two parallel edges. Note that there exists a map $M_n'' \mapsto M_{n+1}''$ for $n \geq 4$ in which the degree-2 vertex u with neighbors t and, say, v_1 ($v_1 = s$ if $n = 4$) is removed and replaced by two new vertices, say u', v', together with four edges $tu', u'v, v'v, v'v_1$. It is enough to show that $t_2(M_n'') = 2$. This equality is easily seen for $n = 3, 4$. Use the map $M_n'' \mapsto M_{n+1}''$ to show by induction on $n \geq 4$ that in each traceable pair of M_n'' the part containing the edge tu is either the v–u section of the cycle C_n or its switching at ${}^2st$. ■

3. Concluding Remarks

More examples of multigraphs M on n vertices (inclusive of the above examples M_n'') with the smallest possible nonzero number of traceable pairs $t_2(M) = 2$, and with $|U| = 2$, are given in author's paper [3] for each $n \geq 7$. Then $p_2(M) = 2$, with vertices in U being the only possible endvertices of decomposition parts. Simple graphs G of each order $n \geq 5$ with $t_2(G) = 4$ and $|U| = 3$ are given in [3], too.

At the other extreme, $t_2(^2P_n) = \frac{1}{2}h_2(^2C_n) = 2^{n-2}$ and this is not the largest value of t_2 among n-vertex simple graphs. It is a challenging problem to find (good estimates of) the largest value of t_2 (and/or h_2) among simple graphs (or multigraphs) on n vertices.

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