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HAMILTON DECOMPOSITIONS OF LINE GRAPHS OF SOME BIPARTITE GRAPHS

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Abstract

Some bipartite Hamilton decomposable graphs that are regular of degree $\delta \equiv 2 \pmod 4$ are shown to have Hamilton decomposable line graphs. One consequence is that every bipartite Hamilton decomposable graph G with connectivity $\kappa(G) = 2$ has a Hamilton decomposable line graph $L(G)$.

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1. Introduction

The *line graph*, denoted by $L(G)$, of a graph G having vertex set $V(G)$ and edge set $E(G)$ is the graph with vertex set $E(G)$, where two vertices of $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges in G are incident with a common vertex in G.

A *Hamilton decomposition* of a δ-regular graph G consists of a set of Hamilton cycles (plus a 1-factor if δ is odd) of G such that these cycles (and the 1-factor when δ is odd) partition the edges of G. If G has a Hamilton decomposition, it is said to be *Hamilton decomposable*.

Investigating Hamilton decompositions of line graphs has been largely motivated by the following conjecture of Bermond [1]:

Conjecture 1. *If G is Hamilton decomposable, then $L(G)$ is Hamilton decomposable.*

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Bermond's conjecture has been shown to hold when G is a Hamilton decomposable graph satisfying any of the following criteria [3, 4, 5, 6, 7, 8, 9, 10, 11]:

1. $\delta \leq 5$,
2. $\delta \equiv 0 \pmod 4$,
3. δ is odd and G is bipartite,
4. δ is even and G has a perfect 1-factorisation,
5. $G = K_n$, or
6. $G = K_{n,n}$.

In the case where G is a Hamilton decomposable graph that is regular of degree $\delta \equiv 2 \pmod 4$, it is known that $L(G)$ can be decomposed into $\delta - 2$ Hamilton cycles plus a 2-factor [7, 12]. However, aside from when G is either complete or else has a perfect 1-factorisation, it is not generally known whether $L(G)$ is Hamilton decomposable.

As our main result, we show that $L(G)$ is Hamilton decomposable for every bipartite graph G with $\delta \equiv 2 \pmod 4$ that has a Hamilton decomposition H such that $\kappa(G - H_1) = 2$ for some Hamilton cycle H_1 of H. An immediate consequence of this result is that every bipartite Hamilton decomposable graph G with connectivity $\kappa(G) = 2$ has a Hamilton decomposable line graph.

2. Preliminary Results

Let $\mathcal{H}_1$ denote the Hamilton cycle $(1, 3, 5, \ldots, 4k+1, 2, 4k+2, 4k, \ldots, 6, 4, 1)$ of K_{4k+2}, where $V(K_{4k+2}) = \{1, 2, \ldots, 4k+2\}$, and let $\mathcal{F}$ denote the 1-factor of K_{4k+2} having the edges $\{2i-1, 2i\}$ for $i = 1, \ldots, 2k+1$. $\mathcal{H}_1$ is illustrated in Figure 1. A double-centred Walecki construction described by Zhan [12] produces the following Hamilton decomposition of K_{4k+2}:

Lemma 1. *K_{4k+2} has a Hamilton decomposition in which $\mathcal{H}_1$ is one of the Hamilton cycles and $\mathcal{F}$ is the 1-factor.*

Proof. For each $i = 2, 3, \ldots, 2k$, let $\mathcal{H}_i = \sigma^{i-1}(\mathcal{H}_1)$ where σ is the permutation $(1)(2)(3, 5, 9, \ldots, 4k+1, 4k, \ldots, 12, 8, 4, 6, 10, \ldots, 4k+2, 4k-1, \ldots, 11, 7)$. Then the Hamilton cycles $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{2k}$ and the 1-factor $\mathcal{F}$ form a Hamilton decomposition of K_{4k+2}. ■

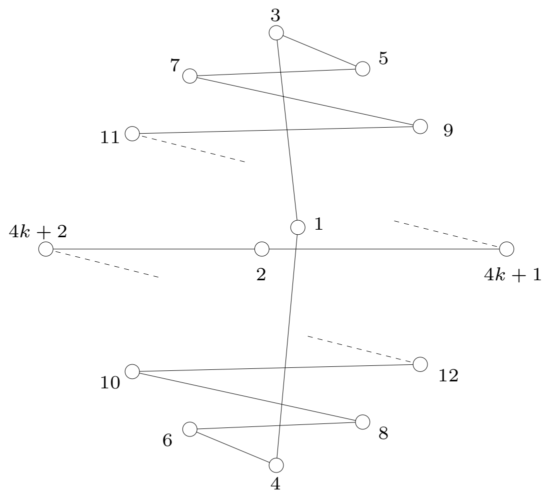

Figure 1. The Hamilton cycle $\mathcal{H}_1$.

In [9] a *stroll* was defined as an alternating sequence of vertices and edges, $v_0e_0v_1e_1 \cdots v_{n-1}e_{n-1}v_n$, such that v_i and v_{i+1} (which are not necessarily distinct) are each end-vertices of the edge e_i, for $0 \leq i \leq n-1$. For closed strolls (in which $v_0 = v_n$), it suffices to state only the sequence of edges. An *Euler stroll* is any closed stroll in a graph G that uses each edge of G exactly once. Two Euler strolls are said to be *compatible* if no pair of adjacent edges (i.e., no 2-path) is contained in both.

Note that an Euler stroll in a graph G naturally corresponds to a Hamilton cycle in $L(G)$, and that any set of pairwise compatible Euler strolls in G corresponds to a set of edge-disjoint Hamilton cycles in $L(G)$.

3. Main Results

Theorem 1. *Let G be a bipartite graph that is regular of degree $\delta = 4k+2$. If there exists a Hamilton decomposition H of G such that, for some Hamilton cycle H_1 of H, $\kappa(G - H_1) = 2$, then $L(G)$ is Hamilton decomposable.*

Proof. Let H be a Hamilton decomposition of G, consisting of the Hamilton cycles $H_1, \ldots, H_{2k+1}$, such that $((G - H_1) - \{u, v\})$ is disconnected for some pair of vertices u and v. Fix a bipartite colouring of the vertices of G using the colours red and blue, such that vertex u is coloured red.

Alternately colour the edges of H_1 with the colours 1 and 2. Since $\{u, v\}$ is a vertex cut in $(G - H_1)$, then there are two components, say $\mathcal{C}$ and $\mathcal{C}'$, in $((G - H_1) - \{u, v\})$.

For each $i = 2, \ldots, 2k+1$, alternately colour the edges of H_i with the colours $(2i-1)$ and $2i$ so that the edge from vertex u into $\mathcal{C}$ has colour $(2i-1)$. Let P_i be the sequence of edges along the cycle H_i of G, beginning with the edge from u to $\mathcal{C}$, ending with the edge from v to $\mathcal{C}'$, and including all edges coloured either $(2i-1)$ or $2i$ that are contained in $\mathcal{C}$. Similarly, let P'_i be the sequence of edges along the cycle H_i, beginning with the edge from $\mathcal{C}$ to u, ending with the edge from $\mathcal{C}'$ to v, and including all edges coloured either $(2i-1)$ or $2i$ that are contained in $\mathcal{C}'$. Together, P_i and P'_i contain all of the edges of H_i (with 2 edges of H_i being contained in $P_i \cap P'_i$).

Let $\mathcal{H}_1, \ldots, \mathcal{H}_{2k}$ be the Hamilton cycles of the Hamilton decomposition of K_{4k+2} described in Lemma 1. These cycles will be used to generate $4k$ mutually compatible Euler strolls in G, and hence $4k$ edge-disjoint Hamilton cycles in $L(G)$.

For each $i = 1, \ldots, 2k$, we wish to use H_1 and $\mathcal{H}_i$ to generate 2 strolls in G. This can be done by noting that $\mathcal{H}_i$ can be broken into two equal-length paths, each going from vertex 1 to vertex 2 of K_{4k+2}. For $\mathcal{H}_1$, let $\mathcal{P}_1$ (resp. $\mathcal{P}'_1$) denote the path with internal vertices having odd (resp. even) labels, so that $\mathcal{P}_1 = (1, 3, 5, 7, 9, 11, \ldots, 4k+1, 2)$ and $\mathcal{P}'_1 = (1, 4, 6, 8, 10, \ldots, 4k+2, 2)$. For $i = 2, 3, \ldots, 2k$, let $\mathcal{P}_i = \sigma^{i-1}(\mathcal{P}_1)$ and $\mathcal{P}'_i = \sigma^{i-1}(\mathcal{P}'_1)$, where σ is the permutation presented in Lemma 1.

For the first stroll generated by $\mathcal{H}_i$, where $i \in \{1, 2, \ldots, 2k\}$, we use path $\mathcal{P}_i$ (resp. $\mathcal{P}'_i$) at each vertex of H_1 that is coloured red (resp. blue) and for the second stroll we use path $\mathcal{P}_i$ (resp. $\mathcal{P}'_i$) at each vertex that is coloured blue (resp. red). We use the paths $\mathcal{P}_i$ and $\mathcal{P}'_i$ to describe how to replace each edge sequence (e, e') of H_1 with an edge sequence $(e, e_1, e_2, \ldots, e_{2k}, e')$ where each of the edges $e_1, \ldots, e_{2k}$ is incident with the vertex of G that is common to e and e'. Specifically, we wish the edge colours of the edges in the sequence $(e, e_1, e_2, \ldots, e_{2k}, e')$ to be the same as the vertex labels along the path $\mathcal{P}_i$ or $\mathcal{P}'_i$ as appropriate. So, for example, for the first stroll generated by $\mathcal{H}_1$, we would replace each 2-path in H_1 from an edge of colour 2 to an edge of colour 1 and having a blue internal vertex with a stroll consisting of edges incident with this blue vertex and having edge colours $(2, 4k+2, \ldots, 10, 8, 6, 4, 1)$, whereas for the second stroll generated by $\mathcal{H}_1$ each such 2-path of H_1 would be replaced by a stroll whose edges are coloured $(2, 4k+1, \ldots, 11, 9, 7, 5, 3, 1)$.

The $4k$ Euler strolls which are generated in this manner will be mutually compatible, and hence correspond to $4k$ edge-disjoint Hamilton cycles in $L(G)$. Let B_{2i-1} and B_{2i} denote the two Hamilton cycles in $L(G)$ that are generated from $\mathcal{H}_i$, for each $i = 1, \dots, 2k$.

If we were to remove the Hamilton cycles $B_1, \dots, B_{4k}$ from $L(G)$ we would then have a 2-factor consisting of $(2k+1)$ disjoint cycles of length $|V(G)|$. Let $A_1, \dots, A_{2k+1}$ denote these $(2k+1)$ cycles in $L(G)$. Note that for each $i = 1, \dots, 2k+1$, there exists a natural correspondence between the cycle A_i in $L(G)$ and the Hamilton cycle H_i of G. With the vertices of $L(G)$ inheriting colours from the edges of G, it follows that the vertices of A_i are alternately coloured with the colours $(2i-1)$ and $2i$.

To achieve a Hamilton decomposition of $L(G)$ we now show that the subgraph of $L(G)$ that is formed from the union of B_1 and $A_1, \dots, A_{2k+1}$ is itself Hamilton decomposable. The structure formed by $B_1 \cup A_1$ is particularly important at this point, and is illustrated in Figure 2, where the outer cycle is B_1 and the inner cycle is A_1.

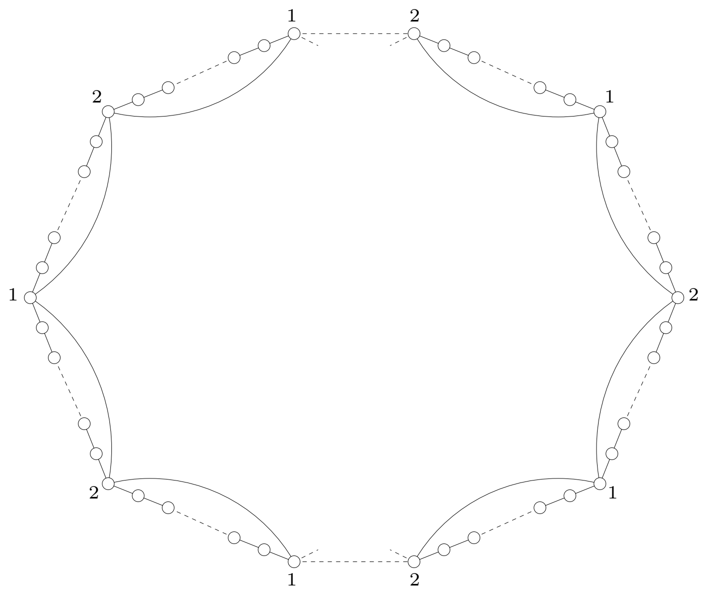

Figure 2. $B_1 \cup A_1$

Note that between each consecutive pair of vertices having colours 1 and 2 is a sequence of vertices whose colours match the labels of the vertices of either $\mathcal{P}_1$ or $\mathcal{P}'_1$. Also, each segment of $B_1 \cup A_1$ (i.e., each set of vertices that is between a consecutive pair of vertices having colours 1 and 2) is a subgraph of a clique of $L(G)$ that was generated by the δ edges incident with a common vertex, say x, of G. We will call this segment (resp. clique) the x *segment* (resp. x *clique*) of $B_1 \cup A_1$ (resp. $L(G)$).

Observe now that the edge sequences $P_2, \ldots, P_{2k+1}$ in G correspond to a set of $2k$ paths in $L(G)$, say $L(P_2), \ldots, L(P_{2k+1})$. Moreover, since each sequence P_i in G begins at an edge incident with u and ends at an edge incident with v, the corresponding path $L(P_i)$ in $L(G)$ will begin in the u segment and finish in the v segment. The internal edges of the sequence P_i pass through the component $\mathcal{C}$ of $((G - H_1) - \{u, v\})$, and so it follows that the set of segments of $B_1 \cup A_1$ through which the path $L(P_2)$ travels is the same set of segments as for each of the paths $L(P_3), \ldots, L(P_{2k+1})$.

Similarly, the paths $L(P'_2), \ldots, L(P'_{2k+1})$ in $L(G)$ start and end in the u and v segments, and go through a common set of segments of $B_1 \cup A_1$ that is the complement of those used by the internal vertices of $L(P_2), \ldots, L(P_{2k+1})$.

We now construct a Hamilton cycle C_1 in $L(G)$, using only edges of $B_1 \cup A_1 \cup \cdots \cup A_{2k}$. Include in C_1 the $(k+1)$ edges that form a maximum matching in the B_1 portion of the u segment of $B_1 \cup A_1$. Also include in C_1 the maximum matching in the v segment of $B_1 \cup A_1$ that contains the edge from A_1. Add to C_1 all of the edges in each of $L(P_2), \ldots, L(P_{2k+1})$. Figure 3 now illustrates the portion of C_1 that we have so far constructed. (Note that there are two cases, depending on whether u and v are in the same part of the bipartition of G.)

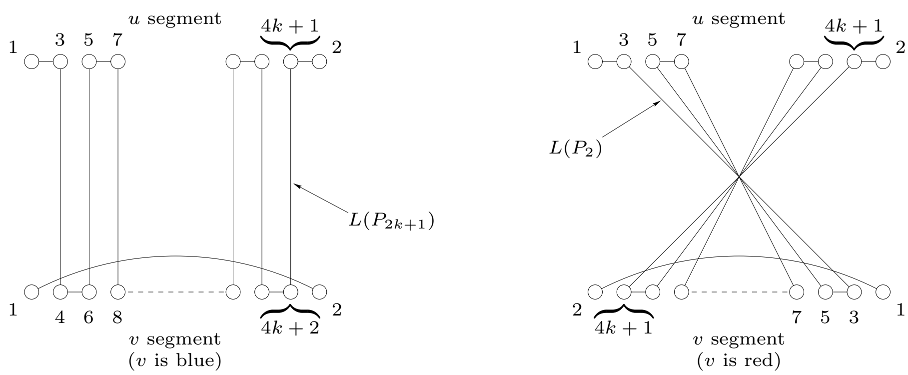

Figure 3. Some of the edges of the Hamilton cycle C_1 in $L(G)$.

Now, in each segment of $B_1 \cup A_1$ that is used by an internal vertex of $L(P_2)$, include in C_1 the edge from A_1. In each segment of $B_1 \cup A_1$ not used by any vertices of $L(P_2)$, include in C_1 all $(2k+1)$ edges from B_1. At this point we find that C_1 is a Hamilton cycle of $L(G)$.

The edges which remain when C_1 is removed from the union of B_1 and $A_1, \ldots, A_{2k+1}$ form a second Hamilton cycle, C_2. C_1 and C_2, together with $B_2, \ldots, B_{4k}$, constitute the $(4k+1)$ Hamilton cycles of a Hamilton decomposition of $L(G)$. ■

It follows from Theorem 1 that if G is a bipartite Hamilton decomposable graph with $\delta \equiv 2 \pmod 4$ and connectivity $\kappa(G) = 2$, then $L(G)$ is Hamilton decomposable. Combined with known results [7, 9], we conclude that every bipartite Hamilton decomposable graph G with $\kappa(G) = 2$ has a Hamilton decomposable line graph.

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