HAMilton Decompositions of Line Graphs of some bipartite Graphs

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Abstract

Some bipartite Hamilton decomposable graphs that are regular of degree \( \delta \equiv 2 \pmod{4} \) are shown to have Hamilton decomposable line graphs. One consequence is that every bipartite Hamilton decomposable graph \( G \) with connectivity \( \kappa(G) = 2 \) has a Hamilton decomposable line graph \( L(G) \).

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1. Introduction

The line graph, denoted by \( L(G) \), of a graph \( G \) having vertex set \( V(G) \) and edge set \( E(G) \) is the graph with vertex set \( E(G) \), where two vertices of \( L(G) \) are adjacent in \( L(G) \) if and only if the corresponding edges in \( G \) are incident with a common vertex in \( G \).

A Hamilton decomposition of a \( \delta \)-regular graph \( G \) consists of a set of Hamilton cycles (plus a 1-factor if \( \delta \) is odd) of \( G \) such that these cycles (and the 1-factor when \( \delta \) is odd) partition the edges of \( G \). If \( G \) has a Hamilton decomposition, it is said to be Hamilton decomposable.

Investigating Hamilton decompositions of line graphs has been largely motivated by the following conjecture of Bermond [1]:

**Conjecture 1.** If \( G \) is Hamilton decomposable, then \( L(G) \) is Hamilton decomposable.

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Bermond’s conjecture has been shown to hold when $G$ is a Hamilton decomposable graph satisfying any of the following criteria [3, 4, 5, 6, 7, 8, 9, 10, 11]:

1. $\delta \leq 5$,
2. $\delta \equiv 0 \pmod{4}$,
3. $\delta$ is odd and $G$ is bipartite,
4. $\delta$ is even and $G$ has a perfect 1-factorisation,
5. $G = K_n$, or
6. $G = K_{n,n}$.

In the case where $G$ is a Hamilton decomposable graph that is regular of degree $\delta \equiv 2 \pmod{4}$, it is known that $L(G)$ can be decomposed into $\delta - 2$ Hamilton cycles plus a 2-factor [7, 12]. However, aside from when $G$ is either complete or else has a perfect 1-factorisation, it is not generally known whether $L(G)$ is Hamilton decomposable.

As our main result, we show that $L(G)$ is Hamilton decomposable for every bipartite graph $G$ with $\delta \equiv 2 \pmod{4}$ that has a Hamilton decomposition $H$ such that $\kappa(G - H_1) = 2$ for some Hamilton cycle $H_1$ of $H$. An immediate consequence of this result is that every bipartite Hamilton decomposable graph $G$ with connectivity $\kappa(G) = 2$ has a Hamilton decomposable line graph.

2. Preliminary Results

Let $\mathcal{H}_1$ denote the Hamilton cycle $(1, 3, 5, \ldots, 4k + 1, 2, 4k + 2, 4k, \ldots, 6, 4, 1)$ of $K_{4k+2}$, where $V(K_{4k+2}) = \{1, 2, \ldots, 4k+2\}$, and let $\mathcal{F}$ denote the 1-factor of $K_{4k+2}$ having the edges $\{2i - 1, 2i\}$ for $i = 1, \ldots, 2k + 1$. $\mathcal{H}_1$ is illustrated in Figure 1. A double-centred Walecki construction described by Zhan [12] produces the following Hamilton decomposition of $K_{4k+2}$:

**Lemma 1.** $K_{4k+2}$ has a Hamilton decomposition in which $\mathcal{H}_1$ is one of the Hamilton cycles and $\mathcal{F}$ is the 1-factor.

**Proof.** For each $i = 2, 3, \ldots, 2k$, let $\mathcal{H}_i = \sigma^{i-1}(\mathcal{H}_1)$ where $\sigma$ is the permutation $(1)(2)(3, 5, 9, \ldots, 4k+1, 4k, \ldots, 12, 8, 4, 6, 10, \ldots, 4k+2, 4k-1, \ldots, 11, 7)$. Then the Hamilton cycles $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{2k}$ and the 1-factor $\mathcal{F}$ form a Hamilton decomposition of $K_{4k+2}$. ■
In [9] a stroll was defined as an alternating sequence of vertices and edges, \(v_0 e_0 v_1 e_1 \cdots v_{n-1} e_{n-1} v_n\), such that \(v_i\) and \(v_{i+1}\) (which are not necessarily distinct) are each end-vertices of the edge \(e_i\), for \(0 \leq i \leq n - 1\). For closed strolls (in which \(v_0 = v_n\)), it suffices to state only the sequence of edges. An Euler stroll is any closed stroll in a graph \(G\) that uses each edge of \(G\) exactly once. Two Euler strolls are said to be compatible if no pair of adjacent edges (i.e., no 2-path) is contained in both.

Note that an Euler stroll in a graph \(G\) naturally corresponds to a Hamilton cycle in \(L(G)\), and that any set of pairwise compatible Euler strolls in \(G\) corresponds to a set of edge-disjoint Hamilton cycles in \(L(G)\).

3. Main Results

**Theorem 1.** Let \(G\) be a bipartite graph that is regular of degree \(\delta = 4k + 2\). If there exists a Hamilton decomposition \(H\) of \(G\) such that, for some Hamilton cycle \(H_1\) of \(H\), \(\kappa(G - H_1) = 2\), then \(L(G)\) is Hamilton decomposable.

**Proof.** Let \(H\) be a Hamilton decomposition of \(G\), consisting of the Hamilton cycles \(H_1, \ldots, H_{2k+1}\), such that \(((G - H_1) - \{u, v\})\) is disconnected for some pair of vertices \(u\) and \(v\). Fix a bipartite colouring of the vertices of \(G\) using the colours red and blue, such that vertex \(u\) is coloured red.
Alternately colour the edges of $H_1$ with the colours 1 and 2. Since $\{u, v\}$ is a vertex cut in $(G - H_1)$, then there are two components, say $C$ and $C'$, in $((G - H_1) - \{u, v\})$.

For each $i = 2, \ldots, 2k + 1$, alternately colour the edges of $H_i$ with the colours $(2i - 1)$ and $2i$ so that the edge from vertex $u$ into $C$ has colour $(2i - 1)$. Let $P_i$ be the sequence of edges along the cycle $H_i$ of $G$, beginning with the edge from $u$ to $C$, ending with the edge from $v$ to $C'$, and including all edges coloured either $(2i - 1)$ or $2i$ that are contained in $C$. Similarly, let $P'_i$ be the sequence of edges along the cycle $H_i$, beginning with the edge from $C$ to $u$, ending with the edge from $C'$ to $v$, and including all edges coloured either $(2i - 1)$ or $2i$ that are contained in $C'$. Together, $P_i$ and $P'_i$ contain all of the edges of $H_i$ (with 2 edges of $H_i$ being contained in $P_i \cap P'_i$).

Let $\mathcal{H}_1, \ldots, \mathcal{H}_{2k}$ be the Hamilton cycles of the Hamilton decomposition of $K_{4k+2}$ described in Lemma 1. These cycles will be used to generate $4k$ mutually compatible Euler strolls in $G$, and hence $4k$ edge-disjoint Hamilton cycles in $L(G)$.

For each $i = 1, \ldots, 2k$, we wish to use $H_1$ and $\mathcal{H}_i$ to generate 2 strolls in $G$. This can be done by noting that $\mathcal{H}_i$ can be broken into two equal-length paths, each going from vertex 1 to vertex 2 of $K_{4k+2}$. For $\mathcal{H}_i$, let $P_1$ (resp. $P'_1$) denote the path with internal vertices having odd (resp. even) labels, so that $P_1 = (1, 3, 5, 7, 9, 11, \ldots, 4k+1, 2)$ and $P'_1 = (1, 4, 6, 8, 10, \ldots, 4k+2, 2)$. For $i = 2, 3, \ldots, 2k$, let $P_i = \sigma^{i-1}(P_1)$ and $P'_i = \sigma^{i-1}(P'_1)$, where $\sigma$ is the permutation presented in Lemma 1.

For the first stroll generated by $\mathcal{H}_i$, where $i \in \{1, 2, \ldots, 2k\}$, we use path $P_i$ (resp. $P'_i$) at each vertex of $H_1$ that is coloured red (resp. blue) and for the second stroll we use path $P_i$ (resp. $P'_i$) at each vertex that is coloured blue (resp. red). We use the paths $P_i$ and $P'_i$ to describe how to replace each edge sequence $(e, e')$ of $H_1$ with an edge sequence $(e, e_1, e_2, \ldots, e_{2k}, e')$ where each of the edges $e_1, \ldots, e_{2k}$ is incident with the vertex of $G$ that is common to $e$ and $e'$. Specifically, we wish the edge colours of the edges in the sequence $(e, e_1, e_2, \ldots, e_{2k}, e')$ to be the same as the vertex labels along the path $P_i$ or $P'_i$ as appropriate. So, for example, for the first stroll generated by $\mathcal{H}_1$, we would replace each 2-path in $H_1$ from an edge of colour 2 to an edge of colour 1 and having a blue internal vertex with a stroll consisting of edges incident with this blue vertex and having edge colours $(2, 4k + 2, \ldots, 10, 8, 6, 4, 1)$, whereas for the second stroll generated by $\mathcal{H}_1$ each such 2-path of $H_1$ would be replaced by a stroll whose edges are coloured $(2, 4k + 1, \ldots, 11, 9, 7, 5, 3, 1)$.  

The $4k$ Euler strolls which are generated in this manner will be mutually compatible, and hence correspond to $4k$ edge-disjoint Hamilton cycles in $L(G)$. Let $B_{2i-1}$ and $B_{2i}$ denote the two Hamilton cycles in $L(G)$ that are generated from $H_i$, for each $i = 1, \ldots, 2k$.

If we were to remove the Hamilton cycles $B_1, \ldots, B_{4k}$ from $L(G)$ we would then have a 2-factor consisting of $(2k + 1)$ disjoint cycles of length $|V(G)|$. Let $A_1, \ldots, A_{2k+1}$ denote these $(2k + 1)$ cycles in $L(G)$. Note that for each $i = 1, \ldots, 2k + 1$, there exists a natural correspondence between the cycle $A_i$ in $L(G)$ and the Hamilton cycle $H_i$ of $G$. With the vertices of $L(G)$ inheriting colours from the edges of $G$, it follows that the vertices of $A_i$ are alternately coloured with the colours $(2i - 1)$ and $2i$.

To achieve a Hamilton decomposition of $L(G)$ we now show that the subgraph of $L(G)$ that is formed from the union of $B_1$ and $A_1, \ldots, A_{2k+1}$ is itself Hamilton decomposable. The structure formed by $B_1 \cup A_1$ is particularly important at this point, and is illustrated in Figure 2, where the outer cycle is $B_1$ and the inner cycle is $A_1$.

![Figure 2. $B_1 \cup A_1$](image)

Note that between each consecutive pair of vertices having colours 1 and 2 is a sequence of vertices whose colours match the labels of the vertices of either $P_1$ or $P'_1$. Also, each segment of $B_1 \cup A_1$ (i.e., each set of vertices
that is between a consecutive pair of vertices having colours 1 and 2) is a
subgraph of a clique of $L(G)$ that was generated by the $\delta$ edges incident
with a common vertex, say $x$, of $G$. We will call this segment (resp. clique)
the $x$ segment (resp. $x$ clique) of $B_1 \cup A_1$ (resp. $L(G)$).

Observe now that the edge sequences $P_2, \ldots, P_{2k+1}$ in $G$ correspond to
a set of $2k$ paths in $L(G)$, say $L(P_2), \ldots, L(P_{2k+1})$. Moreover, since each
sequence $P_i$ in $G$ begins at an edge incident with $u$ and ends at an edge
incident with $v$, the corresponding path $L(P_i)$ in $L(G)$ will begin in the $u$
segment and finish in the $v$ segment. The internal edges of the sequence $P_i$
pass through the component $C$ of $(G - H_1) - \{u, v\}$, and so it follows that
the set of segments of $B_1 \cup A_1$ through which the path $L(P_2)$ travels is the
same set of segments as for each of the paths $L(P_3), \ldots, L(P_{2k+1})$.

Similarly, the paths $L(P'_2), \ldots, L(P'_{2k+1})$ in $L(G)$ start and end in the $u$
and $v$ segments, and go through a common set of segments of $B_1 \cup A_1$ that is
the complement of those used by the internal vertices of $L(P_2), \ldots, L(P_{2k+1})$.

We now construct a Hamilton cycle $C_1$ in $L(G)$, using only edges of
$B_1 \cup A_1 \cup \cdots \cup A_{2k}$. Include in $C_1$ the $(k + 1)$ edges that form a maximum
matching in the $B_1$ portion of the $u$ segment of $B_1 \cup A_1$. Also include in
$C_1$ the maximum matching in the $v$ segment of $B_1 \cup A_1$ that contains the edge from $A_1$. Add to $C_1$ all of the edges in each of $L(P_2), \ldots, L(P_{2k+1})$.

Figure 3 now illustrates the portion of $C_1$ that we have so far constructed.
(Note that there are two cases, depending on whether $u$ and $v$ are in the
same part of the bipartition of $G$.)

Figure 3. Some of the edges of the Hamilton cycle $C_1$ in $L(G)$. 
Now, in each segment of $B_1 \cup A_1$ that is used by an internal vertex of $L(P_2)$, include in $C_1$ the edge from $A_1$. In each segment of $B_1 \cup A_1$ not used by any vertices of $L(P_2)$, include in $C_1$ all $(2k + 1)$ edges from $B_1$. At this point we find that $C_1$ is a Hamilton cycle of $L(G)$.

The edges which remain when $C_1$ is removed from the union of $B_1$ and $A_1, \ldots, A_{2k+1}$ form a second Hamilton cycle, $C_2$. $C_1$ and $C_2$, together with $B_2, \ldots, B_{2k}$, constitute the $(4k + 1)$ Hamilton cycles of a Hamilton decomposition of $L(G)$. $\blacksquare$

It follows from Theorem 1 that if $G$ is a bipartite Hamilton decomposable graph with $\delta \equiv 2 \pmod{4}$ and connectivity $\kappa(G) = 2$, then $L(G)$ is Hamilton decomposable. Combined with known results [7, 9], we conclude that every bipartite Hamilton decomposable graph $G$ with $\kappa(G) = 2$ has a Hamilton decomposable line graph.

References


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