NOTE ON THE SPLIT DOMINATION NUMBER
OF THE CARTESIAN PRODUCT OF PATHS

MACIEJ ZWIERZCHOWSKI

Institute of Mathematics
Technical University of Szczecin
Al. Piastów 48/49, 70–310 Szczecin, Poland

e-mail: mzwierz@ps.pl

Abstract

In this note the split domination number of the Cartesian product of two paths is considered. Our results are related to [2] where the domination number of $P_m \Box P_n$ was studied. The split domination number of $P_2 \Box P_n$ is calculated, and we give good estimates for the split domination number of $P_m \Box P_n$ expressed in terms of its domination number.

Keywords: domination number, split domination number, Cartesian product of graphs.

2000 Mathematics Subject Classification: 05C69.

1. Introduction

In this paper we consider finite undirected simple graphs. For any graph $G$ we denote $V(G)$ and $E(G)$, the vertex set of $G$ and the edge set of $G$, respectively. If $n$ is the cardinality of $V(G)$, then we say that $G$ is of order $n$. By $\langle X \rangle_G$ we mean a subgraph of a graph $G$ induced by a subset $X \subseteq V(G)$. A subset $D \subseteq V(G)$ is a dominating set of $G$, if for every $x \in V(G) - D$, there is a vertex $y \in D$ such that $xy \in E(G)$. We also say that $x$ is dominated by $D$ in $G$ or by $y$ in $G$. A dominating set $D$ of $G$ is a split dominating set of $G$, if the induced subgraph $\langle V(G) - D \rangle_G$ of $G$ is disconnected. The domination number, $[\text{the split domination number}]$ of a graph $G$, denoted $\gamma(G)$, $[\gamma_s(G)]$ is the cardinality of the smallest dominating [the smallest split dominating] set
of $G$. A dominating set $D$ is called a $\gamma(G)$-set if $D$ realizes the domination number. Similarly we define a $\gamma_s(G)$-set. From the definition of a split dominating set it follows immediately that $\gamma(G) \leq \gamma_s(G)$. Additionally note that for a connected graph $G$ a $\gamma_s(G)$-set exists if and only if $G$ is different from a complete graph. More information about a split dominating set and the split domination number can be found in [3]. The Cartesian product of two graphs $G$ and $H$, is a graph $G \square H$ with $V(G \square H) = V(G) \times V(H)$ and $(g_1, h_1)(g_2, h_2) \in E(G \square H)$ if and only if $(g_1 = g_2$ and $h_1h_2 \in E(H))$ or $(g_1g_2 \in E(G)$ and $h_1 = h_2)$.

Any other terms not defined in this paper can be found in [1].

2. Main Results

Theorem 1. For any $n, m \geq 2$

$$\gamma(P_m \square P_n) \leq \gamma_s(P_m \square P_n) \leq \gamma(P_m \square P_n) + 1.$$  

Proof. Let $m, n \geq 2$ and let $D$ be the minimum dominating set of $P_m \square P_n$. According to the definition of a split dominating set we have $\gamma(P_m \square P_n) \leq \gamma_s(P_m \square P_n)$. Thus to prove this theorem we will show that $\gamma_s(P_m \square P_n) \leq \gamma(P_m \square P_n) + 1$. Consider the graph $P_m \square P_n$, as $m$ canonical copies of $P_n$ with vertices labelled $x_{i,j}$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, and with edges $x_{i,j}x_{i+1,j}$ and $x_{i,j}x_{i,j+1}$.

If $x_{1,1} \in D$, then the subset $D' = D - \{x_{1,1}\} \cup \{x_{1,2}, x_{2,1}\}$ is also a dominating set of $P_m \square P_n$. Moreover, since $N_{P_m \square P_n}(x_{1,1}) = \{x_{1,2}, x_{2,1}\} \subset D'$, then $x_{1,1}$ is an isolated vertex of the induced subgraph $\langle V(P_m \square P_n) - D' \rangle_{P_m \square P_n}$ of a graph $P_m \square P_n$. It means that $D'$ is a split dominating set of $P_m \square P_n$, with $|D'| \leq \gamma(P_m \square P_n) + 1$.

If $x_{1,1} \notin D$, then it must be that $x_{1,2} \in D$ or $x_{2,1} \in D$ (otherwise $x_{1,1}$ would not be dominated by $D$ in $P_m \square P_n$). Assume that $x_{1,2} \in D$, then $D' = D \cup \{x_{1,2}\}$ is a split dominating set of $P_m \square P_n$ and $|D'| \leq |D| + 1 = \gamma(P_m \square P_n) + 1$, as desired.

Thus $\gamma_s(P_m \square P_n) \leq \gamma(P_m \square P_n) + 1$, for any $m, n \geq 2$ and the proof is complete. $
$

In [2] it was obtained that $\lim_{n,m \to \infty} \frac{\gamma(P_m \square P_n)}{mn} = \frac{1}{5}$. As a consequence from the above fact and from Theorem 1 we obtain the following
Corollary 2.

\[
\lim_{n,m \to \infty} \frac{\gamma_s(P_m \Box P_n)}{mn} = \frac{1}{5}
\]

The following result was proved in [2].

**Theorem 3** [2]. For \( n \geq 2 \),

\[
\gamma(P_2 \Box P_n) = \left\lceil \frac{n+1}{2} \right\rceil.
\]

Inspired by this result we shall calculate the split domination number of \( P_2 \Box P_n \), for \( n \geq 2 \). Before proceeding we give a few necessary results.

Let \( V(P_2) = \{v_1, v_2\} \) and \( V(P_n) = \{u_1, u_2, \ldots, u_n\} \). For convenience, in the rest of the paper we will write \( x_i \) instead of \( (v_1, u_i) \in V(P_2 \Box P_n) \) and \( y_i \) instead of \( (v_2, u_i) \in V(P_2 \Box P_n) \), for \( i = 1, 2, \ldots, n \). Hence \( V(P_2 \Box P_n) = \{x_i, y_i : i = 1, 2, \ldots, n\} \) and \( E(P_2 \Box P_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, x_i y_n : i = 1, 2, \ldots, n\} \).

**Lemma 4.** If \( n \equiv 2(\text{mod} \ 4), n \geq 2 \), then

\[
D = \{x_i : i \equiv 1(\text{mod} \ 4)\} \cup \{y_j : j \equiv 3(\text{mod} \ 4)\} \cup \{y_n\}
\]

is the \( \gamma_s(P_2 \Box P_n) \)-set with \( |D| = \left\lceil \frac{n+1}{2} \right\rceil \).

**Proof.** Let \( D = \{x_i : i \equiv 1(\text{mod} \ 4)\} \cup \{y_j : j \equiv 3(\text{mod} \ 4)\} \cup \{y_n\} \) be a subset of \( V(P_2 \Box P_n) \).

We show that any vertex of \( P_2 \Box P_n \) is either in \( D \) or it is adjacent to some vertex from \( D \). Let \( r \) be an integer not greater than \( n \).

If \( r = 4q, q \geq 1 \), then the vertex \( x_r \) is adjacent to \( x_{r+1} = x_{4q+1} \in D \) and \( y_r \) is adjacent to \( y_{r-1} = y_{4q-1} \in D \).

If \( r = 4q + 1, q \geq 0 \), then \( x_r \in D \) and \( y_r \) is adjacent to \( x_r \).

If \( r = 4q + 2, q \geq 0 \), then \( x_r \) is adjacent to \( x_{r-1} \in D \). If \( r = n \), then \( y_r = y_n \in D \) and if \( r < n \), then \( y_r \) is adjacent to \( y_{r+1} \in D \).

Finally, if \( r = 4q + 3, q \geq 0 \), then \( y_r \in D \) and \( x_r \) is adjacent to \( y_r \).

All this together gives that \( D \) is a dominating set of \( P_2 \Box P_n \).

Let \( n = 4s + 2, s \geq 0 \). We state that \( |D| = \left\lceil \frac{n+1}{2} \right\rceil \). Indeed, partition \( V(P_2 \Box P_n) \) into subsets \( B_i = \{x_{4i-3}, y_{4i-3}, \ldots, x_{4i}, y_{4i}\} \), for \( i = 1, 2, \ldots, s \).
and \( B_{s+1} = \{x_{n-1}, y_{n-1}, x_n, y_n\} \). Note that \(|D \cap B_i| = 2\), for \( i = 1, 2, \ldots, s + 1\). Thus \(|D| = 2s + 2 = \left\lceil \frac{n+1}{2} \right\rceil = \gamma(P_2 \Box P_n)\), by Theorem 3. Since \( N_{P_2 \Box P_n}(x_n) = \{x_{n-1}, y_n\} \subset D\), hence \( x_n \) is an isolated vertex of \((V(P_2 \Box P_n) - D)P_2 \Box P_n\). Thus this induced subgraph is disconnected. All this together gives that \( D \) is a \( \gamma_s(P_2 \Box P_n)\)-set, since \( D \) is a split dominating set of \( P_2 \Box P_n \) with the minimum cardinality. Hence the result is true. ■

Lemma 5. If \( n \equiv 0(\text{mod} \ 4)\), \( n \geq 2\), then

\[ D = \{x_i : i \equiv 1(\text{mod} \ 4)\} \cup \{y_j : j \equiv 3(\text{mod} \ 4)\} \cup \{x_n\} \]

is the \( \gamma_s(P_2 \Box P_n)\)-set with \(|D| = \left\lceil \frac{n+1}{2} \right\rceil\).

Proof. Let \( D \) be as in the statement of the theorem. Arguing similarly as in the proof of above lemma, it follows that \( D \) is a dominating set of \( P_2 \Box P_n\).

Now, we show that \(|D| = \left\lceil \frac{n+1}{2} \right\rceil\). Put \( n = 4s \) and partition \( V(P_2 \Box P_n) \) into the subsets \( B_i = \{x_{4i-3}, y_{4i-3}, \ldots, x_{4i}, y_{4i}\} \), for \( i = 1, 2, \ldots, s \). It is easy to observe that \(|D \cap B_i| = 2\), for \( i = 1, 2, \ldots, s - 1 \) and \(|D \cap B_s| = 3\). Hence \(|D| = 2(s-1) + 3 = 2s+1 = \left\lceil \frac{n+1}{2} \right\rceil = \gamma(P_2 \Box P_n)\), as desired. Finally, observe that \( y_1 \) is an isolated vertex of \((V(P_2 \Box P_n) - D)P_2 \Box P_n\). This means that the last subgraph is disconnected and as a consequence \( D \) is a split dominating set of \( P_2 \Box P_n\). Since \( D \) is also a \( \gamma(P_2 \Box P_n)\)-set, it is a \( \gamma_s(P_2 \Box P_n)\)-set, as required. ■

Lemma 6. Let \( n \geq 5 \) be odd and let \( D \) be a \( \gamma(P_2 \Box P_n)\)-set. Then exactly one of \( x_1 \) and \( y_1 \) belong to \( D \).

Proof. Let \( n = 2k + 1 \) with \( k \geq 2 \) and let \( D \) be a \( \gamma(P_2 \Box P_n)\)-set. Assume that \( x_1, y_1 \notin D \), then it must be that \( x_2, y_2 \in D \) (otherwise \( x_1 \) or \( y_1 \) would not be dominated by \( D \)). Since \( n \geq 5 \) is odd, then \( \{x_3, y_3\} \subset V(P_2 \Box P_n)\).

Moreover \( x_3, y_3 \notin D \). Indeed, without loss of generality, suppose that \( x_3 \in D \). Then \( D \cup \{y_1\} - \{x_2, y_2\} \) is a dominating set of \( P_2 \Box P_n\), having the cardinality \(|D| - 1\). This contradicts the fact that \( D \) is the minimum dominating set of \( P_2 \Box P_n\).

So, we have \( x_1, y_1, x_3, y_3 \notin D \) and \( x_2, y_2 \in D \). Consider two induced subgraphs of \( P_2 \Box P_n\):

\[ X_1 = \langle \{x_1, y_1, x_2, y_2, x_3, y_3\} \rangle P_2 \Box P_n \quad \text{and} \]
\[ X_2 = \langle \{x_4, y_4, \ldots, x_n, y_n\} \rangle P_2 \Box P_n \].
Since \( X_2 \cong P_2 \square P_{n-3} \), then by Theorem 3 we have \( \gamma(X_2) = \lceil \frac{n-2}{2} \rceil = \lceil \frac{2k-1}{2} \rceil = k \). Further \(|D| = \gamma(X_1) + \gamma(X_2) = 2 + k = \lceil \frac{n+3}{2} \rceil > \lceil \frac{n+1}{2} \rceil = \gamma(P_2 \square P_n) \), a contradiction, since \( D \) is a \( \gamma(P_2 \square P_n) \)-set.

Now, assume that \( x_1 \) and \( y_1 \) \( \in D \), then \( x_2, y_2, x_3, y_3 \notin D \) (otherwise there would exist a dominating set of \( P_2 \square P_n \) with order strictly less than the cardinality of \( D \)). Arguing as above, for \( X_1 = \langle \{x_1, y_1, x_2, y_2\}\rangle P_2 \square P_n \) and \( X_2 = \langle \{x_3, y_3, \ldots, x_n, y_n\}\rangle P_2 \square P_n \), we also come to a contradiction. Hence the proof is complete.

In [2] the following was proved

**Lemma 7** [2]. If \( n \geq 5 \) and \( n \) is odd, then

\[
D = \{x_i : i \equiv 1(\text{mod } 4)\} \cup \{y_j : j \equiv 3(\text{mod } 4)\}
\]

is the \( \gamma(P_2 \square P_n) \)-set with \(|D| = \lceil \frac{n+1}{2} \rceil \).

**Lemma 8.** Let \( n \geq 5 \) be odd and let \( D \) be a \( \gamma(P_2 \square P_n) \)-set. Then

\[
|D \cap \{x_i, y_i, x_{i+1}, y_{i+1}\}| = 1,
\]

for \( i = 1, 2, \ldots, n - 1 \).

**Proof.** We prove this lemma by induction. First consider the base case, when \( n = 5 \). By Lemma 6, either \( x_1 \in D \) or \( y_1 \in D \) and \( x_5 \in D \) or \( y_5 \in D \). Since \( \gamma(P_2 \square P_5) = 3 \), then

\[
|D \cap \{x_2, y_2, x_3, y_3, x_4, y_4\}| = 1.
\]

If \( x_3, y_3 \notin D \), then \( x_3 \) or \( y_3 \) is not dominated by \( D \) in \( P_2 \square P_5 \). So it must be that either \( x_3 \in D \) or \( y_3 \in D \). Thus the result holds for \( n = 5 \).

Assume that the result holds for \( n = 2k + 1 \) and consider \( n = 2k + 3 \). By Lemma 6, either \( x_1 \in D \) or \( y_1 \in D \). If \( x_2, y_2 \notin D \), then by the assumption

\[
|D \cap \{x_i, y_i, x_{i+1}, y_{i+1}\}| = 1,
\]

for \( i = 3, 4, \ldots, n - 1 \). Moreover,

\[
|D \cap \{x_1, y_1, x_2, y_2\}| = 1 \quad \text{and} \quad |D \cap \{x_2, y_2, x_3, y_3\}| = 1.
\]

Thus the result holds for \( n = 2k + 3 \).
If $x_2 \in D$ or $y_2 \in D$, then $D_1 = D \cap \{x_i, y_i : i = 4, \ldots, n\}$ is a $\gamma(P_2 \square P_2k)$-set and $|D_1| = \left\lceil \frac{2k+1}{2} \right\rceil = k + 1$, by Theorem 3. Thus

$$|D| \geq |D_1| + 2 = k + 3 > \left\lceil \frac{2k + 3}{2} \right\rceil = \gamma(P_2 \square P_{2k+3})$$

but this is impossible, since $D$ is a $\gamma(P_2 \square P_{2k+3})$-set.

Hence the result is true for all odd $n \geq 5$.

**Theorem 9.** For $n \geq 2$,

$$\gamma_s(P_2 \square P_n) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil, & \text{if } n \text{ is even or } n = 3, \\ \left\lceil \frac{n+1}{2} \right\rceil + 1, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

**Proof.** Let $n \geq 2$ be even. According to Lemma 4 and Lemma 5 the result is true.

If $n = 3$, then the set $\{x_2, y_2\}$ is the minimum split dominating set of $P_2 \square P_3$, with the required cardinality.

Next, suppose that $n \geq 5$ is odd. Then $n = 2k + 1$, ($k \geq 2$). According to Lemma 8 we have that the set $D$ of Lemma 7 is unique (modulo the automorphism that exchanges paths $P_n$). Moreover, observe that $D$ is not a split dominating set of $P_2 \square P_n$. Thus $\gamma(P_2 \square P_n) < \gamma_s(P_2 \square P_n)$ and by Theorem 1 we obtain that $\gamma_s(P_2 \square P_n) = \gamma(P_2 \square P_n) + 1$.

**Acknowledgement**

The author would like to thank the referees for many helpful corrections.

**References**

