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ON (k, l)-KERNEL PERFECTNESS OF SPECIAL CLASSES OF DIGRAPHS

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Abstract

In the first part of this paper we give necessary and sufficient conditions for some special classes of digraphs to have a (k, l)-kernel. One of them is the duplication of a set of vertices in a digraph. This duplication come into being as the generalization of the duplication of a vertex in a graph (see [4]). Another one is the D-join of a digraph D and a sequence α of nonempty pairwise disjoint digraphs. In the second part we prove theorems, which give necessary and sufficient conditions for special digraphs presented in the first part to be (k, l)-kernel-perfect digraphs. The concept of a (k, l)-kernel-perfect digraph is the generalization of the well-know idea of a kernel perfect digraph, which was considered in [1] and [6].

Keywords: kernel, (k, l)-kernel, kernel-perfect digraph.

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1. Introduction

Let D denote a finite, directed graph (for short: a *digraph*) without loops and multiple arcs, where $V(D)$ is the set of vertices of D and $A(D)$ is the set of arcs of D. By $D[S]$ we denote the subdigraph of D induced by a nonempty subset $S \subseteq V(D)$. A vertex $x \in V(D)$ is a *source* of a digraph D, if for every $y \in V(D)$ there is no arc $\overrightarrow{yx}$ in D. By a *path* from a vertex x_1 to a vertex x_n in D we mean a sequence of distinct vertices

$x_1, x_2, \ldots, x_n$ from $V(D)$ and arcs $\overrightarrow{x_i x_{i+1}} \in A(D)$, for $i = 1, 2, \ldots, n-1$ and $n \geq 2$ for the simplicity we denote it by $P[x_1, x_2, \ldots, x_n]$. A *circuit* is a path with $x_1 = x_n$, for $n \geq 3$. By P_m we denote an elementary path on m vertices meant as a digraph with $V(P_m) = \{x_1, x_2, \ldots, x_m\}$. By $d_D(x, y)$ we denote the length of the shortest path from x to y in D. For any $X, Y \subseteq V(D)$ and $x \in V(D) \setminus X$ we put $d_D(x, X) = \min_{y \in X} d_D(x, y)$, $d_D(X, x) = \min_{y \in X} d_D(y, x)$ and $d_D(X, Y) = \min_{x \in X, y \in Y} d_D(x, y)$. Let k, l be fixed integers, $k \geq 2$ and $l \geq 1$. We say that a subset $J \subseteq V(D)$ is a (k, l)-*kernel of* D if

(i) for each $x, y \in J$ and $x \neq y, d_D(x, y) \geq k$ and

(ii) for each $x \in V(D) \setminus J$, $d_D(x, J) \leq l$.

The concept of a (k, l)-kernel was introduced by M. Kwaśnik in [13] and considered in [7, 8, 12] and [14]. If $k = 2$ and $l = 1$, then we obtain the definition of a kernel or in other words a $(2, 1)$-kernel of a digraph. We call a $(k, k-1)$-kernel a k-kernel. If J satisfies the condition (i), then we say that J is k-*stable* in D. Moreover, we assume that the subset including exactly one vertex also is k-stable in D, for $k \geq 2$. We say that J is l-*dominating in* D, when the condition (ii) is fulfilled. More precisely with respect to the vertex x we say: x is l-*dominated by* J *in* D or J l-*dominates* x *in* D.

A digraph whose every induced subdigraph has a (k, l)-kernel is called a (k, l)-*kernel-perfect digraph* (for short a (k, l)-KP digraph). If $l = k-1$, then we obtain k-kernel perfect digraph. In [11] we can find some results about k-kernel perfectness of special digraphs. The last concept is the generalization of a kernel-perfect digraph, which was considered in [1, 2] and [6].

For concepts not defined here, see [5].

2. The Existence of (k, l)-kernels of the D-join

Let D be a digraph with $V(D) = \{x_1, x_2, \ldots, x_n\}$ and $\alpha = (D_i)_{i \in \{1, 2, \ldots, n\}}$ be a sequence of vertex disjoint digraphs. The D-join of the digraph D and the sequence α is a digraph $\sigma(\alpha, D)$ such that $V(\sigma(\alpha, D)) = \bigcup_{i=1}^{n} V(D_i)$ and

$$A(\sigma(\alpha, D)) = \left(\bigcup_{i=1}^{n} A(D_i) \right) \cup \big\{ \overrightarrow{uv} : u \in V(D_s), v \in V(D_t), s \neq t \\ \text{and } \overrightarrow{x_s x_t} \in A(D) \big\}.$$

It may be noted that if all digraphs from the sequence α have the same vertex set, then from the D-join we obtain the generalized lexicographic product of the digraph D and the sequence of the digraphs D_i, i.e., $\sigma(\alpha, D) = D[D_1, D_2, \ldots, D_n]$, For the reminder, the generalized lexicographic product $G[G_1, G_2, \ldots, G_n]$ of the graph G and the sequence of the graphs G_i was introduced in [3] and its definition was applied to digraphs in [14]. Additionally if all digraphs from the sequence α are isomorphic to the same digraph D', then from the D-join we obtain the lexicographic product $D[D']$ of the digraphs D and D'. The D-join $\sigma(\alpha, D)$ is the special case of a digraph, which was considered with reference to kernels by H. Galeana-Sanchez and V. Neumann-Lara in [9].

Theorem 1. *Let D be a digraph without circuits of length less than k. Let $\alpha = (D_i)_{i\in\{1,2,\ldots,n\}}$ be a sequence of vertex disjoint digraphs. A subset $J^* \subseteq V(\sigma(\alpha, D))$ is k-stable in the D-join $\sigma(\alpha, D)$ if and only if there exists a k-stable set $J \subseteq V(D)$ of the digraph D such that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i)$ and J_i is k-stable in D_i for every $i \in \mathcal{I}$.*

Proof. I. Let J^* be k-stable in the D-join $\sigma(\alpha, D)$. Denote

$$J = \{x_i \in V(D) : J^* \cap V(D_i) \neq \varnothing\}.$$

At first we will prove that J is k-stable in D. Assume on the contrary that there exist distinct vertices $x_i, x_j \in J$ such that $d_D(x_i, x_j) < k$. Since $x_i, x_j \in J$, then $J^* \cap V(D_i) \neq \varnothing$ and $J^* \cap V(D_j) \neq \varnothing$. Additionally the definition of the D-join and the assumption that $d_D(x_i, x_j) < k$ implies that $d_{\sigma(\alpha,D)}(u, v) < k$ for every $u \in V(D_i)$ and $v \in V(D_j)$. This means that J^* is not k-stable in the digraph $\sigma(\alpha, D)$, a contradiction with the assumption. So J is k-stable in the digraph D. The definition of the set J implies that we can depict J^* in the following way: $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$ and $J_i \subseteq V(D_i)$. Of course for every $i \in \mathcal{I}$ we have that J_i is k-stable in D_i, since $J_i \subseteq J^*$ and J^* is k-stable in $\sigma(\alpha, D)$.

II. Let $J \subseteq V(D)$ be a k-stable set of the digraph D. Let $\mathcal{I}$ be a set of indexes of vertices belonging to J and let J_i be k-stable in D_i for every $i \in \mathcal{I}$. We prove that $J^* = \bigcup_{i\in\mathcal{I}} J_i$ is k-stable in the D-join $\sigma(\alpha, D)$. Let $u, v \in J^*$, $u \neq v$. Assume on the contrary that $d_{\sigma(\alpha,D)}(u, v) < k$. Consider two cases:

Case 1. $u, v \in J_i$ for some $i \in \mathcal{I}$. Of course $d_{D_i}(u, v) \geq k$, since J_i is k-stable in D_i. So there exists a path P from u to v in $\sigma(\alpha, D)$ of length less than k such that at least one inner vertex of P does not belong to $V(D_i)$. In other words there exists a vertex $z \in V(D_j)$ for $i \neq j$ such that $P = [u, \ldots, z, \ldots, v]$. The existence of a circuit $C = [x_i, \ldots, x_j, \ldots, x_i]$ in the digraph D of length less than k follows from the definition of the digraph $\sigma(\alpha, D)$, a contradiction with the assumption.

Case 2. $u \in J_i$ and $v \in J_j$, where $i \neq j$. Since $d_{\sigma(\alpha,D)}(u, v) < k$, so the definition of the digraph $\sigma(\alpha, D)$ implies the fact that $d_D(x_i, x_j) < k$, a contradiction with the assumption that x_i, x_j belong to a k-stable set J of the D-join.

Taking two above cases into consideration we obtain that for distinct $u, v \in J^*$, $d_{\sigma(\alpha,D)}(u, v) \geq k$, hence J^* is k-stable in $\sigma(\alpha, D)$. ∎

Theorem 2. *Let $J \subseteq V(D)$, $\mathcal{I} = \{i : x_i \in J\}$ and $J_i \subseteq V(D_i)$ for every $i \in \mathcal{I}$. If J is l-dominating in D and J_i is l-dominating in D_i for every $i \in \mathcal{I}$, then $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is l-dominating in the D-join $\sigma(\alpha, D)$.*

Proof. Assume that J is l-dominating in D, $\mathcal{I} = \{i : x_i \in J\}$ and J_i is l-dominating in D_i for every $i \in \mathcal{I}$. Let $J^* = \bigcup_{i \in \mathcal{I}} J_i$ and $u \in V(\sigma(\alpha, D)) \setminus J^*$. We show that u is l-dominated by J^* in $\sigma(\alpha, D)$. Let i be a positive integer such that $u \in V(D_i)$. If $i \in \mathcal{I}$, then u is l-dominated by $J_i \subseteq J^*$ in the D-join. If $i \notin \mathcal{I}$, then we obtain that $d_D(x_i, J) \leq l$, since J is l-dominating in D. This means that there exists a vertex $x_j \in J$ such that $d_D(x_i, x_j) \leq l$. We obtain that $d_{\sigma(\alpha,D)}(u, v) \leq l$ for every $v \in V(D_j)$ in view of the definition of the digraph $\sigma(\alpha, D)$. Hence $d_{\sigma(\alpha,D)}(u, J_j) \leq l$. Since $J_j \subseteq J^*$, then $d_{\sigma(\alpha,D)}(u, J^*) \leq l$. So we proved that each $u \in V(\sigma(\alpha, D)) \setminus J^*$ is l-dominated by J^* in $\sigma(\alpha, D)$, i.e., J^* is l-dominating in $\sigma(\alpha, D)$. ∎

Remark 1. It is not difficult to observe that the sufficient condition from Theorem 2 is not a necessary condition for the set J^* to be l-dominating in $\sigma(\alpha, D)$. For example, let $D = P_{l+1}$, $V(P_{l+1}) = \{x_1, x_2, \ldots, x_{l+1}\}$ and $D_i = P_2$, where $V(D_i) = \{u_1^i, u_2^i\}$ for every $i = 1, \ldots, l+1$. $J^* = \{u_1^1, u_2^{l+1}\}$ is l-dominating in $\sigma(\alpha, D)$, but $J^* \cap V(D_1)$ is not l-dominating in D_1.

From Theorem 1 and Theorem 2 we obtain the following corollary.

Corollary 1. *Let D be a digraph without circuits of length less than k and let $\alpha = (D_i)_{i \in \{1,2,\ldots,n\}}$ be a sequence of vertex disjoint digraphs. If $J \subseteq V(D)$ is a (k, l)-kernel of D, $\mathcal{I} = \{i : x_i \in J\}$ and J_i is a (k, l)-kernel of D_i for every $i \in \mathcal{I}$, then $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a (k, l)-kernel of the D-join $\sigma(\alpha, D)$.*

Theorem 3. *Let $l \leq k-1$. Let D be a digraph without circuits of length less than k and $\alpha = (D_i)_{i\in\{1,2,\ldots,n\}}$ be a sequence of vertex disjoint digraphs. If J^* is a (k,l)-kernel of the D-join $\sigma(\alpha, D)$, then there exists a k-kernel $J \subseteq V(D)$ of the digraph D such that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i)$ and J_i is a k-kernel of D_i for every $i \in \mathcal{I}$.*

Proof. Let J^* be a (k,l)-kernel of $\sigma(\alpha, D)$, where $l \leq k-1$. From Theorem 1 we get that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $J \subseteq V(D)$ is k-stable in D and $J_i \subseteq V(D_i)$ is k-stable in D_i for every i such that $i \in \mathcal{I}$. We will show that J is l-dominating in D. Let $x_p \in V(D) \setminus J$. Hence $p \notin \mathcal{I}$ and $V(D_p) \cap J^* = \varnothing$. This means that if $u \in V(D_p)$, then $u \in V(\sigma(\alpha, D)) \setminus J^*$. Since J^* is a (k,l)-kernel of $\sigma(\alpha, D)$, hence $d_{\sigma(\alpha,D)}(u, J^*) \leq l$. So there exists $v \in J^*$ such that $d_{\sigma(\alpha,D)}(u, v) \leq l$. Hence $v \in V(D_t)$, where $t \in \mathcal{I}$, i.e., $x_t \in J$ and $d_D(x_p, x_t) \leq l$ in view of the definition of the D-join, so x_p is l-dominated by J in D.

Now we will prove that J_i is l-dominating in D_i for every $i \in \mathcal{I}$. Assume on the contrary that there exists an integer i such that J_i is not l-dominating in the digraph D_i. This means that the existence of a vertex $u \in J_i$ such that $d_{D_i}(u, J_i) > l$ is assured. Because of the assumption that J^* is l-dominating in the digraph $\sigma(\alpha, D)$, there must exist a vertex $v \in J^* \setminus V(D_i)$ such that $d_{\sigma(\alpha,D)}(u, v) \leq l$. From the definition of the D-join we obtain the inequality $d_{\sigma(\alpha,D)}(V(D_i), v) \leq l$ and finally $d_{\sigma(\alpha,D)}(J_i, v) \leq l \leq k-1$, a contradiction with the assumption that J^* is a (k,l)-kernel of the D-join $\sigma(\alpha, D)$. This means that J_i is l-dominating in D_i for every $i \in \mathcal{I}$.

So every (k,l)-kernel J^* of the D-join $\sigma(\alpha, D)$, where $l \leq k-1$ can be described in the form $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where J is a (k,l)-kernel of D and J_i is a (k,l)-kernel of D_i for every $i \in \mathcal{I}$. ■

From Corollary 1 and Theorem 3 we obtain the next corollary.

Corollary 2. *Let D be a digraph without circuits of length less than k and let $\alpha = (D_i)_{i\in\{1,2,\ldots,n\}}$ be a sequence of vertex disjoint digraphs. The subset J^* is a k-kernel of the D-join $\sigma(\alpha, D)$ if and only if there exists a k-kernel $J \subseteq V(D)$ of the digraph D such that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i)$ and J_i is a k-kernel of D_i for every $i \in \mathcal{I}$.*

3. The Existence of a (k,l)-kernel of the Duplication

In [11] was given the definition of the duplication of a subset of vertices of a graph as the generalization of the duplication of a vertex of a graph introduced in [4]. This definition can be apply to digraphs in the following way. Let X be a proper subset of the vertex set of a digraph D and let H be a digraph isomorphic to $D[X]$. A vertex belonging to $V(H)$ and corresponding to a vertex $x \in X$ will be denoted by x'. The duplication of the subset X, $X \subset V(D)$ is the digraph D^X such that $V(D^X) = V(D) \cup V(H)$ and $A(D^X) = A(D) \cup A(H) \cup A_0 \cup A_1$, where

$$\begin{aligned} A_0 &= \left\{\overrightarrow{x'y} : x' \in V(H), y \in V(D) \text{ and } \overrightarrow{xy} \in A(D)\right\} \text{ and} \\ A_1 &= \left\{\overrightarrow{yx'} : x' \in V(H), y \in V(D) \text{ and } \overrightarrow{yx} \in A(D)\right\}. \end{aligned}$$

Denote $X' = V(H)$. A vertex $x' \in X'$ (resp. a subset $S' \subseteq X'$) will be called the copy of the vertex $x \in X$ (resp. the copy of the subset $S \subseteq X$). We will call the vertex x as the original of the vertex x' and the subset $S \subseteq X$ the original of the subset S'. We will prove a necessary and sufficient condition for the duplication D^X to have a (k,l)-kernel. To this end some lemmas will be given. The next one follows directly from the definition of D^X.

Lemma 1. *Let D^X be the duplication of a subset X, $X \subset V(D)$. Let $x, y \in X$, $x', y' \in X'$ and $w, z \in V(D) \setminus X$. Then*

$$d_D(x,y) = d_{D^X}(x,y) = d_{D^X}(x',y') = d_{D^X}(x,y') = d_{D^X}(x',y), \tag{1}$$

$$d_D(w,z) = d_{D^X}(w,z), \tag{2}$$

$$d_D(w,x) = d_{D^X}(w,x) = d_{D^X}(w,x'), \tag{3}$$

$$d_D(x,w) = d_{D^X}(x,w) = d_{D^X}(x',w). \tag{4}$$

The next corollary follows from Lemma 1.

Corollary 3. *Let D^X be the duplication of a subset X, where $X \subset V(D)$. If $x, y \in V(D)$, then $d_D(x,y) = d_{D^X}(x,y)$.*

Lemma 2. *Let $X \subset V(D)$. If $J^* \subseteq V(D^X)$ is k-stable in the duplication D^X, then $(J^* \cap V(D)) \cup S$ is a k-stable set of D, where S is the original of the set $J^* \cap X'$.*

Proof. Assume that $J^* \subseteq V(D^X)$ is k-stable in the duplication D^X and S is the original of $J^* \cap X'$, i.e., $J^* \cap X' = S'$. Put $J = J^* \cap V(D)$. Of course J, S' and S are k-stable in D^X, so J and S are k-stable in D. To show that $J \cup S$ is k-stable in the digraph D it is enough to prove that $d_D(J,S) \geq k$ and $d_D(S,J) \geq k$. Let $x \in J \setminus S$ and $y \in S \setminus J$. From Lemma 1 we obtain that $d_D(x,y) = d_{D^X}(x,y')$ and $d_D(y,x) = d_{D^X}(y',x)$, where $y' \in S' \setminus (J \cap X)'$ is the copy of the vertex y. Since J^* is k-stable in the duplication D^X, then $d_{D^X}(x,y') \geq k$ and $d_{D^X}(y',x) \geq k$. Hence $d_D(x,y) \geq k$ and $d_D(y,x) \geq k$, which means that $d_D(J,S) \geq k$ and $d_D(S,J) \geq k$. Thus the theorem is proved. ■

Theorem 4. *Let D be a digraph and $X \subset V(D)$. If J^* is a (k,l)-kernel of the duplication D^X and $J^* \subseteq V(D^X)$, then $(J^* \cap V(D)) \cup S$ is a (k,l)-kernel of the digraph D, where S is the original of $J^* \cap X'$.*

Proof. Assume that $J^* \subseteq V(D^X)$ is a (k,l)-kernel of D^X. Lemma 2 implies that $J^* \cap V(D) \cup S$ is k-stable in D. We show that $(J^* \cap V(D)) \cup S$ is l-dominating in the digraph D. Let $x \in V(D) \setminus (J^* \cup S)$. Since J^* is l-dominating in D^X, hence $d_{D^X}(x, J^*) \leq l$. This means that there exists $y \in J^*$ such that $d_{D^X}(x,y) \leq l$. Consider two cases.

Case 1. Let $x \in X$. If $y \in J^* \cap V(D)$, then $d_D(x,y) = d_{D^X}(x,y) \leq l$ in view of Corollary 3. If $y \in J^* \cap X'$, then from the condition (1) of Lemma 1 we obtain that $d_D(x,z) = d_{D^X}(x,y) \leq l$, where $z \in S$ is the original of the vertex y.

Case 2. Let $x \in V(D) \setminus X$. If $y \in J^* \cap V(D)$, then Corollary 3 implies that $d_D(x,y) = d_{D^X}(x,y) \leq l$. If $y \in J^* \cap X'$, then from the condition (3) of Lemma 1 we obtain $d_D(x,z) = d_{D^X}(x,y) \leq l$, where $z \in S$ is the original of the vertex y.

Finally $d_D(x, (J^* \cap V(D)) \cup S) \leq l$, which means that $(J^* \cap V(D)) \cup S$ is l-dominating in D and completes the proof. ■

Lemma 3. *Let D be a digraph, in which there exists a subset $X \subset V(D)$ such that D has no circuit of length less than k including vertices from X. Let D^X be the duplication of X. If J is k-stable in D and $(J \cap X)'$ is the copy of $J \cap X$ in D^X, then $J \cup (J \cap X)'$ is k-stable in D^X.*

Proof. Assume that D is a digraph, in which there exists a subset $X \subset V(D)$ such that D has no circuit of length less than k including vertices from X. Let J be an arbitrary subset of vertices of the digraph D and let $(J \cap X)'$ be the copy of $J \cap X$ in the duplication D^X. Assume that $J \cup (J \cap X)'$ is not k-stable in D^X. We will show that J is not a k-stable set of D. Consider two cases.

Case 1. If $J \cap X = \varnothing$, then $J \cup (J \cap X)' = J$. From the assumption the set J is not k-stable in D^X, so J is not k-stable in D.

Case 2. If $J \cap X \neq \varnothing$, then there exist two distinct vertices $x, y \in J \cup (J \cap X)'$ such that $d_{D^X}(x, y) < k$. If $x, y \in J$, then the inequality $d_D(x, y) = d_{D^X}(x, y) < k$ follows from Corollary 3. If $x, y \in (J \cap X)'$, then from the condition (1) of Lemma 1 we obtain that $d_D(z, w) = d_{D^X}(x, y) < k$, where $z, w \in J \cap X$ are the copies of vertices x, y, respectively. If $x \in J$ and $y \in (J \cap X)'$ (resp. $y \in J$ and $x \in (J \cap X)'$), then in view of Lemma 1 we obtain that $d_D(x, w) = d_{D^X}(x, y) < k$ (resp. $d_D(z, y) = d_{D^X}(x, y) < k$), where $w \in J \cap X$ is the original of the vertex y (resp. $z \in J \cap X$ is the original of the vertex x). Of course $w \neq x$ (resp. $z \neq y$). Otherwise, there exists a circuit of length less than k including a vertex from X, a contradiction with the assumption.

To recapitulate, we proved that J is not a k-stable in D. ■

Theorem 5. *Let D be a digraph, in which there exists a subset $X \subset V(D)$ such that D has no circuit of length less than k including vertices from X. Let D^X be the duplication of X. If J is a (k, l)-kernel of D and $(J \cap X)'$ is the copy of $J \cap X$ in D^X, then $J \cup (J \cap X)'$ is a (k, l)-kernel of D^X.*

Proof. Assume that J is a (k, l)-kernel of D and $(J \cap X)'$ is the copy of $J \cap X$ in D^X. We will show that $J \cup (J \cap X)'$ is a (k, l)-kernel of D^X. If $J \cap X = \varnothing$, then $(J \cap X)' = \varnothing$. Hence $J \cup (J \cap X)' = J$. Since J is a (k, l)-kernel of the digraph D, then $d_D(x, y) \geq k$ and $d_D(z, J) \leq l$ for every $x, y \in J$ and $z \in V(D) \setminus J$. So from Lemma 1 it follows that $d_{D^X}(x, y) \geq k$, $d_{D^X}(z, J) \leq l$ and $d_{D^X}(z', J) \leq l$, where z' is the copy of a vertex z, if $z \in X \setminus J$. Hence $J \cup (J \cap X)'$ is a (k, l)-kernel of the duplication D^X in the case when $J \cap X = \varnothing$. Thus assume that $J \cap X \neq \varnothing$. From Lemma 3 we get that $J \cup (J \cap X)'$ is a k-stable in D^X. So we need only prove that this set is l-dominating in the digraph D^X. Since $V(D^X) \setminus (J \cup (J \cap X)') = (V(D) \setminus J) \cup (X' \setminus (J \cap X)')$, so let us consider two cases.

Case 1. If $x \in V(D) \setminus J$, then x is l-dominated by J in the digraph D, because J is a (k,l)-kernel of D. Thus x is l-dominated by J in the duplication D^X.

Case 2. If $x \in X' \setminus (J \cap X)'$, then its original $y \in X \setminus J$ is l-dominated by J in D. Therefore there exists a path from the vertex y to some vertex $z \in J$ in D of length not greater than l, i.e., $d_D(y,z) \leq l$. If $z \in J \cap X$, then the condition (1) of Lemma 1 implies that $d_{D^X}(x,z') = d_D(y,z) \leq l$, where $z' \in (J \cap X)'$. This means that $d_{D^X}(x,(J \cap X)') \leq l$. If $z \in J \cap (V(D) \setminus X)$, then from the condition (3) of Lemma 1 we obtain that $d_{D^X}(x,z) \leq l$. So $d_{D^X}(x,J) \leq l$.

Therefore x is l-dominated by $J \cup (J \cap X)'$ in the duplication D^X. Because of the fact that $J \cup (J \cap X)'$ is k-stable in D^X we obtain that $J \cup (J \cap X)'$ is a (k,l)-kernel of the duplication D^X. ∎

The next corollary follows from Theorem 4 and Theorem 5.

Corollary 4. *Let D be a digraph, in which there exists a subset $X \subset V(D)$ such that D has no circuit of length less than k including vertices from X. Then the duplication D^X possesses a (k,l)-kernel if and only if the digraph D has a (k,l)-kernel.*

4. The Existence of a k-kernel of the Digraph $D(a,P_m)$

Let D be an arbitrary digraph and P_m be a path meant as a digraph for $m \geq 2$, where $V(P_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(D) \cap V(P_m) = \varnothing$. If $a = \overrightarrow{pq}$ is an arc of the digraph D, then $D(a,P_m)$ is a digraph such that $V(D(a,P_m)) = V(D) \cup V(P_m)$ and $A(D(a,P_m)) = A(D) \cup A(P_m) \cup \{\overrightarrow{px_1}, \overrightarrow{x_m q}\}$.

The following theorem gives a necessary and sufficient condition for the existence of a k-kernel of $D(a,P_m)$.

Theorem 6. *Let D be a digraph without circuits of length less than k. Let $a = \overrightarrow{pq} \in A(D)$ and $n \geq 1$. J^* is a k-kernel of the digraph $D(a,P_{nk})$ if and only if there exists a k-kernel J of D such that $J^* = J \cup J'$, where $J' = \{x_{1+s}, x_{1+k+s}, \ldots, x_{1+(n-1)k+s}\} \subset V(P_{nk})$ and $s = d_D(q,J)$.*

Proof. I. Let $a = \overrightarrow{pq} \in A(D)$ and let J^* be a k-kernel of the digraph $D(a,P_{nk})$. We will prove that $J^* \cap V(P_{nk}) = J'$ and $J^* \cap V(D)$ is a k-kernel of D. Put $J = J^* \cap V(D)$. Let $s = d_D(q, J)$. It is not difficult to observe that $J^* \cap V(P_{nk}) = \{x_{1+s}, x_{1+k+s}, \ldots, x_{1+(n-1)k+s}\}$, i.e., $J^* \cap V(P_{nk}) = J'$. Otherwise, J^* is not k-stable or $(k-1)$-dominating in $D(a, P_{nk})$.

Of course J and $J^* \cap V(P_{nk})$ are k-stable in $D(a, P_{nk})$, so J is k-stable in D. So it remains to show that J is $(k-1)$-dominating in D. Let $x \in V(D) \setminus J^*$. Since J^* is a k-kernel of $D(a, P_{nk})$, hence $d_{D(a,P_{nk})}(x, J^*) \leq k-1$. It is enough to prove that if x is $(k-1)$-dominated by J' in $D(a, P_{nk})$, then it is $(k-1)$-dominated by $J^* \cap V(D)$ in D. Let x be $(k-1)$-dominated in $D(a, P_{nk})$ by a vertex belonging to J'. Hence $d_{D(a,P_{nk})}(x, x_{1+s}) \leq k-1$. At the same time $d_{D(a,P_{nk})}(x, x_{1+s}) = d_D(x, p) + d_{D(a,P_{nk})}(p, x_{1+s}) = d_D(x, p) + s + 1$. Thus $d_D(x, p) \leq k - s - 2$. On the other hand from the assumption we have that $d_D(q, J) = s$. So we get that

$$\begin{aligned} d_D(x, J) &\leq d_D(x, p) + d_D(p, q) + d_D(q, J) \\ &= d_D(x, p) + 1 + s \leq k - 1, \end{aligned}$$

which means that x is $(k-1)$-dominated by J in D. Finally, J is a k-kernel of D, what completes this part of the proof.

II. Let J be a k-kernel of D and $J' = \{x_{1+s}, x_{1+k+s}, \ldots, x_{1+(n-1)k+s}\} \subset V(P_{nk})$, where $s = d_D(q, J)$. We prove that $J \cup J'$ is a k-kernel of $D(a, P_{nk})$. Since J is a k-kernel of D, then every $x \in V(D) \setminus J$ is $(k-1)$-dominated by J in D, which means that x is $(k-1)$-dominated by $J \cup J'$ in $D(a, P_{nk})$. To show that $J \cup J'$ is $(k-1)$-dominating in $D(a, P_{nk})$, it is enough to prove that vertices from $V(P_{nk})$ not belonging to $J \cup J'$ are $(k-1)$-dominated by $J \cup J'$ in the digraph $D(a, P_{nk})$. Let $x_i \in V(P_{nk}) \setminus J'$. If $1 \leq i \leq 1 + (n-1)k + s$, then $d_{P_{nk}}(x_i, J') \leq k-1$. Hence $d_{D(a,P_{nk})}(x_i, J \cup J') \leq k-1$. If $2 + (n-1)k + s \leq i \leq nk$, then

$$\begin{aligned} d_{D(a,P_{nk})}(x_i, J) &= d_{P_{nk}}(x_i, q) + d_D(q, J) = nk + 1 - i + s \\ &\leq nk + 1 - (2 + (n-1)k + s) + s = k - 1. \end{aligned}$$

So $J \cup J'$ is $(k-1)$-dominating in $D(a, P_{nk})$. Moreover, the definition of the digraph $D(a, P_{nk})$ implies that J and J' are k-stable in $D(a, P_{nk})$. To prove that $J \cup J'$ is k-stable in $D(a, P_{nk})$ it is enough to show that $d_{D(a,P_{nk})}(J', J) \geq k$ and $d_{D(a,P_{nk})}(J, J') \geq k$. Since $d_D(q, J) = s$, then

$$\begin{aligned} d_{D(a,P_{nk})}(x_{1+(n-1)k+s}, J) &= d_{P_{nk}}(x_{1+(n-1)k+s}, q) + d_D(q, J) \\ &= (k - s) + s = k. \end{aligned}$$

Hence $d_{D(a,P_{nk})}(J',J) \geq k$. So we need only to prove that $d_{D(a,P_{nk})}(J,J') \geq k$. Assume on the contrary that $d_{D(a,P_{nk})}(J,J') < k$. Hence there exists a vertex $y \in J$ such that there is a path $[y,\ldots,p,\ldots,x_{1+s}]$ of length less than k in D. This means that there exists a path $[y,\ldots,p]$ of length less than $k-s-1$ in the digraph D. At the same time, since $s = d_D(q,J)$, then there exists $z \in J$ such that $d_D(q,z) = s$. So we can conclude that if $y \neq z$, then J is not k-stable in D or if $y = z$, then there is a circuit $[y,\ldots,p,q,\ldots,z=y]$ in D of length less than k, a contradiction with the assumptions. Finally $d_{D(a,P_{nk})}(J,J') \geq k$. The facts proved above imply that $J \cup J'$ is a k-kernel of $D(a,P_{nk})$, which completes the part II of the proof. Thus theorem is proved. ∎

Theorem 6 implies the next corollary.

Corollary 5. *Let D be a digraph without circuits of length less than k. The digraph $D(a,P_{nk})$ has a k-kernel for an arbitrary $a \in A(D)$ and $n \geq 1$ if and only if the digraph D possesses a k-kernel.*

5. (k,l)-kernel Perfect Digraphs

This section includes necessary and sufficient conditions for special classes of digraphs considered above to be (k,l)-kernel perfect digraphs. The definition of a (k,l)-KP digraph implies the next propositions.

Proposition 1. *If D is a (k,l)-KP digraph, then every induced subdigraph of D is a (k,l)-KP digraph.*

Proposition 2. *The disjoint union of D_1 and D_2 is a (k,l)-KP digraph if and only if digraphs D_1 and D_2 are (k,l)-KP digraphs.*

Theorem 7. *Let D be a digraph, in which there exists $X \subset V(D)$ such that D has no circuit of length less than k including vertices from X. Then the duplication D^X is a (k,l)-KP digraph if and only if D is a (k,l)-KP digraph.*

Proof. I. If the duplication D^X is a (k,l)-KP digraph, then the induced subdigraph $D^X[V(D)]$ is a (k,l)-KP digraph and it is isomorphic to D. So D is a (k,l)-KP digraph.

II. Let D be a (k,l)-KP digraph, in which there exists $X \subset V(D)$ such that D has no circuit of length less than k including vertices from X.

We will prove that D^X is a (k,l)-KP digraph. Let $Y \subseteq V(D^X)$. We show that $D^X[Y]$ has a (k,l)-kernel. If $Y \subseteq V(D)$ or $Y \subseteq X'$, where X' is the copy of X in the duplication D^X, then the induced subdigraph $D^X[Y]$ possesses a (k,l)-kernel, because it is isomorphic to some induced subdigraph of the digraph D. Now assume that $Y \cap V(D) \neq \varnothing$, $Y \cap X' \neq \varnothing$ and denote $Y_D = Y \cap V(D)$, $Z' = Y \cap X'$. Of course $Y = Y_D \cup Z'$. Let Z denotes the original of $Y \cap X'$.

Since D is a (k,l)-KP digraph, then the induced subdigraph $D[Y_D \cup Z]$ has a (k,l)-kernel, say J. Let $K = J \cap Z$ and let K' be the copy of K, i.e., $K' = (J \cap Z)'$. If $K = \varnothing$, then we assume that $K' = \varnothing$. We show that $J^* = (J \cap Y_D) \cup K'$ is a (k,l)-kernel of $D^X[Y]$. First, we prove that J^* is l-dominating in $D^X[Y]$. Let $x \in V(D^X[Y]) \setminus J^*$. Since

$$V\left(D^X[Y]\right) \setminus J^* = Y \setminus J^* = \left(Y_D \cup Z'\right) \setminus J^* = (Y_D \setminus J^*) \cup \left(Z' \setminus J^*\right),$$

then consider two cases.

Case 1. If $x \in Y_D \setminus J^*$, then $d_{D[Y_D \cup Z]}(x, J) \leq l$, because J is l-dominating in $D[Y_D \cup Z]$. This means that there exists a path $P = [x, \ldots, y]$ of length not greater than l in the digraph $D[Y_D \cup Z]$, where $y \in J$. Replacing all vertices of the path P belonging to Z with their copies from Z' we get the path P' from the vertex x to some vertex from J^* of length not greater than l in $D^X[Y]$, hence $d_{D^X[Y]}(x, J^*) \leq l$.

Case 2. If $x \in Z' \setminus J^* = Z' \setminus K'$ and $y \in Z$ is the original of x, then $d_{D[Y_D \cup Z]}(y, J) \leq l$, since J is a (k,l)-kernel of $D[Y_D \cup Z]$. Arguing like in Case 1 we obtain that $d_{D^X[Y]}(x, J^*) \leq l$.

So we proved that for every $x \in V(D^X[Y]) \setminus J^*$, $d_{D^X[Y]}(x, J^*) \leq l$, which means that J^* is l-dominating in $D^X[Y]$.

Now we will show the k-stability of J^* in the digraph $D^X[Y]$. Of course $J \cap Y_D$ and K are k-stable in $D^X[Y_D \cup Z]$ in view of the k-stability of J in $D[Y_D \cup Z]$ and the definition of D^X. Assume on the contrary that $J \cap Y_D$ (resp. K') is not k-stable in $D^X[Y]$. This means that there exists a path $P = [x, \ldots, y]$ in $D^X[Y]$ of length less than k, where $x, y \in J \cap Y_D$ (resp. $x, y \in K'$). Exchanging all vertices of the path P belonging to Z' for their originals from Z we obtain a path P' from x to y (resp. from w to z, where w, z are the originals of vertices x, y and $w, y \in K$) in the digraph $D[Y_D \cup Z]$ of length less than k, a contradiction with the fact given above that $J \cap Y_D$ and K are k-stable in $D[Y_D \cup Z]$. This means that $J \cap Y_D$ and K' are

k-stable in $D^X[Y]$. Since $J^* = (J \cap Y_D) \cup K'$, we need only show that $d_{D^X[Y]}(J \cap Y_D, K') \geq k$ and $d_{D^X[Y]}(K', J \cap Y_D) \geq k$. Let $x \in J \cap Y_D$ and $y' \in K'$. If $x \in X \cap J \cap Y_D$, then there exists its copy x'. Since vertices x', y' are not necessary distinct, consider two cases.

Case (a). Let $x \in X \cap J \cap Y_D$ and $x' \neq y'$ or $x \notin X$. If $d_{D^X[Y]}(x, y') < k$, then there is a path $P = [x, \ldots, y']$ of length less than k in $D^X[Y]$. Replacing all vertices of the path P belonging to Z' with their originals from Z we get the path P' from the vertex $x \in J \cap Y_D$ to the vertex $y \in K = J \cap Z$ of length less than k in $D[Y_D \cup Z]$, a contradiction with the assumption that J is a (k,l)-kernel of $D[Y_D \cup Z]$. Hence $d_{D^X[Y]}(x, y') \geq k$. Analogously it can be proved that $d_{D^X[Y]}(y', x) \geq k$.

Case (b). Let $x \in X \cap J \cap Y_D$ and $x' = y'$. This means that $d_{D^X[Y]}(x, y') \geq k$ and $d_{D^X[Y]}(y', x) \geq k$. Otherwise, there exists a circuit in D of length less than k including vertices from X, a contradiction with the assumption.

So J^* is k-stable in $D^X[Y]$ and finally J^* is a (k,l)-kernel of $D^X[Y]$. This means that the duplication D^X is a (k,l)-KP digraph. ■

The definition of the D-join implies the next result.

Proposition 3. *Every induced subdigraph of the D-join $\sigma(\alpha, D)$ is:*

(1) *the D_0-join $\sigma(\alpha_0, D_0)$, where D_0 is an induced subdigraph of D with the vertex set $V(D_0) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ and α_0 is a sequence of digraphs $\{D_{i_1}, D_{i_2}, \ldots, D_{i_m}\}$ or*

(2) *an induced subdigraph of D_i for some $1 \leq i \leq n$ or*

(3) *the disjoint union of digraphs from items* (1) *or* (2).

Theorem 8. *Let D be a digraph without circuits of length less than k and $V(D) = \{x_1, x_2, \ldots, x_n\}$. Let $\alpha = (D_i)_{i \in \{1,2,\ldots,n\}}$ be a sequence of vertex disjoint digraphs. The D-join $\sigma(\alpha, D)$ is a (k,l)-KP digraph if and only if the digraph D and the digraphs $D_1, D_2, \ldots, D_n$ are (k,l)-KP digraphs.*

Proof. I. If the digraph $\sigma(\alpha, D)$ is a (k,l)-KP digraph, then a subdigraph of the digraph $\sigma(\alpha, D)$ induced by $V(D_i)$ is a (k,l)-KP digraph for $i = 1, 2, \ldots, n$. The definition of the D-join implies that the induced subdigraph $\sigma(\alpha, D)[V(D_i)]$ is isomorphic to D_i. Hence digraph D_i is a (k,l)-KP digraph for $i = 1, 2, \ldots, n$. Now consider a subset X of the vertex set of $\sigma(\alpha, D)$

including exactly one vertex from $V(D_i)$ for every $i = 1, 2, \ldots, n$. From the definition of the D-join we obtain that the induced subdigraph $\sigma(\alpha, D)[X]$ is isomorphic to the digraph D. So the digraph D is a (k,l)-KP digraph.

II. Let D and $D_1, D_2, \ldots, D_n$ be (k,l)-KP digraphs. Corollary 1 implies that the D-join $\sigma(\alpha_0, D_0)$, where D_0 is an induced subdigraph of the digraph D with the vertex set $V(D_0) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ and α_0 is a sequence of induced subdigraphs of digraphs $\{D_{i_1}, D_{i_2}, \ldots, D_{i_m}\}$, has a (k,l)-kernel. So from Proposition 2 and Proposition 3 we get that the digraph $\sigma(\alpha, D)$ is a (k,l)-KP digraph. ■

For $k = 2$ and $l = 1$ Theorem 8 is similar to result given in [9].

We give the necessary and sufficient condition for the digraph $D(a, P_m)$ to be a k-KP digraph. But first we prove some useful lemmas.

Let D be a digraph and P_m be a path meant as a digraph for $m \geq 2$, where $V(P_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(D) \cap V(P_m) = \emptyset$. If x is a vertex of the digraph D, then symbols $D(x^+, P_m)$ and $D(x^-, P_m)$ will denote digraphs such that $V(D(x^+, P_m)) = V(D(x^-, P_m)) = V(D) \cup V(P_m)$, and $A(D(x^+, P_m)) = A(D) \cup A(P_m) \cup \{\overrightarrow{xx_1}\}$ $A(D(x^-, P_m)) = A(D) \cup A(P_m) \cup \{\overrightarrow{x_m x}\}$.

From the definition of digraphs $D(x^+, P_m)$ and $D(x^-, P_m)$ we get immediately the following proposition.

Proposition 4. *Every induced subdigraph of the digraph $D(x^+, P_m)$ (resp. $D(x^-, P_m)$), where $x \in V(D)$, is:*

(1) *a digraph in the form $D_0(x^+, P_s)$ (resp. $D_0(x^-, P_s)$), where D_0 is an induced subdigraph of the digraph D and $2 \leq s \leq m$ or*

(2) *an induced subdigraph of the digraph D or*

(3) *induced subdigraph of the path P_m or*

(4) *the disjoint sum of digraphs from items* (1), (2) *or* (3).

Since every k-kernel J of the digraph D can be easily extended to a k-kernel of the digraph $D(x^-, P_m)$ by adding to J some vertices from $V(P_m)$, then on basis of Proposition 4 and Proposition 2 we can formulate the following result.

Proposition 5. *A digraph D is a k-KP digraph if and only if $D(x^-, P_m)$ is a k-KP digraph, for every $x \in V(D)$, where $m \geq 2$.*

Theorem 9. *Let D_1, D_2 and D be digraphs such that $V(D_1) \cap V(D_2 = \{x\}$ and $D = D_1 \cup D_2$, where x is a source of digraphs D_1 and D_2. The digraph D is a k-KP digraph if and only if D_1 and D_2 are k-KP digraphs.*

Proof. The necessary condition follows from Proposition 1. Assume that D_i is a k-KP digraph for $i = 1, 2$. We will show that D is a k-KP digraph. Let $X \subseteq V(D)$.

If $X \subseteq V(D_1)$ or $X \subseteq V(D_2)$, then an induced subdigraph $D[X]$ has a k-kernel, since digraphs D_1 and D_2 are k-KP digraphs.

If $x \in V(D) \setminus X$ and $X \cap V(D_i) \neq \varnothing$ for $i = 1, 2$, then

$$d_{D[X]}\left(X \cap V\left(D_1\right), X \cap V\left(D_2\right)\right) \geq k,$$

since x is a source of digraphs D_1 and D_2. This means that $J_1 \cup J_2$, where J_i is a k-kernel of $D_i[X \cap V(D_i)]$, for $i = 1, 2$, is a k-kernel of the digraph $D[X]$.

So assume that $x \in X$ and $X \cap V(D_i) \neq \varnothing$ for $i = 1, 2$. Let J_i be a k-kernel of the subdigraph of $D[X]$ induced by $X \cap V(D_i) \setminus \{x\}$ for $i = 1, 2$. The existence of a k-kernel J_i follows from the assumption that D_i is a k-KP digraph.

If $d_{D[X]}(x, J_1 \cup J_2) \leq k - 1$, then $J_1 \cup J_2$ is a $(k-1)$-dominating in the digraph $D[X]$. Of course $J_1 \cup J_2$ is a k-stable in $D[X]$, since x is a source of digraphs D_1 and D_2. So $J_1 \cup J_2$ is a k-kernel of the digraph $D[X]$.

If $d_{D[X]}(x, J_1 \cup J_2) \geq k$, then $J_1 \cup J_2 \cup \{x\}$ is k-stable and $(k-1)$-dominating in $D[X]$. This means that $J_1 \cup J_2 \cup \{x\}$ is a k-kernel of $D[X]$. Hence D is a k-KP digraph. ■

For $k = 2$ Theorem 9 is a special case of a result given by H. Jacob in [10].

Theorem 10 [10]**.** *Let D_1, D_2 and D be digraphs such that $V(D_1) \cap V(D_2) = \{x\}$ and $D = D_1 \cup D_2$. Then D is a KP digraph if and only if D_1 and D_2 are KP digraphs.*

Assuming that x is a source of the digraph D, from Theorem 9 we obtain the next corollary.

Corollary 6. *If $x \in V(D)$ is a source of D, then $D(x^+, P_m)$ is a k-KP digraph if and only if D is a k-KP digraph.*

The definition of the digraph $D(a, P_m)$ implies the following proposition.

Proposition 6. *Every induced subdigraph of the digraph $D(a, P_m)$, where $a \in A(D)$ and $a = \overrightarrow{pq}$ is:*

(1) *a digraph in the form $D_0(a, P_m)$, where D_0 is an induced subdigraph of D or*

(2) *an induced subdigraph of D or*

(3) *an induced subdigraph of P_m or*

(4) *an induced subdigraph of $D(p^+, P_m)$ or an induced subdigraph of $D(q^-, P_m)$ or*

(5) *the disjoint sum of digraphs from items* (1), (2), (3) *or* (4).

Taking Proposition 5, Proposition 6 and Corollary 5, Corollary 6 into consideration we get the next theorem.

Theorem 11. *Let D be a digraph without circuits of length less than k for $k \geq 2$. If $a \in A(D)$ and the initial vertex of the arc a is a source of D, then the digraph D is a k-KP digraph if and only if the digraph $D(a, P_{nk})$ is a k-KP digraph, for $n \geq 1$.*

References

[1] C. Berge and P. Duchet, *Perfect graphs and kernels*, Bulletin of the Institute of Mathematics Academia Sinica **16** (1988) 263–274.

[2] M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel, *Some operations preserving the existence of kernels*, Discrete Math. **205** (1999) 211–216.

[3] M. Borowiecki and A. Szelecka, *One-factorizations of the generalized Cartesian product and of the X-join of regular graphs*, Discuss. Math. Graph Theory **13** (1993) 15–19.

[4] M. Burlet and J. Uhry, *Parity graphs*, Annals of Discrete Math. **21** (1984) 253–277.

[5] R. Diestel, Graph Theory (Springer-Verlag New-York, Inc., 1997).

[6] P. Duchet, *A sufficient condition for a digraph to be kernel-perfect*, J. Graph Theory **11** (1987) 81–85.

[7] H. Galeana-Sánchez, *On the existence of (k, l)-kernels in digraphs*, Discrete Math. **85** (1990) 99–102.

[8] H. Galeana-Sánchez, *On the existence of kernels and h-kernels in directed graphs*, Discrete Math. **110** (1992) 251–255.

[9] H. Galeana-Sanchez and V. Neumann-Lara, *On the dichromatic number in kernel theory*, Math. Slovaca **48** (1998) 213–219.

[10] H. Jacob, Etude theórique du noyau d' un graphe (Thèse, Univesitè Pierre et Marie Curie, Paris VI, 1979).

[11] M. Kucharska, *On (k,l)-kernels of orientations of special graphs*, Ars Combin. **60** (2001) 137–147.

[12] M. Kucharska and M. Kwaśnik, *On (k,l)-kernels of special superdigraphs of P_m and C_m*, Discuss. Math. Graph Theory **21** (2001) 95–109.

[13] M. Kwaśnik, On (k,l)-kernels in graphs and their products (PhD Thesis, Technical University of Wrocław, Wrocław, 1980).

[14] A. Włoch and I. Włoch, *On (k,l)-kernels in generalized products*, Discrete Math. **164** (1997) 295–301.

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