Discussiones Mathematicae
Graph Theory 25 (2005) 13–28

GRAPHS WITH LARGE DOUBLE DOMINATION NUMBERS

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Abstract

In a graph G, a vertex dominates itself and its neighbors. A subset $S \subseteq V(G)$ is a double dominating set of G if S dominates every vertex of G at least twice. The minimum cardinality of a double dominating set of G is the double domination number $\gamma_{\times 2}(G)$. If $G \neq C_5$ is a connected graph of order n with minimum degree at least 2, then we show that $\gamma_{\times 2}(G) \le 3n/4$ and we characterize those graphs achieving equality.

Keywords: bounds, domination, double domination, minimum degree two.

2000 Mathematics Subject Classification: 05C69.

1. Introduction

In this paper we continue the study of double domination in graphs started by Harary and Haynes [5] and studied further in [1, 2, 3, 4, 8, 9] and elsewhere.

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [6, 7]. For a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V$ is a *dominating set* if each vertex in $V - S$ is adjacent to at least one vertex of S. Equivalently,

S is a dominating set of G if for every vertex $v \in V$, $|N[v] \cap S| \geq 1$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set.

In [5] Harary and Haynes defined a generalization of domination as follows: a subset S of V is a *k-tuple dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \geq k$, that is, v is in S and has at least $k-1$ neighbors in S or v is in $V - S$ and has at least k neighbors in S. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of a k-tuple dominating set of G, if such a set exists. Clearly, $\gamma(G) = \gamma_{\times 1}(G) \leq \gamma_{\times k}(G)$, while $\gamma_t(G) \leq \gamma_{\times 2}(G)$ where $\gamma_t(G)$ denotes the total domination number of G (see [6, 7]). For a graph to have a k-tuple dominating set, its minimum degree is at least $k-1$. Hence for trees, $k \leq 2$. A k-tuple dominating set where $k = 2$ is called a *double dominating set* (DDS). A DDS of cardinality $\gamma_{\times 2}(G)$ we call a $\gamma_{\times 2}(G)$-set. The redundancy involved in k-tuple domination makes it useful in many applications.

For notation and graph theory terminology we in general follow [6]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. The minimum degree among the vertices of G is denoted by $\delta(G)$. A *support vertex* is a vertex adjacent to a vertex of degree one.

A *daisy* with $k \geq 2$ *petals* is a connected graph that can be constructed from $k \geq 2$ disjoint cycles by identifying a set of k vertices, one from each cycle, into one vertex. In particular, if the k cycles have lengths $n_1, n_2, \ldots, n_k$, we denote the daisy by $D(n_1, n_2, \ldots, n_k)$.

2. Known Results

The value of $\gamma_{\times 2}(C_n)$ for a cycle C_n is established in [5].

Proposition 1 (Harary, Haynes [5]). *For $n \geq 3$, $\gamma_{\times 2}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.*

As an immediate consequence of a result of Blidia et al. [2], we obtain the following upper bound on the double domination number of a connected graph in terms of the order of the graph, the number of vertices of degree one and the number of support vertices in the graph.

Theorem 2 ([2]). *If G is a connected graph of order $n \geq 3$ with ℓ vertices of degree one and s support vertices, then $\gamma_{\times 2}(G) \leq (2n + \ell + s)/3$.*

In particular, we have the following upper bound on the double domination number of a connected graph in terms of its order.

Corollary 3 ([2]). *If G is a connected graph of order $n \geq 2$, then $\gamma_{\times 2}(G) \leq n$ with equality if and only if every vertex of G has degree one or is a support vertex.*

If we restrict the minimum degree to be at least two, then Blidia et al. [1] showed that the upper bound in Corollary 3 on the double domination number can be improved to eleven-thirteens its order.

Theorem 4 ([1]). *If G is a graph of order n with $\delta(G) \geq 2$, then $\gamma_{\times 2}(G) \leq 11n/13$.*

3. Main Result

Our aim in this paper is to improve the upper bound in Theorem 4 on the double domination number from eleven-thirteens its order to three-fourths its order when $G \neq C_5$, and to characterize those graphs achieving equality.

In order to characterize the connected graphs with minimum degree at least two that have maximum possible double domination number we introduce a family $\mathcal{H}$ of graphs as follows. We define a *unit* to be a graph that is isomorphic to a cycle C_4. By *attaching a unit* to a vertex v of a graph, we mean adding a path P_3 to that graph and joining v to both end-vertices of the P_3. The resulting 4-cycle containing v we call a *unit* of the graph and we call v the *link vertex* of the unit. Let $\mathcal{H}$ be the family of a graphs that can be obtained from a connected graph by attaching a unit to every vertex of that graph. A graph in the family $\mathcal{H}$ with four units is shown in Figure 1.

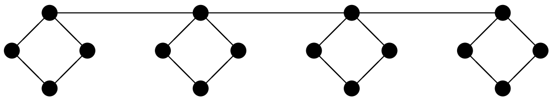

Figure 1. A graph in the family $\mathcal{H}$

Let F_1 be the graph obtained from an 8-cycle by adding an edge between two vertices at maximum distance 4 apart on the cycle. The graph F_1 is shown in Figure 2.

F_1 :

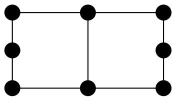

Figure 2. The graph F_1

Our main result establishes an upper bound on the double domination number of a connected graph with minimum degree at least two that is not a 5-cycle and characterizes those graphs achieving this upper bound. A proof of Theorem 5 is given in Section 1.

Theorem 5. *If* $H \neq C_5$ *is a connected graph of order* n *with* $\delta(H) \geq 2$*, then* $\gamma_{\times 2}(H) \leq 3n/4$ *with equality if and only if* $H \in \{F_1, C_8\} \cup \mathcal{H}$*.*

4. $\frac{3}{4}$-Minimal Graphs

The key to our proof of Theorem 5 is a characterization of what we call $\frac{3}{4}$-minimal graphs. We will refer to a graph G as a $\frac{3}{4}$*-minimal graph* if G is edge-minimal with respect to satisfying the following three conditions:

(i) $\delta(G) \geq 2$,
(ii) G is connected, and
(iii) $\gamma_{\times 2}(G) \geq 3n/4$, where n is the order of G.

As a consequence of Proposition 1, we can establish which cycles are $\frac{3}{4}$-minimal graphs.

Corollary 6. *A cycle* G *is a* $\frac{3}{4}$*-minimal graph if and only if* $G \in \{C_4, C_5, C_8\}$*.*

Next we establish which daisies are $\frac{3}{4}$-minimal graphs.

Proposition 7. *If* G *is a daisy of order* n*, then* $\gamma_{\times 2}(G) \leq (2n+1)/3$*.*

Proof. We proceed by induction on the order n of the daisy. If $n = 5$, then $G = D(3,3)$ and $\gamma_{\times 2}(G) = 3$, while if $n = 6$, then $G = D(3,4)$ and $\gamma_{\times 2}(G) = 4$. Hence if $n \in \{5,6\}$, $\gamma_{\times 2}(G) < (2n+1)/3$. This establishes the base cases. Assume, then, that $n \geq 7$ and that if G' is a daisy of order n', where $n' < n$, then $\gamma_{\times 2}(G') \leq (2n'+1)/3$. Let G be a daisy of order n and let v denote the vertex of maximum degree in G. Let $F\colon v, v_1, v_2, \ldots, v_{n_1}, v$ be a cycle containing v. Thus, $F \cong C_{n_1+1}$. Let $S_1 = \{v_i \mid i \equiv 0 \text{ or } 2 \pmod 3\}$. Then, $|S_1| \leq 2n_1/3$. Let $G' = G - (V(F) - \{v\})$ and let G' have order n', and so $n' = n - n_1$.

Suppose first that G' is a cycle, i.e., $G' = C_{n'}$. Let S' be a $\gamma_{\times 2}(G')$-set that contains v. By Proposition 1, $|S'| \leq 2(n'+1)/3 = 2(n - n_1 + 1)/3$. Since $S_1 \cup S'$ is a DDS of G, $\gamma_{\times 2}(G) \leq |S_1 \cup S'| \leq 2(n+1)/3$.

Suppose secondly that G' is a daisy. Applying the inductive hypothesis to G', $\gamma_{\times 2}(G') \le (2n'+1)/3 = (2n-2n_1+1)/3$. Let S' be a $\gamma_{\times 2}(G')$-set. The restriction of S' to the vertices of at least one cycle in G' must be a DDS in that cycle. Hence we may choose S' to contain the vertex v. Then, $S_1 \cup S'$ is a DDS, and so $\gamma_{\times 2}(G) \le |S_1 \cup S'| \le (2n+1)/3$. ∎

Since $(2n+1)/3 < 3n/4$ for $n \ge 5$, we have the following immediate consequence of Proposition 7.

Corollary 8. *No daisy is a* $\frac{3}{4}$*-minimal graph.*

Let $\mathcal{G}$ be the collection of graphs that can be obtained from a *tree* by attaching a unit to every vertex of the tree. Hence the family $\mathcal{G}$ is a subfamily of the family $\mathcal{H}$. The following observation about graphs in the family $\mathcal{G}$ will prove to be useful.

Observation 9. Each graph in the family $\mathcal{G}$ has double domination number three-fourths its order and is a $\frac{3}{4}$-minimal graph. Further, there is a $\gamma_{\times 2}(G)$-set that contains any specified vertex of G.

The following key result, a proof of which is given in Section 1, characterizes $\frac{3}{4}$-minimal graphs.

Theorem 10. *A graph* G *is a* $\frac{3}{4}$*-minimal graph if and only if* $G \in \{C_5, C_8\} \cup \mathcal{G}$.

5. Proof of Theorem 5

Theorem 5, our main result, is simply a corollary of Theorem 10. Since the double domination number of a graph cannot decrease if edges are removed, it follows from Observation 9 and Theorem 10 that the double domination number of H is at most three-fourths its order. Further suppose H has double domination number exactly three-fourths its order. Then by removing edges of H, if necessary, we produce a $\frac{3}{4}$-minimal graph H'. If $H' = C_5$, then $H = C_5$, a contradiction. Hence by Theorem 10, $H' = C_8$ or $H' \in \mathcal{G}$. If $H' = C_8$, then $H \in \{F_1, C_8\}$.

Suppose $H' \in \mathcal{G}$. We show that each vertex of H' that is not a link vertex must have degree 2 in H, whence $H \in \mathcal{H}$. If $n = 4$, then $H = C_4$. Hence we may assume $n \ge 8$. Let $C_v\colon v, w, x, y, v$ be a unit of H' with link vertex v and let $vv' \in E(H')$ where v' is a link vertex of H' different from v. If $xx' \in E(H)$ where x' is a vertex not in the unit C_v (possibly, $v' = x'$), then the set $\{v, v'\} \cup \{x, x'\}$ can easily be extended to a DDS of H that contains three vertices from every unit different from C_v and two vertices from the unit C_v, and so $\gamma_{\times 2}(H) < 3n/4$, a contradiction. If $vx \in E(H)$, then the set $\{v, x\}$ can be extended to a DDS of H that contains three vertices from every unit different from C_v and two vertices from the unit C_v, a contradiction. Hence each vertex of H' that is neither a link vertex nor adjacent to a link vertex has degree 2 in H. If $ww' \in E(H)$ where w' is a vertex not in the unit C_v (possibly, $v' = w'$), then the set $\{x, y, v'\} \cup \{w'\}$ can easily be extended to a DDS of H that contains three vertices from every unit different from C_v and two vertices from the unit C_v, a contradiction. If $wy \in E(H)$, then the set $\{w, y\}$ can be extended to a DDS of H that contains three vertices from every unit different from C_v and two vertices from the unit C_v, a contradiction. It follows that every vertex of H' that is not a link vertex has degree 2 in H. Thus, $H \in \mathcal{H}$.

6. Proof of Theorem 10

The sufficiency follows from Corollary 6 and Observation 9. To prove the necessary, we proceed by induction on the order $n \geq 3$ of a $\frac{3}{4}$-minimal graph. If G is a $\frac{3}{4}$-minimal graph of order n, $3 \leq n \leq 5$, then $G \in \{C_4, C_5\}$. This establishes the base case. For our inductive hypothesis, let $n \geq 6$ and assume that for $n' < n$, a graph G' is a $\frac{3}{4}$-minimal graph if and only if $G' \in \{C_5, C_8\} \cup \mathcal{G}$. This implies (see the proof of Theorem 5) the following result.

Observation 11. If $H \neq C_5$ is a connected graph of order $n' < n$ with $\delta(H) \geq 2$, then $\gamma_{\times 2}(H) \leq 3n'/4$ with equality if and only if $H \in \{F_1, C_8\} \cup \mathcal{H}$.

Let $G = (V, E)$ be a $\frac{3}{4}$-minimal graph of order n. Before proceeding further, we prove a few results that will be useful in what follows. If e is an edge of G, then $\gamma_{\times 2}(G - e) \geq \gamma_{\times 2}(G)$. Hence, by the minimality of G, we have the following observation.

Observation 12. If $e \in E$, then either e is a bridge of G or $\delta(G - e) = 1$.

The next result is a consequence of the inductive hypothesis.

Observation 13. If G' is a connected subgraph of G of order $n' < n$ with $\delta(G') \geq 2$, then either $G' \in \{C_5, C_8\} \cup \mathcal{G}$ or $\gamma_{\times 2}(G') < 3n'/4$.

Suppose $G = C_n$ (and still $n \geq 6$). Then, by Corollary 6, $G = C_8$. So we may assume that G is not a cycle. Hence, G contains at least one vertex of degree at least 3. Let $S = \{v \in V \mid \deg(v) \geq 3\}$. Each vertex of $V - S$ therefore has degree 2 in G. If $|S| = 1$, then G is a daisy, contradicting Corollary 8. Hence, $|S| \geq 2$.

For each $v \in S$, we define the 2-*graph of* v to be the component of $G - (S - \{v\})$ that contains v. So each vertex of the 2-graph of v has degree 2 in G, except for v. Furthermore, the 2-graph of v consists of edge-disjoint cycles through v, which we call 2-*graph cycles*, and paths emanating from v, which we call 2-*graph paths.*

We show next that there in no long path in G whose internal vertices have degree 2 in G. A proof of the following result is given in Subsection 6..1

Lemma 14. *There is no path on six vertices the internal vertices of which have degree* 2 *in* G.

For integers $n_1 \geq n_2 \geq 3$ and $k \geq 0$, we define a *dumb-bell* $D_b(n_1, n_2, k)$ to be the graph of order $n = n_1 + n_2 + k$ obtained from the cycles C_{n_1} and C_{n_2} by joining a vertex of C_{n_1} to a vertex of C_{n_2} and subdividing the resulting edge k times.

By Lemma 14, every 2-graph path in G has length one, two or three, while every 2-graph cycle has length at most five. Hence it now a simple exercise to verify the following result.

Observation 15. If G is a dumb-bell, then $G = D_b(4, 4, 0) \in \mathcal{G}$.

By Observation 15, we may assume that G is not a dumb-bell. Using the inductive hypothesis, we shall prove the following lemma, a proof of which is given in Subsection 6.2 to Subsection 6.6.

Lemma 16. *If* S *is an independent set, then the graph* G *has the following five properties:*

- (a) *There is no* 2*-graph cycle in* G*.*
- (b) *Every* 2*-graph path in* G *has length one or two.*
- (c) $|S| \geq 3$.

(d) *Each vertex of S is within distance 3 from at least two other vertices of S.*

(e) *Let u and v be two vertices of S that are joined by a path the internal vertices of which are in $V - S$. Then, u has exactly one neighbor that does not belong to any 2-graph path of v or v has exactly one neighbor that does not belong to any 2-graph path of u.*

Using Lemma 16, we prove the following result, a proof of which is presented in Subsection 6..7

Lemma 17. *The set S is not an independent set.*

As a consequence of Lemma 17, we have the following result, a proof of which is given in Subsection 6.8, which completes the proof of Theorem 10.

Lemma 18. *$G \in \mathcal{G}$.*

6..1 Proof of Lemma 14

Suppose that $v_1, v_2, \ldots, v_6$ is a path in G where $\deg_G(v_i) = 2$ for $2 \le i \le 5$. Let G' be the graph of order $n' = n - 3$ obtained from G by deleting the three vertices v_3, v_4 and v_5 and adding the edge v_2v_6, i.e., $G' = (G - \{v_3, v_4, v_5\}) \cup \{v_2v_6\}$. By assumption $G \ne C_8$, and so $G' \ne C_5$. By Observation 11, $\gamma_{\times 2}(G') \le 3n'/4$. Let D' be a $\gamma_{\times 2}(G')$-set. If $v_2 \notin D'$, let $D = D' \cup \{v_3, v_4\}$. If $\{v_1, v_2\} \subseteq D'$, let $D = D' \cup \{v_4, v_5\}$. If $\{v_2, v_6\} \subseteq D'$, let $D = D' \cup \{v_3, v_5\}$. In all cases, the set D is a DDS of G, and so $\gamma_{\times 2}(G) \le |D| = \gamma_{\times 2}(G') + 2 \le 3n'/4 + 2 = (3n - 1)/4$, contradicting the fact that G is a $\frac{3}{4}$-minimal graph.

6..2 Proof of Lemma 16(a)

Suppose, to the contrary, that there is a 2-graph cycle in G. Let $v \in S$ and suppose that C_v is a 2-graph cycle of v of length $n_1 + 1$. By Lemma 14, $2 \le n_1 \le 4$. We consider two possibilities.

Case 1. $\deg_G(v) \ge 4$. Let $G_2 = G - (V(C_v) - \{v\})$. Then, G_2 is a connected graph with minimum degree at least 2 and of order $n_2 = n - n_1$. Since $|S| \ge 2$, G_2 is not a cycle. By our assumption that S is an independent set, $G_2 \notin \mathcal{G}$. Hence by Observation 13, $\gamma_{\times 2}(G_2) < 3n_2/4 = 3(n - n_1)/4$. If v belongs to some $\gamma_{\times 2}(G_2)$-set, then such a DDS of G_2 can be extended to

a DDS of G by adding at most $2n_1/3$ vertices from the path $C_v - v$, whence $\gamma_{\times 2}(G) \leq 2n_1/3 + \gamma_{\times 2}(G_2) < 2n_1/3 + 3n_2/4 < 3n/4$, a contradiction. Hence the vertex v belongs to no $\gamma_{\times 2}(G_2)$-set.
Since S is an independent set, every neighbor of v has degree 2. Since v belongs to no $\gamma_{\times 2}(G_2)$-set, it follows that every $\gamma_{\times 2}(G_2)$-set contains every neighbor of v in G_2 and every vertex at distance 2 from v in G_2. Further it follows that there is no 4-cycle or 5-cycle in G_2 containing v. Let G' be the graph of order $n' = n_2 - 1$ obtained from $G_2 - v$ by joining a neighbor of v in G_2 to every other neighbor of v in G_2. Then G' is a connected graph with $\delta(G') \geq 2$. Since G_2 is not a cycle, neither is G'. Since v belongs to no $\gamma_{\times 2}(G_2)$-set, $G' \neq F_1$. Since v belongs to neither a 4-cycle nor a 5-cycle in G_2, no vertex in $N(v)$ in G' belongs to a 4-cycle. Hence, $G' \notin \mathcal{H}$. Thus by Observation 11, $\gamma_{\times 2}(G') < 3n'/4 = 3(n - n_1 - 1)/4$.

If $n_1 = 2$, then a $\gamma_{\times 2}(G')$-set can be extended to a DDS of G by adding to it the vertex v and one of its two neighbors in C_v. Hence, $\gamma_{\times 2}(G) \leq 2 + \gamma_{\times 2}(G') < 2 + 3(n-3)/4 < 3n/4$, a contradiction.

If $n_1 = 3$, then a $\gamma_{\times 2}(G')$-set can be extended to a DDS of G by adding to it the vertex v and its two neighbors in C_v. Hence, $\gamma_{\times 2}(G) \leq 3 + \gamma_{\times 2}(G') < 3 + 3(n-4)/4 = 3n/4$, a contradiction.

If $n_1 = 4$, then a $\gamma_{\times 2}(G_2)$-set can be extended to a DDS of G by adding to it the vertex v and the two vertices in C_v that are not adjacent to v. Hence, $\gamma_{\times 2}(G) \leq 3 + \gamma_{\times 2}(G_2) < 3 + 3(n-4)/4 = 3n/4$, a contradiction.

Case 2. $\deg_G(v) = 3$. Let $P\colon v, v_1, \ldots, v_k, w$ be the path from v to the vertex w of $S - \{v\}$ every internal vertex of which belongs to $V - S$. Since S is independent, $k \geq 1$. Furthermore, by Lemma 14, $k \leq 3$.

Let $G' = G - V(C_v) - [V(P) - \{w\}]$. Then, G' is a connected graph of order $n' = n - n_1 - k - 1$ with $\delta(G') \geq 2$. By our assumption that G is not a dumb-bell, G' is not a cycle. Thus by Observation 11, $\gamma_{\times 2}(G') \leq 3n'/4$.

Let G^* be the graph of order $n^* = n' - 1$ obtained from $G' - w$ by joining a neighbor of w in G' to every other neighbor of w in G'. Then, G^* is a connected graph with $\delta(G^*) \geq 2$. It follows from our assumption that G is not a dumb-bell that G^* is not a cycle. Thus by Observation 11, $\gamma_{\times 2}(G^*) \leq 3n^*/4$. Since all neighbors of w in G have degree 2, it follows that every DDS of G^* must contain at least one neighbor of w.

Case 2.1. $k = 3$. Let $F = (G - \{v_1, v_2, v_3\}) \cup \{vw\}$. Then, F is a connected graph of order $n - 3$ with $\delta(F) \geq 2$. Since F is not a cycle, $\gamma_{\times 2}(F) \leq 3(n-3)/4$ by Observation 11. Let S_F be a $\gamma_{\times 2}(F)$-set. If $n_1 = 2$, we can choose S_F to contain v and a neighbor of v in C_v, while if $n_1 = 3$, we can choose S_F to contain v and its two neighbors in C_v. Thus if $n_1 \in \{2,3\}$, then $S_F \cup \{v_2, v_3\}$ is a DDS of G, and so $\gamma_{\times 2}(G) \le 3(n-3)/4 + 2 < 3n/4$, a contradiction. If $n_1 = 4$, then we can clearly choose S_F to contain the two vertices in C_v that are not adjacent to v and the two vertices v and w. Thus, $S_F \cup \{v_1, v_3\}$ is a DDS of G, and so $\gamma_{\times 2}(G) \le 3n'/4 + 2 < 3n/4$, a contradiction.

Case 2.2. $k = 2$ and $n_1 = 4$. Any $\gamma_{\times 2}(G')$-set can be extended to a DDS of G by adding to it the two vertices in C_v that are not adjacent to v and the three vertices v, v_1 and v_2. Hence, $\gamma_{\times 2}(G) \le 5 + \gamma_{\times 2}(G') \le 5 + 3(n-7)/4 < 3n/4$, a contradiction.

Case 2.3. $k = 2$ and $n_1 \in \{2,3\}$. If $n_1 = 2$, then any $\gamma_{\times 2}(G^*)$-set can be extended to a DDS of G by adding to it a neighbor of v in C_v and the vertices in the set $\{v, v_2, w\}$. Hence, $\gamma_{\times 2}(G) \le 4 + \gamma_{\times 2}(G^*) \le 4 + 3(n-6)/4 < 3n/4$, a contradiction. If $n_1 = 3$, then any $\gamma_{\times 2}(G^*)$-set can be extended to a DDS of G by adding to it the two neighbors of v in C_v and the vertices in the set $\{v, v_2, w\}$. Hence, $\gamma_{\times 2}(G) \le 5 + \gamma_{\times 2}(G^*) \le 5 + 3(n-7)/4 < 3n/4$, a contradiction.

Case 2.4. $k = 1$ and $n_1 \in \{2,4\}$. If $n_1 = 4$, then any $\gamma_{\times 2}(G')$-set can be extended to a DDS of G by adding to it the two vertices in C_v that are not adjacent to v and the two vertices v and v_1. Hence, $\gamma_{\times 2}(G) \le 4 + \gamma_{\times 2}(G') \le 4 + 3(n-6)/4 < 3n/4$, a contradiction. Suppose $n_1 = 2$. If w belongs to some $\gamma_{\times 2}(G')$-set, then such a $\gamma_{\times 2}(G')$-set can be extended to a DDS of G by adding to it v and a neighbor of v in C_v, whence $\gamma_{\times 2}(G) \le 2 + \gamma_{\times 2}(G') \le 2 + 3(n-4)/4 < 3n/4$, a contradiction. On the other hand, if w belongs to no $\gamma_{\times 2}(G')$-set, then it follows from Observations 9 and 13 that $\gamma_{\times 2}(G') < 3n'/4$. Now any $\gamma_{\times 2}(G')$-set can be extended to a DDS of G by adding to it v, v_1 and a neighbor of v in C_v, whence $\gamma_{\times 2}(G) \le 3 + \gamma_{\times 2}(G') < 3 + 3(n-4)/4 = 3n/4$, a contradiction.

Case 2.5. $k = 1$ and $n_1 = 3$. Any $\gamma_{\times 2}(G^*)$-set can be extended to a DDS of G by adding to it the two neighbors of v in C_v and the two vertices in the set $\{v, w\}$, whence $\gamma_{\times 2}(G) \le 4 + \gamma_{\times 2}(G^*) \le 4 + 3(n-6)/4 < 3n/4$, a contradiction.

6..3 Proof of Lemma 16(b)

By Lemma 14, every 2-graph path in G has length one, two or three. Suppose there is a 2-graph path of length three. Let $v \in S$ and suppose v, v_1, v_2, v_3 is a 2-graph path of v. Let w be the vertex of S adjacent to v_3. Then, $\deg_G(v_i) = 2$ for $i = 1, 2, 3$.

Let $F = G - \{v_1, v_2, v_3\}$. Then, $\delta(F) \geq 2$. If $F = C_5$, then $n = 8$ and $\gamma_{\times 2}(G) \leq 5 < 3n/4$, a contradiction. Hence, $F \neq C_5$. Further since there is no 2-graph cycle in G by Lemma 16(a), if F is disconnected, then neither component of F is a cycle. Hence by Observation 11 applied to F, if F is connected, or to the two components of F, if F is disconnected, $\gamma_{\times 2}(F) \leq 3(n-3)/4$. If there exists a $\gamma_{\times 2}(F)$-set that contains w, then such a set can be extended to a DDS of G by adding to it the vertices in the set $\{v_1, v_2\}$, whence $\gamma_{\times 2}(G) \leq 2 + \gamma_{\times 2}(F) \leq 2 + 3(n-3)/4 < 3n/4$, a contradiction. Hence no $\gamma_{\times 2}(F)$-set contains w. Similarly, no $\gamma_{\times 2}(F)$-set contains v.

Let F' be the graph of order $n' = n - 4$ obtained from $F - w$ by joining a neighbor w' of w in $V(F)$ to every other neighbor of w in $V(F)$. Then, $\delta(F') \geq 2$. If $F' = C_5$, then $\gamma_{\times 2}(G) < 3n/4$, a contradiction. Hence, $F' \neq C_5$. If F' is disconnected, then since there is no 2-graph cycle in G, neither component of F' is a cycle. Hence it follows by Observation 11 that $\gamma_{\times 2}(F') \leq 3n'/4$ (irrespective of whether F' is connected or disconnected). Since all neighbors of w in G have degree 2, every DDS of F' must contain at least one neighbor of w. Hence any $\gamma_{\times 2}(F')$-set can be extended to a DDS of G by adding to it the vertices in the set $\{v_1, v_2, w\}$, whence $3n/4 = \gamma_{\times 2}(G) \leq 3 + \gamma_{\times 2}(F') \leq 3 + 3(n-4)/4 = 3n/4$. Thus we must have equality throughout this inequality chain. In particular, $\gamma_{\times 2}(F') = 3n'/4$.

Suppose F' is disconnected. Then each component of F' has double domination number three-fourths its order. As observed earlier, neither component of F' is a cycle. Further since S is an independent set, the component of F' containing v cannot be in $\mathcal{G}$. But then by Observation 13, the component of F' containing v has double domination number less than three-fourths its order, a contradiction. Hence, F' is connected.

As observed earlier, $F' \neq C_5$. Thus by Observation 11, $F' \in \{C_8, F_1\} \cup \mathcal{H}$. By our assumption that S is an independent set in G, and since there is no 2-graph cycle in G by Lemma 16(a), it follows that $F' \notin \mathcal{H}$. Suppose $F' = F_1$. Since S is an independent set in G, it follows from the way in which F' is constructed that w' is one of the two vertices of degree 3 in F'

and that the new edges added to $F - w$ to produce F' are the two edges joining w' to its two neighbors of degree 2 in F'. Thus, F is obtained from F_1 by subdividing once the edge joining the two vertices of degree three in F_1 (where w is one of the resulting two vertices of degree 3 in F). But then there exists a $\gamma_{\times 2}(F)$-set that contains w, a contradiction. Hence, $F' \neq F_1$, and so $F' = C_8$. But then $F = C_9$ and once again there exists a $\gamma_{\times 2}(F)$-set that contains w, a contradiction. We deduce, therefore, that there is no 2-graph path of length 3 in G.

6..4 Proof of Lemma 16(c)

Suppose $|S| = 2$. Let $S = \{u, v\}$. By Lemma 16(b), every 2-graph path in G has length one or two, and so $G - S = \ell_1 K_1 \cup \ell_2 K_2$ where $\ell_1 + \ell_2 \geq 3$. If $\ell_2 = 0$, then $n \geq 5$ and adding a vertex of $V - S$ to the set S produces a DDS of G, and so $\gamma_{\times 2}(G) = 3 < 3n/4$, a contradiction. If $\ell_2 = 1$, then $n \geq 6$ and adding the two vertices of the P_2-component of $G - S$ to the set S produces a DDS of G, and so $\gamma_{\times 2}(G) = 4 < 3n/4$, a contradiction. Hence, $\ell_2 \geq 2$. Adding to the set S one vertex from each P_2-component of $G - S$ in such a way that both u and v are adjacent to at least one added vertex produces a DDS of G, and so $\gamma_{\times 2}(G) \leq 2 + \ell_2 < 3(1 + \ell_2)/2 \leq 3n/4$, a contradiction. Hence, $|S| \geq 3$.

6..5 Proof of Lemma 16(d)

Suppose some vertex $v \in S$ is within distance 3 from only one other vertex w of S. Thus, w is adjacent to an end-vertex from every 2-graph path emanating from v. Since $|S| \geq 3$ and G is connected, at least one neighbor of w does not belong to any 2-graph path of v.

Suppose w has at least two neighbors that do not belong to any 2-graph path of v. Let G_w be the subgraph of G induced by w and the vertices on all 2-graph paths of v. Then, $G_w - v - w = \ell_1 K_1 \cup \ell_2 K_2$ where $\ell_1 + \ell_2 \geq 3$. An identical proof to that of Lemma 16(c) shows that there exists a DDS D_w of G_w that contains v and w and such that $|D_w| < 3|V(G_w)|/4$. Let G' be the graph of order $n' = n - |V(G_w)|$ obtained from $G - V(G_w)$ by joining a neighbor w' of w in $V(G) - V(G_w)$ to every other neighbor of w in $V(G) - V(G_w)$. Then, G' is a connected graph with $\delta(G') \geq 2$. The degree of each vertex of $S - \{v, w\}$ is unchanged in G and G', and so G' has at least one vertex of degree at least 3 in G'. In particular, G' is not a cycle. By Observation 11, $\gamma_{\times 2}(G') \leq 3n'/4$. Any DDS of G' can be extended to a DDS of G by adding to it the vertices in the set D_w. Hence, $\gamma_{\times 2}(G) \leq |D_w| + \gamma_{\times 2}(G') < 3(n-n')/4 + 3n'/4 = 3n/4$, a contradiction. Thus exactly one neighbor of w does not belong to any 2-graph path of v. Let x denote such a neighbor of w.

Let P be the 2-graph path of w that contains x. Suppose that P is a 2-graph path of length 2. Let w, x, y denote this path, and let z denote the vertex of S adjacent to y. Let G_1 and G_2 be the two components of $G - yz$, where $y \in V(G_1)$. For $i = 1, 2$, let $|V(G_i)| = n_i$. The graph G_2 is connected with $\delta(G_2) \geq 2$. Since G has no 2-graph cycle, G_2 is not a cycle. Hence by Observation 11, $\gamma_{\times 2}(G_2) \leq 3n_2/4$. We now consider the component G_1. Let $X = \{v, w, x, y\}$. The graph $G_1 - X = \ell_1 K_1 \cup \ell_2 K_2$ where $\ell_1 + \ell_2 \geq 3$. If $\ell_2 = 0$, then $n \geq 7$ and adding a common neighbor of v and w to the set X produces a DDS of G_1, and so $\gamma_{\times 2}(G_1) \leq 5 < 3n_1/4$. If $\ell_2 = 1$, then $n \geq 8$ and adding the neighbor of v on the 2-graph path of v of length 2 to the set X produces a DDS of G_1, and so $\gamma_{\times 2}(G_1) \leq 5 < 3n_1/4$. If $\ell_2 \geq 2$, then adding to the set X a neighbor of v from each 2-graph path of v of length 2 produces a DDS of G_1, and so $\gamma_{\times 2}(G_1) \leq 4 + \ell_2$. In this case, $n_1 = 4 + \ell_1 + 2\ell_2$, whence it follows that $\gamma_{\times 2}(G_1) < 3n_1/4$. Hence in all cases, $\gamma_{\times 2}(G_1) < 3n_1/4$. Thus, $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G_1) + \gamma_{\times 2}(G_2) < 3n_1/4 + 3n_2/4 = 3n/4$, a contradiction. Hence P is a 2-graph path of length 1, i.e., P is the path w, x. An almost identical proof used when P has length 2 (here we take $X = \{v, w, x\}$) shows that $\gamma_{\times 2}(G) < 3n/4$, once again a contradiction.

6..6 Proof of Lemma 16(e)

Suppose that u has at least two neighbors that do not belong to any 2-graph path of v and v has at least two neighbors that do not belong to any 2-graph path of u. Let F be the subgraph of G induced by u and v and the vertices on all u–v paths every internal vertex of which is in $V - S$.

Let G' be the graph of order $n' = n - |V(F)|$ obtained from $G - V(F)$ by joining a neighbor u' of u in $V(G) - V(F)$ to every other neighbor of u in $V(G) - V(F)$ and joining a neighbor v' of v in $V(G) - V(F)$ to every other neighbor of v in $V(G) - V(F)$. Then, $\delta(G') \geq 2$ and either G' is connected or G' has exactly two components, namely one component containing u' and the other v'. The degree of each vertex of $S - \{u, v\}$ is unchanged in G and G'. Since S is an independent set, and since G has no 2-graph cycle (by Lemma 16(a)) and since $|S| \geq 3$ (by Lemma 16(c)), it follows readily that no component of G' is a cycle or belongs to the family $\mathcal{H}$ and that if G' is connected, then $G' \neq F_1$. If G' is disconnected and has a component isomorphic to F_1, then we would contradict Lemma 16(b) and Lemma 16(d). Hence no component of G' is isomorphic to F_1. Thus it follows by Observation 11 that $\gamma_{\times 2}(G') < 3n'/4$.

Since all neighbors of u (respectively, v) in G have degree 2, every DDS of G' must contain at least one neighbor of u and at least one neighbor of v. Hence any $\gamma_{\times 2}(G')$-set can be extended to a DDS of G by adding to it the vertices u and v, and one vertex from each P_2-component of $F - u - v$ if any. Thus, $\gamma_{\times 2}(G) \leq 3|V(F)|/4 + \gamma_{\times 2}(G') < 3(n - n')/4 + 3n'/4 = 3n/4$, a contradiction.

6..7 Proof of Lemma 17

Suppose, to the contrary, that S is an independent set. Let u and v be two vertices of S that are joined by a path the internal vertices of which are in $V - S$. By Lemma 16(e), we may assume that u has exactly one neighbor u' that does not belong to any 2-graph path of v. Thus, v is adjacent to the end-vertex of every 2-graph path of u except for the 2-graph path of u that contains u'. By Lemma 16(d), v has a neighbor v' that does not belong to any 2-graph path of u.

Let F be the subgraph of G induced by u and v and the vertices on all u–v paths every internal vertex of which is in $V - S$. Let G' be the graph of order $n' = n - |V(F)|$ obtained from $G - V(F)$ by joining v' to every other neighbor of v in $V(G) - V(F)$ and joining v' to the vertex u'. Then, G' is a connected graph with $\delta(G') \geq 2$. The degree of each vertex of $S - \{u, v\}$ is unchanged in G and G', and so at least one vertex of G' different from v' has degree at least 3 in G'. Since S is an independent set, and since G has no 2-graph cycle (by Lemma 16 (a)) and since $|S| \geq 3$ (by Lemma 16(c)), it follows readily that G' is neither a cycle nor isomorphic to F_1 nor in the family $\mathcal{H}$. Thus by Observation 11, $\gamma_{\times 2}(G') < 3n'/4$.

Let $D_F \subset V(F) - \{u, v\}$ be defined as follows. If every vertex of $F - \{u, v\}$ is isolated, let $D_F = \{u'\}$; otherwise, let D_F consist of a neighbor of u from every P_2-component in $F - \{u, v\}$. Since all neighbors of v in G have degree 2, every DDS of G' must contain at least one neighbor of v. Hence any $\gamma_{\times 2}(G')$-set can be extended to a DDS of G by adding to it the set $D_F \cup \{u, v\}$. Thus, $\gamma_{\times 2}(G) \leq 3|V(F)|/4 + \gamma_{\times 2}(G') < 3(n - n')/4 + 3n'/4 = 3n/4$.

6..8 Proof of Lemma 18

By Lemma 17, there is an edge $e = uv$ where $u, v \in S$. By Observation 12, e must be a bridge of G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of $G - e$ where $u \in V_1$. For $i = 1, 2$, let $|V_i| = n_i$. Each G_i satisfies $\delta(G_i) \geq 2$ and is connected. Hence by Observation 13, $G_i \in \{C_5, C_8\} \cup \mathcal{G}$ or $\gamma_{\times 2}(G_i) < 3n_i/4$ for $i = 1, 2$.

Suppose $\gamma_{\times 2}(G_1) < 3n_1/4$. If $G_2 \neq C_5$, then $\gamma_{\times 2}(G_2) \leq 3n_2/4$, and so $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G_1) + \gamma_{\times 2}(G_2) < 3n/4$, a contradiction. Hence, $G_2 = C_5$. If $\gamma_{\times 2}(G_1) < (3n_1 - 1)/4$, then $\gamma_{\times 2}(G) < 3n/4$, a contradiction. Hence, $\gamma_{\times 2}(G_1) = (3n_1 - 1)/4 = 3n/4 - 4$ (and so, $n_1 \equiv 3 \pmod 4$). If u is in some $\gamma_{\times 2}(G_1)$-set, then such a set can be extended to a DDS of G by adding to it v and the two vertices at distance 2 from v in G_2, and so $\gamma_{\times 2}(G) \leq 3n/4 - 1$, a contradiction. Hence, u belongs to no $\gamma_{\times 2}(G_1)$-set. Let G_1' be the graph obtained from $G_1 - u$ by adding all edges between neighbors of u in G_1. Then G_1' is a connected graph with $\delta(G_1') \geq 2$. Since u is in no $\gamma_{\times 2}(G_1)$-set, it follows that G_1, and therefore G_1', is not a cycle. Hence, by Observation 11, $\gamma_{\times 2}(G_1') \leq 3n'/4 = 3(n - 6)/4$. Any $\gamma_{\times 2}(G_1')$-set can be extended to a DDS of G by adding to it both u and v and the two vertices at distance 2 from v in G_2, and so $\gamma_{\times 2}(G) \leq 3(n - 6)/4 + 4 < 3n/4$, a contradiction. Hence, $\gamma_{\times 2}(G_1) \geq 3n_1/4$. Similarly, $\gamma_{\times 2}(G_2) \geq 3n_2/4$. Hence, by Observation 13, $G_i \in \{C_5, C_8\} \cup \mathcal{G}$ for $i = 1, 2$.

If $G_i \in \{C_5, C_8\}$ for $i = 1, 2$, then $\gamma_{\times 2}(G) < 3n/4$, a contradiction. Hence we may assume $G_1 \in \mathcal{G}$. Let D_1 be a $\gamma_{\times 2}(G_1)$-set that contains the vertex u (such a set exists by Observation 9). If now $G_2 \in \{C_5, C_8\}$, then D_1 can be extended to a DDS of G by adding to it $\gamma_{\times 2}(G_2) - 1$ vertices of G_2, and so $\gamma_{\times 2}(G) < 3n/4$, a contradiction. Hence, $G_2 \in \mathcal{G}$.

Suppose $G \notin \mathcal{G}$. Then we may assume that G_1 contains at least two units and that u is not a link vertex of G_1. By Observation 12, u is the vertex at distance 2 from the link vertex in its unit in G_1 (for otherwise the edge joining u and the link vertex in its unit does not satisfy Observation 12). But then the set $\{u, v\}$ can easily be extended to a DDS of G that contains two vertices from the unit of G_1 containing u (namely, u and the link vertex of the unit) and three vertices from every other unit of G_1 and G_2, whence $\gamma_{\times 2}(G) < 3n/4$, a contradiction. Hence, $G \in \mathcal{G}$.

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Received 25 August 2003
Revised 20 May 2004